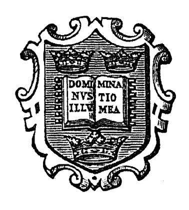
A TREATISE

ON

ANALYTICAL MECHANICS.

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ON

ANALYTICAL MECHANICS.

BY

BARTHOLOMEW PRICE, M.A., F.R.S., F.R.A.S.,

SEDLEIAN PROFESSOR OF NATURAL PHILOSOPHY, OXFORD.

VOL. II.

DYNAMICS OF A MATERIAL SYSTEM.

SECOND EDITION.

Oxford:

AT THE CLARENDON PRESS.

M.DCCC.LXXXIX.

A TREATISE

ON

INFINITESIMAL CALCULUS;

CONTAINING

DIFFERENTIAL AND INTEGRAL CALCULUS,
CALCULUS OF VARIATIONS, APPLICATIONS TO ALGEBRA AND GEOMETRY,
AND ANALYTICAL MECHANICS.

BY

BARTHOLOMEW PRICE, M.A., F.R.S., F.R.A.S.,
SEDLEIAN PROFESSOR OF NATURAL PHILOSOPHY, OXFORD.

VOL. IV.

DYNAMICS OF MATERIAL SYSTEMS.

SECOND EDITION.

"Les progrès de la science ne sont vraiment fructueux, que quand ils amènent aussi le progrès des Traités élémentaires."—CH. DUPIN.

Oxford:
AT THE CLARENDON PRESS.
M.DCCC.LXXXIX.

PREFACE TO THE FIRST EDITION.

ALTHOUGH I have entitled the present Volume, "Dynamics of Material Systems;" yet the investigations contained in it are far from comprising all which a complete treatise on that subject requires. They are indeed almost wholly confined to those particular systems in which the internal forces, brought into action, either effectively or potentially, by means of the external forces, enter in equal and opposite pairs; so that they disappear in the equations of motion formed on D'Alembert's principle. I say almost wholly, because, in the last Chapter but one of the Volume, the motion of the particles of an elastic body is to a certain extent discussed: and herein the elastic forces, which are internal forces, do not disappear, but enter as effective forces, the action of which is determined by Hooke's law, or by an equivalent assumption of a property of such matter. In all other cases, in a rigid body, in a rigid system which is maintained in a state of relative rest by rigid rods and similar modes of constraint, in systems wherein every mutual action of attraction or repulsion is accompanied by an equal

and opposite reaction, the internal forces disappear from the equations of motion.

The expediency, nay, almost the necessity, of giving a geometrical image to complicated mechanical laws demanded the insertion of a preliminary Chapter, which should contain the required geometrical theorems. The importance of familiarity with the symbols of this Chapter and their symmetrical manipulation, with the linear and angular directions, with the geometrical forms, which being in tridimensional space are difficult of imagination, cannot be over-estimated. A system of notation has also been hereby obtained, and this is preserved uniformly throughout the Treatise.

The motion of a system is of course more complicated than that of a single particle, and thus greater prominence has been given to the distinction between kinematics and dynamics than was necessary in the preceding Volume. A force surely cannot be perfectly apprehended as to its effects on the motion of a system, unless the effects have been previously examined, and, I may say, examined in all their generality. Hence arises the importance of the Chapter on kinematics, in which it is shewn that the most general motion of a rigid system is compounded of, and may be resolved into, a translation of any particle and a rotation about an axis which passes through that particle. The applications of this analysis of motion in the subsequent parts of the Treatise are many and various. The Analytical Table of Contents will sufficiently indicate the course of inquiry:

it will manifest the logical sequence of the several parts; the first formation of the equations of motion by means of D'Alembert's principle: the theorems deduced from particular forms of these equations: the more general theorems and principles which they involve: the transformation of the equations of rotation into angular velocities: the consequent geometry of masses, and the theory of principal axes, and their distribution in space: the motion of a body subject to constraint either of a fixed axis or of a fixed point; that of a body perfectly free from all constraint; the theory of relative motion; and the theory of machines in motion and of *work* done thereby. I may, too, observe, that we are herein led to some curious properties of mechanical units, and to the mode of reducing all force-action to an uniform standard of mechanical work.

The concluding Chapter is the work of Mr. W. F. Donkin, M.A., F.R.S., of University College, and Savilian Professor of Astronomy, Oxford; "Theoretical Dynamics" is the subject of it; and the theory is discussed which assigns the number and the order of the differential equations of motion in the most general problem; the possibility of the solution of some or all of them; and the forms of the resulting integrals. The Lagrangian and Hamiltonian equations are investigated; and theorems, important in reference to these equations, discovered by Poisson, Jacobi, Professor Donkin himself, M. Bour, M. Liouville, are demonstrated. Perhaps no one is better able to expound the difficulties of the theory than the

accomplished mathematician who has contributed the Chapter; he has studied the subject, and has made real advances in it.

Investigations of problems of a special character are introduced more generally than in the preceding Volume. Thus the subject of the precession and nutation of the earth's axis has been considered at some length; the apparent effects due to the earth's diurnal rotation on the action of the pendulum-experiment devised by Foucault, and of the gyroscope, on the deviation of heavy bodies, whether falling freely or projected with high velocities, are discussed with considerable minuteness. This course has been to a certain extent unavoidable; because cosmical phenomena, and machines devised to illustrate them, are the most simple and most appropriate examples of general mechanical processes; consequently it has been unnecessary to devise hard problems for the purpose of exhibiting the power of the equations, when they are best illustrated by the movements in which we ourselves daily take part. It is thought also that the utility of the work is hereby increased.

The general principle on which the equations of motion are formed is the same as that which is so frequently and so prominently stated in the preceding Volume; viz. the equality of the impressed and the expressed momentum on a single particle. This principle is indeed directly applicable to the determination of the motion of a material system, only when the internal forces which act on the several molecules are taken account of; and as the nature,

the laws, and the action of these forces are generally unknown, some other mode of estimating the general results is required. If the system is so organized that the internal forces enter in equal and opposite pairs, they disappear in the equations of motion, and the circumstances are expressed without difficulty in a sufficient number of equations.

All the incidents of motion which arise out of continuous laws are expressed as infinitesimals. This is indeed the reason why the present and the preceding Volumes, in which mechanical subjects are treated, are included in a course of Infinitesimal Calculus. Infinitesimals are, as heretofore, stated and applied in their barest forms; and subject to the axiomatic properties of Art. 9, Vol. I. Infinitesimals and finite quantities are the *materies* of calculation according to the same laws. And it is submitted that continuous laws can only thus be adequately expressed symbolically.

The object of the Author has been the construction of an uniform scientific Treatise, pervaded by one idea, and applying one principle. Thus, at the outset, a certain form is given to the equations of motion by D'Alembert's principle; and in that form they are applied to all subsequent purposes and processes; for they are directly applicable to all classes of dynamical problems. In some cases indeed they are conveniently applied in the transformed state, as they are known by the name of Euler's Equations; generally however all special artifices, however ingenious they may be, and whatever abridgement of work they may

introduce, are avoided; the circumstances of a problem are resolved into their simplest elements, and these are expressed by the general equations and their integrals.

The object again is not to make new discoveries or to open new lines of research; but to use the present knowledge and the present materials; to digest, to arrange, to consolidate all into one harmonious Treatise; to make such additions as are necessary for the process; and to present all to a student on an uniform plan. The Treatise has arisen out of the want which the Author himself has frequently experienced in his professional employment; and the attempt to supply that want has given to the work its didactic character and its colloquial style.

The Author is of course under obligations to many writers on Mechanics and kindred subjects. These obligations he has attempted to acknowledge from time to time, as well as to specify the treatises wherein certain subjects have been originally or fully treated. It has however been impossible to satisfy the claims of such writers in all cases. In many cases, the Author has found, that theorems, to which he was led in the course of his investigation, had been previously discovered, and he is also bound to say, that many theorems which are attributed to certain authors have been known and proved long before the time of the writer to whom the credit is commonly given. This is a disappointment to which an inquirer in any branch of science must be liable. He will rejoice however to find that truth has ad-

vanced, although his share in the work may not be as large as he might expect. The benefit of the progress will be permanent, his disappointment will be temporary; and if he will take heed to use it aright, it will be an inducement to greater industry and further research.

PREFACE TO THE SECOND EDITION.

I HAVE little to add in the way of Preface to what has already been said in the former Edition. The Work has been corrected and revised throughout, and in some parts enlarged; but the principle and arrangement remain the same. I have seen no reason for changing these. Many friends have favoured me with criticisms, and to them I give my best thanks. Two require especial mention, viz. the late Mr. Isaac Todhunter, M.A., F.R.S., Fellow of St. John's College, Cambridge, who most carefully commented on the former edition from beginning to end, and Mr. John Purser, M.A., Professor of Mathematics in the Queen's College, Belfast, to whom I am indebted for many observations on the Problem of the Gyroscope; these have induced me to carry the investigation to higher terms than in the former edition.

OXFORD, *July* 19, 1889.

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PART III.

DYNAMICS; THE MOTION OF A MATERIAL SYSTEM.

CHAPTER I.

PRELIMINARY GEOMETRICAL INVESTIGATIONS.

ARTICLE 1.] IN following the course suggested by the nature of the science of mechanics, the subject next for discussion is the motion of a material system; that is, of a system of material particles which are related to each other by means of certain forces of attraction, tension, and such like. These will be explained hereafter. This motion I shall consider in its greatest generality, and by the light of the best processes which modern science has discovered: we shall hereby be enabled to apply our principles to problems of great interest and of practical importance, and to their solution by most elegant methods. I shall also enunciate and explain certain very general principles, which in their mathematical expression include all Dynamical problems. These will be introduced towards the close of our treatise; because I think that such and similar general propositions are more adequately apprehended, when they have been previously applied as it were piecemeal to particular problems. This is the course which I have taken heretofore, and which I shall still take, in the conviction that it is that which is best suited to a didactic treatise.

The general motion of a material system takes place in space; and is capable of determination only by means of properties of space; by means, that is, of systems of coordinates, or of some other equivalent mode of reference. It is necessary therefore for us to be prepared with a sufficient knowledge of these properties.

Moreover in the course of our treatise we shall often have occasion to translate mechanical results into analogous geometrical theorems, whereby we shall obtain a fertile interpretation

of mechanical laws from a geometrical point of view. Now although the theorems which we require may be found scattered here and there in treatises on geometry, yet it is almost necessary to collect them, that all may be seen in one conspectus. And there is also another reason for this introductory section; we shall preserve an uniform notation throughout: and this in complex formulæ is a matter of importance no less than of elegance; for when an algebraical equation or a geometrical form is difficult of comprehension, it is surely undesirable to increase the difficulty by a confused symbolism.

2.] I shall assume in my readers a knowledge of the ordinary equations of the straight line, and of the plane; and of their common properties: and I shall begin with the investigation of the transformation of coordinates from one rectangular system of axes to another rectangular system, both of which originate at the same point. Let us suppose a point P to be (x, y, z), (ξ, η, ζ) in reference to these two systems respectively; and the systems to be connected by direction-cosines indicated in the following scheme:

	x	y	z
ξ	a_1	a_2	a_3
η	b_1	b_2	b_3
ζ	c_1	c_2	c_3

(1)

so that

$$\left.\begin{aligned} x &= a_1\xi + b_1\eta + c_1\zeta, \\ y &= a_2\xi + b_2\eta + c_2\zeta, \\ z &= a_3\xi + b_3\eta + c_3\zeta; \end{aligned}\right\} \qquad (2)$$

and inversely,

$$\left.\begin{aligned} \xi &= a_1 x + a_2 y + a_3 z, \\ \eta &= b_1 x + b_2 y + b_3 z, \\ \zeta &= c_1 x + c_2 y + c_3 z. \end{aligned}\right\} \qquad (3)$$

As the systems are rectangular, we have

$$a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2 = a_3^2 + b_3^2 + c_3^2 = 1, \qquad (4)$$

$$a_1^2 + a_2^2 + a_3^2 = b_1^2 + b_2^2 + b_3^2 = c_1^2 + c_2^2 + c_3^2 = 1, \qquad (5)$$

$$a_2 a_3 + b_2 b_3 + c_2 c_3 = a_3 a_1 + b_3 b_1 + c_3 c_1 = a_1 a_2 + b_1 b_2 + c_1 c_2 = 0, \quad (6)$$

$$b_1 c_1 + b_2 c_2 + b_3 c_3 = c_1 a_1 + c_2 a_2 + c_3 a_3 = a_1 b_1 + a_2 b_2 + a_3 b_3 = 0. \quad (7)$$

From the last two equations of (6) we have,

$$\left.\begin{array}{l} a_2 a_1 + b_2 b_1 + c_2 c_1 = 0 \\ a_3 a_1 + b_3 b_1 + c_3 c_1 = 0 \end{array}\right\}; \tag{8}$$

$$\begin{aligned} \therefore \quad \frac{a_1}{b_2 c_3 - c_2 b_3} = \frac{b_1}{c_2 a_3 - a_2 c_3} &= \frac{c_1}{a_2 b_3 - b_2 a_3} \\ &= \pm \frac{\{a_1^2 + b_1^2 + c_1^2\}^{\frac{1}{2}}}{\{(b_2 c_3 - c_2 b_3)^2 + (c_2 a_3 - a_2 c_3)^2 + (a_2 b_3 - b_2 a_3)^2\}^{\frac{1}{2}}} \\ &= \pm 1. \end{aligned} \tag{9}$$

As to the double sign in the right-hand member of this last equation I would observe, that it indicates the two orders in which the positive axes of the two systems of coordinates may be taken: that is, if both are taken in the same order, when ξ and η coincide with x and y respectively, ζ will also coincide with z: but if they are taken in a contrary order, ζ will coincide with $-z$ when ξ and η coincide with x and y. Let us assume the order of the two systems to be the same: so that when the two systems are coincident the following simultaneous conditions must be satisfied:

$$\left.\begin{array}{lll} a_1 = 1, & a_2 = 0, & a_3 = 0, \\ b_1 = 0, & b_2 = 1, & b_3 = 0, \\ c_1 = 0, & c_2 = 0, & c_3 = 1; \end{array}\right\} \tag{10}$$

and for the fulfilment of these conditions the upper sign must be taken in (9): so that from (9), and similarly from the other pairs of equations of (6), we have

$$\frac{b_2 c_3 - c_2 b_3}{a_1} = \frac{c_2 a_3 - a_2 c_3}{b_1} = \frac{a_2 b_3 - b_2 a_3}{c_1} = 1, \tag{11}$$

$$\frac{b_3 c_1 - c_3 b_1}{a_2} = \frac{c_3 a_1 - a_3 c_1}{b_2} = \frac{a_3 b_1 - b_3 a_1}{c_2} = 1, \tag{12}$$

$$\frac{b_1 c_2 - c_1 b_2}{a_3} = \frac{c_1 a_2 - a_1 c_2}{b_3} = \frac{a_1 b_2 - b_1 a_2}{c_3} = 1. \tag{13}$$

These results may also be deduced from (5) and (7), which are equations inverse to (4) and (6).

3.] In the preceding Article the new system of coordinate axes of (ξ, η, ζ) is connected with the original system of (x, y, z) by means of nine direction-cosines: those, that is, of (1): these however are related by six equations of condition, either (4) and (6), or (5) and (7): so that only three of the nine direction-

cosines are independent. It is however to be observed that the transformation is thus effected by means of symmetrical linear equations; in many cases the advantage of employing such formulæ is greater than the inconvenience of introducing many variables which are not independent; but in other cases it is more convenient to introduce as few variables as possible; and I proceed therefore to explain Euler's process of transformation, in which only three new quantities are required.

Let, as heretofore, x, y, z refer to the original system, and ξ, η, ζ to the transformed system.

(1) Let the system of axes be turned about the axis of z in a positive direction through an angle ψ: see Fig. 1; and let x', y', z' be the values of x, y, z when this rotation has taken place; so that

$$\left.\begin{aligned} x &= x'\cos\psi - y'\sin\psi, \\ y &= x'\sin\psi + y'\cos\psi, \\ z &= z'. \end{aligned}\right\} \qquad (14)$$

(2) Let the system of x', y', z' be turned through an angle θ about the line ON, which is the axis of x'; and let x'', y'', z'' be the coordinates when this rotation has taken place; so that

$$\left.\begin{aligned} x' &= x'', \\ y' &= y''\cos\theta - z''\sin\theta, \\ z' &= y''\sin\theta + z''\cos\theta. \end{aligned}\right\} \qquad (15)$$

(3) Let the system of x'', y'', z'' be turned about the axis of z'' in a positive direction through an angle ϕ; and let ξ, η, ζ be the coordinates when this rotation has taken place; so that

$$\left.\begin{aligned} x'' &= \xi\cos\phi - \eta\sin\phi, \\ y'' &= \xi\sin\phi + \eta\cos\phi, \\ z'' &= \zeta. \end{aligned}\right\} \qquad (16)$$

Then by these successive transformations the system of axes will be transformed in the most general manner possible; and substituting in (14) from (15) and (16), we have

$$x = \xi(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta) + \eta(-\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta) + \zeta\sin\psi\sin\theta, \qquad (17)$$

$$y = \xi(\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta) + \eta(-\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta) - \zeta\cos\psi\sin\theta, \qquad (18)$$

$$z = \xi\sin\phi\sin\theta + \eta\cos\phi\sin\theta + \zeta\cos\theta, \qquad (19)$$

whereby the relations between the old and new coordinates are expressed in terms of three undetermined quantities θ, ϕ, and ψ.

4.] The comparison of (17), (18) and (19) with (2) indicates the following equivalences:

$$\left.\begin{aligned} a_1 &= \cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta, \\ b_1 &= -\sin\phi\cos\psi - \cos\phi\sin\psi\cos\theta, \\ c_1 &= \sin\psi\sin\theta, \end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned} a_2 &= \cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta, \\ b_2 &= -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\theta, \\ c_2 &= -\cos\psi\sin\theta, \end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned} a_3 &= \sin\phi\sin\theta, \\ b_3 &= \cos\phi\sin\theta, \\ c_3 &= \cos\theta; \end{aligned}\right\} \quad (22)$$

these equations also satisfy the conditions (4) ... (7); and from them we have

$$\cos\theta = c_3, \qquad \tan\phi = \frac{a_3}{b_3}, \qquad \tan\psi = -\frac{c_1}{c_2}; \quad (23)$$

so that the nine direction-cosines are expressed by means of three quantities θ, ϕ, ψ.

5.] In the course of our work we shall frequently require for illustration quadric surfaces, or surfaces of the second degree. The most general form of the equation of which is

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 + 2\text{D}yz + 2\text{E}zx + 2\text{F}xy + 2\text{G}x + 2\text{H}y + 2\text{J}z + \text{K} = 0: \quad (24)$$

but as we shall need only central surfaces, and these referred to the centre as the origin, and to rectangular coordinates, it is convenient to reduce (24) to the most simple form which the equation of such surfaces admits of.

Let us transform the equation to a new origin (x', y', z'); and let the new origin be the centre; then substituting $x+x'$, $y+y'$, $z+z'$ severally for x, y, z, (24) becomes

$$\begin{aligned} &\text{A}x^2 + \text{B}y^2 + \text{C}z^2 + 2\text{D}yz + 2\text{E}zx + 2\text{F}xy \\ &+ 2(\text{A}x' + \text{F}y' + \text{E}z' + \text{G})x + 2(\text{F}x' + \text{B}y' + \text{D}z' + \text{H})y \\ &\qquad\qquad + 2(\text{E}x' + \text{D}y' + \text{C}z' + \text{J})z \\ &+ \text{A}x'^2 + \text{B}y'^2 + \text{C}z'^2 + 2\text{D}y'z' + 2\text{E}z'x' + 2\text{F}x'y' \\ &\qquad\qquad + 2\text{G}x' + 2\text{H}y' + 2\text{J}z' + \text{K} = 0; \quad (25) \end{aligned}$$

as (x', y', z') is the centre, this equation is to be unaltered when

for x, y, z we substitute $-x$, $-y$, $-z$; therefore the coefficients of x, y, z must vanish; so that

$$\left.\begin{aligned} \text{A}x'+\text{F}y'+\text{E}z'+\text{G} &= 0, \\ \text{F}x'+\text{B}y'+\text{D}z'+\text{H} &= 0, \\ \text{E}x'+\text{D}y'+\text{C}z'+\text{J} &= 0; \end{aligned}\right\} \tag{26}$$

whence we have finite values for x', y', z'; unless

$$\text{ABC}-\text{AD}^2-\text{BE}^2-\text{CF}^2+2\text{DEF} = \nabla \text{ (say)} = 0, \tag{27}$$

in which case the values of x', y', z' are infinite. Let us however suppose ∇ to be finite: then the equation to the surface becomes

$$\text{A}x^2+\text{B}y^2+\text{C}z^2+2\text{D}yz+2\text{E}zx+2\text{F}xy+\text{K}' = 0; \tag{28}$$

wherein K' is the constant term, and represents the last two lines of (25): and as is evident from (26),

$$\text{K}' = \text{G}x'+\text{H}y'+\text{J}z'+\text{K}; \tag{29}$$

and if we substitute the values of x', y', z', which are determined by (26), we have

$$\begin{aligned} \text{K}'\nabla = {} & \text{G}^2(\text{D}^2-\text{BC})+\text{H}^2(\text{E}^2-\text{CA})+\text{J}^2(\text{F}^2-\text{AB}) \\ & +2\text{HJ}(\text{AD}-\text{EF})+2\text{JG}(\text{BE}-\text{FD})+2\text{GH}(\text{CF}-\text{DE})-\text{K}\nabla, \quad (30) \\ = {} & \nabla' \text{ (say)}. \quad (31) \end{aligned}$$

In passing I would observe, that ∇ is the determinant of the three equations (26), when the last terms are omitted; and that ∇' is, omitting a factor, the determinant of the four equations (26) and (29). This condition has been already determined in Ex. 3, Art. 355, Vol. I. Ed. 2.

If $\nabla = 0$, the coordinates of the centre are infinite: the surface in this case is non-central, and is a paraboloid, or one of its degenerate varieties.

If $\nabla' = 0$, the equation to the surface is

$$\text{A}x^2+\text{B}y^2+\text{C}z^2+2\text{D}yz+2\text{E}zx+2\text{F}xy = 0, \tag{32}$$

and the centre is on the surface. The surface is therefore a cone, or one of its degenerate varieties.

6.] We can further reduce the general equation to central surfaces by means of another transformation of rectangular coordinate axes. Let the centre still be the origin, and let another system of rectangular axes originate at it. Let us omit the accent on K' in (28), and for x, y, z let us substitute the values given in (2). Then (28) becomes

$$\begin{aligned}&\xi^2\{\text{A}a_1^2+\text{B}a_2^2+\text{C}a_3^2+2\text{D}a_2a_3+2\text{E}a_3a_1+2\text{F}a_1a_2\}\\&+\eta^2\{\text{A}b_1^2+\text{B}b_2^2+\text{C}b_3^2+2\text{D}b_2b_3+2\text{E}b_3b_1+2\text{F}b_1b_2\}\\&+\zeta^2\{\text{A}c_1^2+\text{B}c_2^2+\text{C}c_3^2+2\text{D}c_2c_3+2\text{E}c_3c_1+2\text{F}c_1c_2\}\\&+2\eta\zeta\{\text{A}b_1c_1+\text{B}b_2c_2+\text{C}b_3c_3\\&\qquad+\text{D}(b_2c_3+b_3c_2)+\text{E}(b_3c_1+b_1c_3)+\text{F}(b_1c_2+b_2c_1)\}\\&+2\zeta\xi\{\text{A}c_1a_1+\text{B}c_2a_2+\text{C}c_3a_3\\&\qquad+\text{D}(c_2a_3+c_3a_2)+\text{E}(c_3a_1+c_1a_3)+\text{F}(c_1a_2+c_2a_1)\}\\&+2\xi\eta\{\text{A}a_1b_1+\text{B}a_2b_2+\text{C}a_3b_3\\&\qquad+\text{D}(a_2b_3+a_3b_2)+\text{E}(a_3b_1+a_1b_3)+\text{F}(a_1b_2+a_2b_1)\}+\text{K}=0.\end{aligned} \tag{33}$$

In this equation nine direction-cosines are involved, and these thus far are subject to only six conditions; viz. (4) and (6), or (5) and (7) of Art. 2. Three other equations therefore are necessary for their complete determination; assuming the following conditions to be possible and sufficient, let us suppose the coefficients of $\eta\zeta$, $\zeta\xi$, and $\xi\eta$ in (33) to vanish: so that we have

$$\left.\begin{aligned}\text{A}b_1c_1+\text{B}b_2c_2+\text{C}b_3c_3+\text{D}(b_2c_3+b_3c_2)+\text{E}(b_3c_1+b_1c_3)+\text{F}(b_1c_2+b_2c_1)=0,\\\text{A}c_1a_1+\text{B}c_2a_2+\text{C}c_3a_3+\text{D}(c_2a_3+c_3a_2)+\text{E}(c_3a_1+c_1a_3)+\text{F}(c_1a_2+c_2a_1)=0,\\\text{A}a_1b_1+\text{B}a_2b_2+\text{C}a_3b_3+\text{D}(a_2b_3+a_3b_2)+\text{E}(a_3b_1+a_1b_3)+\text{F}(a_1b_2+a_2b_1)=0.\end{aligned}\right\} \tag{34}$$

Also let the new coefficients of ξ^2, η^2, ζ^2 in the transformed equation severally be A', B', C'; so that

$$\left.\begin{aligned}\text{A}a_1^2+\text{B}a_2^2+\text{C}a_3^2+2\text{D}a_2a_3+2\text{E}a_3a_1+2\text{F}a_1a_2&=\text{A}',\\\text{A}b_1^2+\text{B}b_2^2+\text{C}b_3^2+2\text{D}b_2b_3+2\text{E}b_3b_1+2\text{F}b_1b_2&=\text{B}',\\\text{A}c_1^2+\text{B}c_2^2+\text{C}c_3^2+2\text{D}c_2c_3+2\text{E}c_3c_1+2\text{F}c_1c_2&=\text{C}';\end{aligned}\right\} \tag{35}$$

whereby (if these equations are possible) the transformed equation is

$$\text{A}'\xi^2+\text{B}'\eta^2+\text{C}'\zeta^2+\text{K}=0. \tag{36}$$

Now the last two equations of (34) may be put into the following forms:

$$\left.\begin{aligned}(\text{A}a_1+\text{F}a_2+\text{E}a_3)c_1+(\text{F}a_1+\text{B}a_2+\text{D}a_3)c_2+(\text{E}a_1+\text{D}a_2+\text{C}a_3)c_3=0,\\(\text{A}a_1+\text{F}a_2+\text{E}a_3)b_1+(\text{F}a_1+\text{B}a_2+\text{D}a_3)b_2+(\text{E}a_1+\text{D}a_2+\text{C}a_3)b_3=0;\end{aligned}\right\} \tag{37}$$

and our hypothesis requires these to coexist with the second and third of (7), Article 2; viz. with

$$\left.\begin{aligned}a_1c_1+a_2c_2+a_3c_3=0,\\a_1b_1+a_2b_2+a_3b_3=0.\end{aligned}\right\} \tag{38}$$

But if we have two pairs of equations of the forms

$$\left.\begin{array}{l} lx+my+nz=0 \\ l\xi+m\eta+n\zeta=0 \end{array}\right\}, \qquad \left.\begin{array}{l} Lx+My+Nz=0 \\ L\xi+M\eta+N\zeta=0 \end{array}\right\},$$

from these we have

$$\frac{l}{y\zeta-\eta z}=\frac{m}{z\xi-\zeta x}=\frac{n}{x\eta-\xi y};$$

$$\frac{L}{y\zeta-\eta z}=\frac{M}{z\xi-\zeta x}=\frac{N}{x\eta-\xi y};$$

$$\therefore \quad \frac{l}{L}=\frac{m}{M}=\frac{n}{N}. \tag{39}$$

As (37) and (38) are pairs of equations of the same form as these, we have

$$\frac{Aa_1+Fa_2+Ea_3}{a_1}=\frac{Fa_1+Ba_2+Da_3}{a_2}=\frac{Ea_1+Da_2+Ca_3}{a_3} \tag{40}$$

$$=\frac{Aa_1^2+Ba_2^2+Ca_3^2+2Da_2a_3+2Ea_3a_1+2Fa_1a_2}{a_1^2+a_2^2+a_3^2} \tag{41}$$

$$=A', \text{ from } (35); \tag{42}$$

(41) being inferred from (40) by operating on the numerators and denominators of (40) severally with the factors a_1, a_2, a_3, and by adding numerators and denominators.

Similarly from the third and first, and from the first and second of (34), we have

$$\frac{Ab_1+Fb_2+Eb_3}{b_1}=\frac{Fb_1+Bb_2+Db_3}{b_2}=\frac{Eb_1+Db_2+Cb_3}{b_3}=B',$$

$$\frac{Ac_1+Fc_2+Ec_3}{c_1}=\frac{Fc_1+Bc_2+Dc_3}{c_2}=\frac{Ec_1+Dc_2+Cc_3}{c_3}=C'.$$

As these last equations are of precisely the same form as (42), let us take a type-expression of all; and assume X to be the type of A', B', C'; and t_n to be the type of a_n, b_n, c_n : so that we have the following typical form :

$$\left.\begin{array}{l} (A-X)t_1+Ft_2+Et_3=0, \\ Ft_1+(B-X)t_2+Dt_3=0, \\ Et_1+Dt_2+(C-X)t_3=0; \end{array}\right\} \tag{43}$$

now from these three equations combined with the condition

$$t_1^2+t_2^2+t_3^2=1$$

four unknown quantities are to be determined, viz. t_1, t_2, t_3, and X. These quantities we proceed to discover.

Eliminating t_1, t_2, t_3 from (43) by cross-multiplication, we have

$$(A-X)(B-X)(C-X)-D^2(A-X)-E^2(B-X)-F^2(C-X)+2DEF=0; \quad (44$$

and this is the condition of the coexistence of the three equations of (43); and is also the equation for determining X.

As (44) is a cubic in X it has three roots; and these are the values of A', B', C' which are the coefficients of ξ^2, η^2, ζ^2 in equation (36): so that

$$(A-X)(B-X)(C-X)-D^2(A-X)-E^2(B-X)-F^2(C-X)+2DEF$$
$$= (A'-X)(B'-X)(C'-X) = 0; \quad (45)$$

multiplying out each side of this equation, and equating coefficients of like terms, we have

$$\left.\begin{aligned} A'+B'+C' &= A+B+C, \\ B'C'+C'A'+A'B' &= BC+CA+AB-D^2-E^2-F^2, \\ A'B'C' &= ABC-AD^2-BE^2-CF^2+2DEF; \end{aligned}\right\} \quad (46)$$

as the left-hand members are fixed quantities, the right-hand members are Invariants. Their geometrical meaning is well known; the mechanical interpretation of certain similar expressions will be given hereafter. It will be observed that the right-hand member of the last is the determinant ∇, see (27), which is the condition of co-existence of the three partial x-, y-, z-derived functions of the equation to the surface (28).

7.] As the roots of the cubic (44) are the coefficients of ξ^2, η^2, ζ^2 in the reduced equation of the surface, and as these coefficients must be real quantities, the possibility of the preceding reduction depends on the reality of these roots; and this is demonstrated by the following process, due to Cauchy.

Let the equation to the cubic be put into the form

$$(C-X)\{(A-X)(B-X)-F^2\}-D^2(A-X)-E^2(B-X)+2DEF = 0;$$

let r_1 and r_2 be the two roots of

$$(A-X)(B-X)-F^2 = 0,$$

so that
$$r_1 = \frac{A+B}{2} + \left\{\left(\frac{A-B}{2}\right)^2 + F^2\right\}^{\frac{1}{2}},$$
$$r_2 = \frac{A+B}{2} - \left\{\left(\frac{A-B}{2}\right)^2 + F^2\right\}^{\frac{1}{2}};$$

r_1 and r_2 being evidently real quantities. In (44) let us substitute for X,

(1) $+\infty$; the result is negative;

(2) r_1 ; the result is positive;

(3) r_2 ; the result is negative;

(4) $-\infty$; the result is positive;

therefore the roots of (44) lie respectively between $+\infty$ and r_1; between r_1 and r_2; between r_2 and $-\infty$; and are all real. Thus the assumptions made in (34) are demonstrated to be legitimate; and A', B', C' are real quantities, which are determined by being the roots of the equation (44). Henceforth we shall suppose them to be known.

8.] Also from (43) another form of the cubic equation may be found, which is for many purposes more useful than (44). The several equations of (43) may be put into the following form:

$$\frac{t_1}{D}+\frac{t_2}{E}+\frac{t_3}{F} = \frac{t_1}{EF}\left\{X-A+\frac{EF}{D}\right\}$$
$$= \frac{t_2}{FD}\left\{X-B+\frac{FD}{E}\right\} = \frac{t_3}{DE}\left\{X-C+\frac{DE}{F}\right\}.$$

To simplify these expressions, let

$$A-\frac{EF}{D}=\alpha, \quad B-\frac{FD}{E}=\beta, \quad C-\frac{DE}{F}=\gamma;$$

then

$$\frac{t_1}{D}+\frac{t_2}{E}+\frac{t_3}{F} = \frac{\frac{t_1}{D}}{\frac{EF}{D(X-\alpha)}} = \frac{\frac{t_2}{E}}{\frac{FD}{E(X-\beta)}} = \frac{\frac{t_3}{F}}{\frac{DE}{F(X-\gamma)}}; \quad (47)$$

whence adding numerators and denominators, and dividing both sides of the resulting equation by $\frac{t_1}{D}+\frac{t_2}{E}+\frac{t_3}{F}$, we have

$$\frac{EF}{D(X-\alpha)}+\frac{FD}{E(X-\beta)}+\frac{DE}{F(X-\gamma)}=1; \quad (48)$$

and multiplying up,

$$(X-\alpha)(X-\beta)(X-\gamma)$$
$$-\frac{EF}{D}(X-\beta)(X-\gamma)-\frac{FD}{E}(X-\gamma)(X-\alpha)-\frac{DE}{F}(X-\alpha)(X-\beta)=0, \quad (49)$$

which is a cubic equation in X, and is the same expression as (44), but in a different form.

The reality of the roots is at once apparent: for of the three quantities α, β, γ, one must be greater than the other two, and one must be less: to fix our thoughts let us suppose $\alpha>\beta>\gamma$; so that α is the greatest and γ is the least: then if $X=\infty$, the result is positive; $X=\alpha$, the result is negative; $X=\beta$, the result is positive; $X=\gamma$, the result is negative;

consequently one root is greater than a, another root lies between a and β, and the remaining root lies between β and γ.

9.] The values of A', B', C' having been found, the direction-cosines of the corresponding radii vectores may then be found. Let us take the radius vector corresponding to A': then from (42) we have

$$\left.\begin{array}{rrrl} (A-A')a_1 + & Fa_2 & + \; Ea_3 & = 0, \\ Fa_1 & +(B-A')a_2 + & Da_3 & = 0, \\ Ea_1 & + \; Da_2 & +(C-A')a_3 & = 0; \end{array}\right\} \qquad (50)$$

and taking these equations two and two together, we have

$$\frac{a_1}{(B-A')(C-A')-D^2} = \frac{a_2}{DE-F(C-A')} = \frac{a_3}{FD-E(B-A')},$$

$$\frac{a_1}{DE-F(C-A')} = \frac{a_2}{(C-A')(A-A')-E^2} = \frac{a_3}{EF-D(A-A')},$$

$$\frac{a_1}{FD-E(B-A')} = \frac{a_2}{EF-D(A-A')} = \frac{a_3}{(A-A')(B-A')-F^2};$$

from any one of which the direction-cosines may be found. Also we have

$$\frac{{a_1}^2}{(B-A')(C-A')-D^2} = \frac{a_1 a_2}{DE-F(C-A')} = \frac{a_1 a_3}{FD-E(B-A')}$$

$$= \frac{{a_2}^2}{(C-A')(A-A')-E^2} = \frac{{a_3}^2}{(A-A')(B-A')-F^2} \quad (51)$$

$$= \frac{1}{(B-A')(C-A')+(C-A')(A-A')+(A-A')(B-A')-D^2-E^2-F^2} \quad (52)$$

$$= \frac{1}{(B'-A')(C'-A')}; \quad (53)$$

the denominators of (52) and (53) being equal, as may thus be shewn. The derived function of the discriminating cubic which is given in (45), is

$$(B-X)(C-X)+(C-X)(A-X)+(A-X)(B-X)-D^2-E^2-F^2$$
$$= (B'-X)(C'-X)+(C'-X)(A'-X)+(A'-X)(B'-X); \quad (54)$$

let $X = A'$; then the two members become the denominators of (52) and (53) respectively.

Also from (50) we have

$$a_1 : a_2 : a_3 = \frac{1}{EF-D(A-A')} : \frac{1}{FD-E(B-A')} : \frac{1}{DE-F(C-A')}; \quad (55)$$

this last system is also evident by (47).

As the principal planes of the quadric are the central planes perpendicular to the principal axes, the equation to the A-plane is

$$\frac{x}{(\text{A}'-\text{A})\text{D}+\text{EF}}+\frac{y}{(\text{A}'-\text{B})\text{E}+\text{FD}}+\frac{z}{(\text{A}'-\text{C})\text{F}+\text{DE}}=0, \quad (56)$$

and the equations to the other principal planes are similar.

We have thus found systems of symmetrical equations for determining the values of a_1, a_2, a_3 which correspond to A′. In a similar way symmetrical systems may be determined in terms of B′ and C′, whereby the corresponding values of b_1, b_2, b_3, c_1, c_2, c_3 will be found; and therefore generally as these values will be determinate, so will the position of the three lines perpendicular to each other to which A′, B′, C′ correspond be also determinate; and the equation to the surface will be of the form

$$\text{A}'\xi^2+\text{B}'\eta^2+\text{C}'\zeta^2+\text{K}=0. \quad (57)$$

This is the most simple form to which the equation of a central surface of the second degree can be reduced. The three rectangular axes to which it is referred are called principal axes. These names are specially given to those parts of the coordinate axes which are intercepted between the centre and the surface. The three planes passing through the centre, which are perpendicular to the principal axes, are called principal planes: they are the three coordinate planes of the surface (57).

10.] As the equation of a central surface of the second degree will be applied hereafter for the purpose of illustrating certain mechanical laws, it is necessary also to demonstrate other properties of principal axes and principal planes. In the first place I shall shew that the central radii vectores of these surfaces which coincide with the principal axes have critical values; that is, are maxima or minima, either totally or partially.

Let us take the equation (28) to be the equation to central surfaces; and let (x, y, z) on its surface be the extremity of a central radius vector r; then

$$r^2 = x^2+y^2+z^2;$$

and as r is to have a singular value,

$$r\,\text{D}r = x\,dx+y\,dy+z\,dz = 0: \quad (58)$$

but the differentials of these variables are connected also by the differential of (28), whereby we have

$$(\text{A}x+\text{F}y+\text{E}z)\,dx+(\text{F}x+\text{B}y+\text{D}z)\,dy+(\text{E}x+\text{D}y+\text{C}z)\,dz=0; \quad (59)$$

and therefore from (58) and (59),

$$\frac{\text{A}x+\text{F}y+\text{E}z}{x} = \frac{\text{F}x+\text{B}y+\text{D}z}{y} = \frac{\text{E}x+\text{D}y+\text{C}z}{z}. \tag{60}$$

Let l, m, n be the direction-cosines of the critical radius vector r: so that

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = r;$$

and (60) becomes

$$\frac{\text{A}l+\text{F}m+\text{E}n}{l} = \frac{\text{F}l+\text{B}m+\text{D}n}{m} = \frac{\text{E}l+\text{D}m+\text{C}n}{n}. \tag{61}$$

Now these equations are in form identical with (40); and therefore the singular radii vectores are coincident with the principal axes; that is, with those lines for which, when taken as coordinate axes, the terms in the equation to the surface involving $\eta\zeta$, $\zeta\xi$, $\xi\eta$ disappear.

Let each term of (61) be equal to s; then we have

$$\text{A}l^2+\text{B}m^2+\text{C}n^2+2\text{D}mn+2\text{E}nl+2\text{F}lm = s,$$

$$\text{and}\quad \left.\begin{aligned} (\text{A}-s)l+\text{F}m+\text{E}n &= 0,\\ \text{F}l+(\text{B}-s)m+\text{D}n &= 0,\\ \text{E}l+\text{D}m+(\text{C}-s)n &= 0;\end{aligned}\right\} \tag{62}$$

whence we have the cubic equation

$$(\text{A}-s)(\text{B}-s)(\text{C}-s)-\text{D}^2(\text{A}-s)-\text{E}^2(\text{B}-s)-\text{F}^2(\text{C}-s)+2\text{DEF}=0, \tag{63}$$

which is identical with (44): and of which therefore the three roots are real and are A', B', C', and the corresponding values of l, m, n are a_1, b_1, c_1; a_2, b_2, c_2; a_3, b_3, c_3, because the equations for the determination of the three different values of l, m, n which correspond to the three roots of (63) are the same as those by which, in the preceding Article, the direction-cosines of the principal axes have been determined.

It is also evident from the form of the equation that the three singular radii vectores are at right angles to each other. Let us take the cubic which arises from (62) in the form given in (48): and let us take the equations which correspond to B' and C'; whereby, replacing α, β, γ by their values, we have

$$\frac{\text{EF}}{\text{D}(\text{B}'-\text{A})+\text{EF}} + \frac{\text{FD}}{\text{E}(\text{B}'-\text{B})+\text{FD}} + \frac{\text{DE}}{\text{F}(\text{B}'-\text{C})+\text{DE}} - 1 = 0,$$

$$\frac{\text{EF}}{\text{D}(\text{C}'-\text{A})+\text{EF}} + \frac{\text{FD}}{\text{E}(\text{C}'-\text{B})+\text{FD}} + \frac{\text{DE}}{\text{F}(\text{C}'-\text{C})+\text{DE}} - 1 = 0;$$

and subtracting the latter from the former, we have

$$(\text{C}'-\text{B}')\left\{\frac{1}{\{\text{D}(\text{B}'-\text{A})+\text{EF}\}\,\{\text{D}(\text{C}'-\text{A})+\text{EF}\}}\right.$$
$$+\frac{1}{\{\text{E}(\text{B}'-\text{B})+\text{FD}\}\,\{\text{E}(\text{C}'-\text{B})+\text{FD}\}}$$
$$\left.+\frac{1}{\{\text{F}(\text{B}'-\text{C})+\text{DE}\}\,\{\text{F}(\text{C}'-\text{C})+\text{DE}\}}\right\}=0;$$

and by reason of the equations which are analogous to (55) this becomes

$$(\text{C}'-\text{B}')\{b_1c_1+b_2c_2+b_3c_3\} = 0;$$

and as B' is not generally equal to C', we must have

$$b_1c_1+b_2c_2+b_3c_3 = 0;$$

and therefore the two corresponding singular radii vectores are perpendicular to each other: in the same manner it may be shewn that the other singular radius vector is at right angles to each of these: so that the three form a system of rectangular axes; and if the equation to the surface is referred to them as its coordinate axes, its equation is (57).

It is also evident that the normals at the points where these principal axes meet the surface are coincident with the axes.

11.] The theory of principal axes and planes may also be derived from another property of surfaces of the second degree. I shall in the first place demonstrate that the locus of the middle points of a system of parallel chords is a plane.

Let us take (28) to be the equation to the surface; and let the equations to one of a system of parallel chords be

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}=r,$$

where (x, y, z) is a point on the surface, (x', y', z') is a point through which the chord passes, and r is the distance between these two points. For x, y, z in (28) let us substitute $x'+lr$, $y'+mr$, $z'+nr$ respectively, and let us arrange the result in terms of r; then (28) becomes

$$r^2\{\text{A}l^2+\text{B}m^2+\text{C}n^2+2\text{D}mn+2\text{E}nl+2\text{F}lm\}$$
$$+2r\{(\text{A}l+\text{F}m+\text{E}n)x'+(\text{F}l+\text{B}m+\text{D}n)y'+(\text{E}l+\text{D}m+\text{C}n)z'\}$$
$$+\text{A}x'^2+\text{B}y'^2+\text{C}z'^2+2\text{D}y'z'+2\text{E}z'x'+2\text{F}x'y'+\text{K} = 0. \qquad (64)$$

This is a quadratic equation in terms of r, and has two roots; and therefore from a point (x', y', z') two radii vectores can be drawn to a surface of the second degree along the same straight line.

Let us suppose (x', y', z') to be the middle point of the chord: then the two values of r are equal and of opposite directions and signs, and consequently the term involving the first power of r in (64) must vanish: hence we have the condition

$$(\text{A}l+\text{F}m+\text{E}n)x'+(\text{F}l+\text{B}m+\text{D}n)y'+(\text{E}l+\text{D}m+\text{C}n)z'=0. \quad (65)$$

As the chords are all parallel to each other, and to a central radius vector whose equations are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}, \quad (66)$$

l, m, n are constant: therefore (65) is the equation to a plane passing through the centre, of which the current coordinates are x', y', z': and therefore the middle points of a system of parallel chords is a plane passing through the centre of the surface.

The plane and the line whose equations are (65) and (66) respectively are called relatively to each other a conjugate plane and a conjugate diameter.

Now it is evident that generally a diameter will not be perpendicular to its conjugate plane. Let us examine whether this relation between them is ever possible; and, if so, the circumstances under which it may exist.

If (66) is perpendicular to the plane (65),

$$\begin{aligned}\frac{\text{A}l+\text{F}m+\text{E}n}{l}=\frac{\text{F}l+\text{B}m+\text{D}n}{m}&=\frac{\text{E}l+\text{D}m+\text{C}n}{n}\\ &=\text{A}l^2+\text{B}m^2+\text{C}n^2-2\text{D}mn+2\text{E}nl+2\text{F}lm\\ &=s, \qquad (67)\end{aligned}$$

where s is the coefficient of r^2 in (64).

As these equations are identical with (61), they involve similar conclusions.

There are therefore three diameters which are respectively perpendicular to their conjugate planes; and these are the principal diameters, and their conjugate planes are the principal planes of the surface.

We have thus considered the properties of principal axes under three different aspects: (1) if the surface is referred to them as coordinate axes, its equation takes the form (36), and has no term containing the products of the variables: (2) they are critical radii vectores: (3) they are diameters which are perpendicular to their conjugate planes.

Principal axes may be defined by either one of these properties; and all three mutually involve each other; and are in fact identical in the geometrical conception of infinitesimals.

12.] Let us next consider those cases in which the roots of the cubic equation (44) have particular values.

(1) Let two roots be equal: say, let $A' = B'$; then (54), which is the derived function of (44), also vanishes when $X = A' = B'$; and we have

$$(B-A')(C-A')+(C-A')(A-A')+(A-A')(B-A')-D^2-E^2-F^2 = 0. \quad (68)$$

Also as $a_1, a_2, a_3, b_1, b_2, b_3$ cannot be infinite, by reason of (52) we must have

$$(B-A')(C-A')-D^2 = (C-A')(A-A')-E^2 = (A-A')(B-A')-F^2 = 0; \quad (69)$$

that is,
$$A-A' = \frac{EF}{D}, \qquad B-A' = \frac{FD}{E}, \qquad C-A' = \frac{DE}{F}; \quad (70)$$

and consequently,

$$A' = A-\frac{EF}{D} = B-\frac{FD}{E} = C-\frac{DE}{F}; \quad (71)$$

in which case the direction-cosines a_1, a_2, a_3, and similarly the direction-cosines b_1, b_2, b_3 are indeterminate; but c_1, c_2, c_3, which correspond to the unequal root C', are determinate as heretofore. In this case the equation to the surface is

$$A'(\xi^2+\eta^2)+C'\zeta^2+K = 0. \quad (72)$$

The principal axis of ζ being determinate, any two axes in the plane of (ξ, η) perpendicular to each other are the other principal axes. Equation (72), in this case, represents a surface of revolution, whose axis of revolution is the ζ-axis. If A' were the unequal root of (44), and $B' = C'$, then the reduced equation would be

$$A'\xi^2+C'(\eta^2+\zeta^2)+K = 0, \quad (73)$$

which also represents a surface of revolution: the axis of ξ being the determinate axis and the axis of revolution of the surface, the position of the other two axes being indeterminate.

It will be observed that (70) are affected with an ambiguity of sign; but we have taken the only one possible, as otherwise the direction-cosines as given in (53) and (55) would not be indeterminate.

(2) Let all the roots of (44) be equal: that is, let $A' = B' = C'$: then taking the A'-differential of (69), we have

$$2A' = B+C = C+A = A+B;$$

$$\therefore \quad A = B = C = A' = B' = C', \tag{74}$$

and
$$\therefore \quad D = E = F = 0. \tag{75}$$

Thus all the direction-cosines are indeterminate, and any system of rectangular axes originating at the centre is a system of principal axes. And the equation to the surface is

$$\xi^2 + \eta^2 + \zeta^2 + \frac{K}{A'} = 0; \tag{76}$$

and if $K = -a^2 A'$, this is the equation to a sphere whose radius is a.

(3) Let one root of (44) be zero; say, $C' = 0$; then we have

$$ABC - AD^2 - BE^2 - CF^2 + 2DEF = 0; \tag{77}$$

and the equation to the surface becomes

$$A'\xi^2 + B'\eta^2 + K = 0; \tag{78}$$

and therefore $\zeta = \frac{0}{0}$. As (77) is the same expression as (27), the centre of the surface is at an infinite distance. Also (78) is the equation of a central conic in the plane of (ξ, η); therefore the surface is a cylinder whose axis is the axis of ζ, and whose trace on the plane of (ξ, η) is the conic (78).

(4) Let two roots of (44) be zero; say, $B' = C' = 0$; then, besides the condition (77), we have

$$BC + CA + AB - D^2 - E^2 - F^2 = 0; \tag{79}$$

and the equation to the surface becomes

$$A'\xi^2 + K = 0; \tag{80}$$

and as $\eta = \zeta = \frac{0}{0}$, it represents two planes parallel to the plane of (η, ζ).

(5) Let all the roots of (44) be zero; so that $A' = B' = C' = 0$; then the equation to the surface becomes

$$K = 0; \tag{81}$$

which represents a plane at an infinite distance.

13.] We may express these several equations in a more convenient form.

If ∇ does not vanish, but if $\nabla' = 0$, in which case the surface is central, and the constant K in the reduced equation (57) vanishes, then we have

$$A'\xi^2 + B'\eta^2 + C'\zeta^2 = 0; \tag{82}$$

and if A', B', C' are all positive, the only values of ξ, η, ζ which

satisfy the equation are $\xi = \eta = \zeta = 0$; that is, the equation represents a point at the origin.

If one coefficient, say C', is negative, and A' and B' are positive, then if

$$\text{A}' = \frac{1}{a^2}, \qquad \text{B}' = \frac{1}{b^2}, \qquad \text{C}' = -\frac{1}{c^2},$$

(82) becomes
$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 0; \tag{83}$$

which is the equation to an elliptical cone, all plane sections of it perpendicular to the axis of ζ being ellipses; and if $a = b$, the surface is a right circular cone, whose axis of revolution is the axis of ζ.

If however ∇' does not vanish, the equation to the surface is

$$\text{A}'\xi^2 + \text{B}'\eta^2 + \text{C}'\zeta^2 + \text{K} = 0. \tag{84}$$

If A', B', C' and K are all positive, the equation does not admit of geometrical interpretation. Let us however assume K to be negative: so that with obvious substitutions, and with all the varieties of sign which the quantities admit of, the equation may take either of the forms,

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1, \tag{85}$$

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 1, \tag{86}$$

$$\frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 1; \tag{87}$$

which severally represent an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. Let us assume $a > b > c$; then of (85) degenerate species are, (1) an oblate spheroid, when $a = b$; (2) a prolate spheroid, when $b = c$; (3) a sphere, when $a = b = c$. And if $a = b$, the hyperboloids of revolution are particular species of (86) and (87). We have not however space or occasion to enter into all these particulars, or into the nature and forms of the surfaces. This information must be obtained from treatises wherein the properties of these surfaces are specially treated of. The omission of this and similar matter is necessarily incidental to preliminary chapters which must be incomplete. It is our intention to demonstrate for the most part only those geometrical theorems which will be required in the sequel; not because other theorems are in themselves unimportant, but because the interpretation of our mechanical results will not require them.

14.] I will first take the cone whose equation is (83); but will, for convenience of symbols, use x, y, z for the coordinates of any point on its surface; so that its equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0; \tag{88}$$

and I shall observe with respect to it that the vertex is its centre, and that, like other surfaces of the second order which are represented by (84), it has three principal axes; which are respectively the axis of the cone, commonly so called, or, as we may call it, the internal axis; and two lines through the vertex which are respectively parallel to the major and minor axes of those elliptic sections whose planes are perpendicular to the axis of the cone; these are called the external axes.

The plane which contains the internal axis and the major axis of the principal elliptic sections is the plane of greatest section of the cone; and that which contains the internal axis and the minor axis of the elliptic sections is the plane of least section. The principal axes of a cone of the second degree, when its equation is given in the most general form, are determined by the process of Art. 6; for this is applicable to equation (28) whether $\kappa'=0$ or not.

15.] At the vertex of the cone (88) let straight lines be drawn perpendicular to the tangent planes, these will all lie in a second cone which is called the supplementary or reciprocal cone; its equation may thus be found. The equation to the tangent plane of (88) is

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}-\frac{z\zeta}{c^2}=0; \tag{89}$$

and the equations to the line through the vertex perpendicular to it are

$$\frac{a^2\xi}{x}=\frac{b^2\eta}{y}=-\frac{c^2\zeta}{z}; \tag{90}$$

therefore squaring these, and multiplying the terms of (88) respectively by them, we have

$$a^2\xi^2+b^2\eta^2-c^2\zeta^2=0; \tag{91}$$

which is the equation required, and represents a cone of the second degree, which has the same internal axis as (88), but whose major and minor external axes are respectively the minor and major external axes of (88).

For the construction of the second cone (91) it is evident that

as every generating line is perpendicular to a tangent plane of (88), so is every tangent plane of (91) perpendicular to a generating line of (88). Thus the cones have corresponding or reciprocal properties, and to a tangent plane of the first cone and to its line of contact correspond a generating line of the second and the tangent plane along that line.

Again, to the first cone let two tangent planes be drawn, and let the corresponding lines on the second cone be taken; the plane containing these lines is perpendicular to the line of intersection of the two tangent planes of the first cone; and the tangent planes of the second cone along these lines are perpendicular to the lines on the first cone; and their line of intersection is perpendicular to the plane through the lines of contact on the first cone, so that to a line and its polar plane on the first cone correspond a plane and its polar relatively to the second cone.

It is evident therefore that properties relative to the angles contained by certain planes and right lines on the first cone will give rise to properties of corresponding right lines and planes in the second cone; in other words the properties of cones of the second degree are double; and the principle of duality is applicable to them.

16.] The plane sections of the cone whose equation is (88) are generally conics, and may be either elliptical, parabolic, or hyperbolic: they may, however, in certain cases be circular, as we proceed to prove. The planes of circular section are called cyclic planes.

Let us suppose a^2 to be greater than b^2; the relative magnitude of each of these to c^2 is of no importance: and let (88) be put into the form

$$\frac{x^2+y^2+z^2}{a^2}+y^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)-z^2\left(\frac{1}{a^2}+\frac{1}{c^2}\right)=0, \qquad (92)$$

which may be put into the form

$$x^2+y^2+z^2+\left\{\frac{y}{b}(a^2-b^2)^{\frac{1}{2}}-\frac{z}{c}(a^2+c^2)^{\frac{1}{2}}\right\}\left\{\frac{y}{b}(a^2-b^2)^{\frac{1}{2}}+\frac{z}{c}(a^2+c^2)^{\frac{1}{2}}\right\}=0. \qquad (93)$$

$$\left.\begin{array}{ll}\text{Let} & \frac{y}{b}(a^2-b^2)^{\frac{1}{2}}-\frac{z}{c}(a^2+c^2)^{\frac{1}{2}}=k,\\ \text{then} & x^2+y^2+z^2+k\left\{\frac{y}{b}(a^2-b^2)^{\frac{1}{2}}+\frac{z}{c}(a^2+c^2)^{\frac{1}{2}}\right\}=0;\end{array}\right\} \qquad (94)$$

of which the former represents a plane parallel to the x-axis,

and making with the plane of (x, y) an angle whose tangent is $\frac{c}{b}\left\{\frac{a^2-b^2}{a^2+c^2}\right\}^{\frac{1}{2}}$; and the latter represents a sphere passing through the origin, whose centre is in the plane of (y, z), and whose radius $= \frac{ka}{2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)^{\frac{1}{2}}$; the section is consequently a circle, and the plane whose equation is

$$\frac{y}{b}(a^2-b^2)^{\frac{1}{2}} - \frac{z}{c}(a^2+c^2)^{\frac{1}{2}} = k \tag{95}$$

is a cyclic plane of the cone (88). As k is indeterminate, (95) represents a series of parallel planes, all being parallel to the plane

$$\frac{y}{b}(a^2-b^2)^{\frac{1}{2}} - \frac{z}{c}(a^2+c^2)^{\frac{1}{2}} = 0, \tag{96}$$

which contains the x-axis.

Similarly in (93), if $\frac{y}{b}(a^2-b^2)^{\frac{1}{2}} + \frac{z}{c}(a^2+c^2)^{\frac{1}{2}} = k'$,
we have

$$\left.\begin{array}{l} \\ x^2+y^2+z^2+k'\left\{\frac{y}{b}(a^2-b^2)^{\frac{1}{2}} - \frac{z}{c}(a^2+c^2)^{\frac{1}{2}}\right\} = 0: \end{array}\right\} \tag{97}$$

which also represent a plane and a sphere; so that the resulting section is also a circle. The plane is parallel to the x-axis, and makes with the plane of (x, y) an angle whose tangent is $-\frac{c}{b}\left\{\frac{a^2-b^2}{a^2+c^2}\right\}^{\frac{1}{2}}$; and the sphere passing through the origin has its centre in the plane of (y, z), and its radius equal to $\frac{k'a}{2}\left(\frac{1}{b^2}+\frac{1}{c^2}\right)^{\frac{1}{2}}$: also as k' is indeterminate, there is a series of parallel planes, each of which is a cyclic plane, and all are parallel to the plane whose equation is

$$\frac{y}{b}(a^2-b^2)^{\frac{1}{2}} + \frac{z}{c}(a^2+c^2)^{\frac{1}{2}} = 0. \tag{98}$$

Hence there are two series of cyclic planes, parallel to the x-axis, equally inclined to the plane of (x, y), and lying on opposite sides of the y-axis. Hence also the line of intersection of two cyclic planes, one being of each system, is parallel to the major axis of that section of the cone which is perpendicular to the internal axis.

Now if through the vertex of the cone (88) two lines are drawn perpendicular to these cyclic planes, as the line of intersection of the cyclic planes is perpendicular to the plane of least section of (88), so will these lines lie in the plane of the greatest

section of (91); and because every plane perpendicular to one of these right lines cuts the reciprocal cone in a conic, one of whose foci is on this right line, these lines are called focal lines. The analytical proof of this property of focal lines is contained in the preceding equations; a geometrical proof will be found in the Memoir of M. Chasles, entitled, "Sur les propriétés des cones du second degré," and contained in the VIth volume of the Memoirs of the Royal Academy of Brussels.

17.] The following are properties of the ellipsoid which will be required hereafter for illustration.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{99}$$

where $a^2>b^2>c^2$; so that a is the greatest, and c is the least of all central radii vectores; b, however, has also a critical value of a peculiar kind, for it is the semi-axis minor of the elliptic trace in the plane of (x, y), and for that plane is a minimum, but it is the semi-axis major of the elliptic trace in the plane of (y, z), and for that plane is a maximum; we shall immediately determine the limits of these plane elliptical sections through the b-axis, for which b has a maximum or a minimum value respectively; these will be the cyclic planes of the ellipsoid; and the surface will be by them divided into four parts, two equal and opposite pairs of parts, each part possessing distinct properties of its own. This division will, as shewn hereafter, indicate very important mechanical properties.

All plane sections of the ellipsoid are evidently closed curves of the second degree, and accordingly ellipses; these may however in certain cases be circles, as we propose to prove: the planes of such circular sections are called cyclic planes.

Let the equation to the ellipsoid be put into the form

$$\frac{x^2+y^2+z^2}{b^2}-x^2\left(\frac{1}{b^2}-\frac{1}{a^2}\right)+z^2\left(\frac{1}{c^2}-\frac{1}{b^2}\right)=1, \tag{100}$$

which may be expressed as $\qquad x^2+y^2+z^2-b^2$

$$+\left\{\frac{z}{c}(b^2-c^2)^{\frac{1}{2}}-\frac{x}{a}(a^2-b^2)^{\frac{1}{2}}\right\}\left\{\frac{z}{c}(b^2-c^2)^{\frac{1}{2}}+\frac{x}{a}(a^2-b^2)^{\frac{1}{2}}\right\}=0; \tag{101}$$

$$\left.\begin{aligned} &\text{let} \quad \frac{z}{c}(b^2-c^2)^{\frac{1}{2}}-\frac{x}{a}(a^2-b^2)^{\frac{1}{2}}=k, \\ &\text{then} \quad x^2+y^2+z^2-b^2+k\left\{\frac{z}{c}(b^2-c^2)^{\frac{1}{2}}+\frac{x}{a}(a^2-b^2)^{\frac{1}{2}}\right\}=0, \end{aligned}\right\} \tag{102}$$

of which the former represents a plane, and the latter a sphere: consequently in combination they express a circle, which is the circle of intersection of the ellipsoid by the plane; the plane is parallel to the y-axis, and makes with the plane of (x, y) an angle whose tangent is $\frac{c}{a}\left(\frac{a^2-b^2}{b^2-c^2}\right)^{\frac{1}{2}}$; the centre of the sphere is in the plane of (z, x), and its radius $= \frac{b}{2ac}\{k^2(a^2-b^2)+4a^2c^2\}^{\frac{1}{2}}$. Consequently the plane whose equation is

$$\frac{z}{c}(b^2-c^2)^{\frac{1}{2}} - \frac{x}{a}(a^2-b^2)^{\frac{1}{2}} = k \tag{103}$$

is a cyclic plane of the ellipsoid (99). As k is indeterminate, (103) represents a series of parallel planes, all being parallel to the plane

$$\frac{z}{c}(b^2-c^2)^{\frac{1}{2}} - \frac{x}{a}(a^2-b^2)^{\frac{1}{2}} = 0, \tag{104}$$

which contains the y-axis, that is, the mean principal axis of the ellipsoid.

Similarly in (101), if

$$\left.\begin{aligned} &\frac{z}{c}(b^2-c^2)^{\frac{1}{2}} + \frac{x}{a}(a^2-b^2)^{\frac{1}{2}} = k', \\ &\text{we have} \\ &x^2+y^2+z^2-b^2+k'\left\{\frac{z}{c}(b^2-c^2)^{\frac{1}{2}} - \frac{x}{a}(a^2-b^2)^{\frac{1}{2}}\right\} = 0, \end{aligned}\right\} \tag{105}$$

of which the former represents a plane and the latter a sphere: consequently in combination they express a circle which is the circle of intersection of the ellipsoid by the plane; the plane is parallel to the y-axis, and makes with the plane of (x, y) an angle whose tangent is $-\frac{c}{a}\left(\frac{a^2-b^2}{b^2-c^2}\right)^{\frac{1}{2}}$: the centre of the sphere is in the plane of (z, x) and its radius is

$$\frac{b}{2ac}\{k'^2(a^2-b^2)+4a^2c^2\}^{\frac{1}{2}}.$$

Also, as k' is indeterminate, there is a series of parallel planes, each of which is a cyclic plane, and all are parallel to the plane whose equation is

$$\frac{z}{c}(b^2-c^2)^{\frac{1}{2}} + \frac{x}{a}(a^2-b^2)^{\frac{1}{2}} = 0. \tag{106}$$

Hence there are two systems of cyclic planes, each system being of parallel planes, parallel to the y-axis of the ellipsoid, equally inclined to the plane of (x, y), and lying on opposite sides of the

z-axis; and if θ is the angle between the planes and the plane of (x, y),

$$\tan\theta = \pm\frac{c}{a}\left(\frac{a^2-b^2}{b^2-c^2}\right)^{\frac{1}{2}} \tag{107}$$

$$= \pm\left\{\frac{\text{B}'-\text{A}'}{\text{C}'-\text{B}'}\right\}^{\frac{1}{2}}, \tag{108}$$

according to the notation of equation (84).

If in (102) $k = 0$, $x^2+y^2+z^2 = b^2$, and the cyclic plane contains the b-axis, and the radius of the circular section $= b$; the results are also similar, if $k' = 0$: so that in these circumstances the mean principal axis of the ellipsoid is the radius of the circle, and the cyclic planes are related to the ellipsoid in the manner indicated by the lines in Fig. 2, where

$$\text{UOA} = \text{U}'\text{OA}' = \theta; \quad \text{OU} = \text{OU}' = b.$$

Thus all the central radii vectores in these two cyclic planes are equal; and are equal to the mean semi-axis of the ellipsoid. For all parts of the surface of the ellipsoid contained between these two planes towards the maximum axis, the radii vectores are greater than b; and for all parts towards the minimum axis, the radii vectores are less than b; these cyclic planes therefore divide the ellipsoid into four parts, corresponding to two of which at B b is a minimum, and corresponding to the other two it is a maximum.

If planes are drawn touching the ellipsoid and parallel to the cyclic planes, the indicatrix at each point of contact will be a circle, and the four points of contact will be umbilics.

The systems of cyclic planes thus determined are the only ones; for if we operate on the equation (99) with respect to the a- or to the c-axis in the same way as we have operated with respect to the b-axis, the results are impossible.

Hence there are two and only two systems of cyclic planes, and only four umbilics on the ellipsoid.

The centres of the circles lie on the lines drawn from the umbilics to the centre.

Hence also in two ways, and in no more than two ways, can an ellipsoid be generated by the motion of a circle.

18.] Let us in the next place investigate the lengths and the direction-cosines of the maximum and minimum radii vectores of a central plane section of an ellipsoid, with the object of determining certain properties which depend on them.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \tag{109}$$

and let the plane of section be

$$lx+my+nz=0, \tag{110}$$

where $l^2+m^2+n^2=1$: and let r be the critical radius vector lying in this plane, so that

$$x^2+y^2+z^2=r^2. \tag{111}$$

Then differentiating these, we have

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0, \tag{112}$$

$$\left.\begin{aligned} l\,dx+m\,dy+n\,dz &= 0, \\ x\,dx+y\,dy+\ z\,dz &= r\,dr=0: \end{aligned}\right\} \tag{113}$$

hence, multiplying the second equation by λ, the third by μ, adding and equating to zero the several coefficients of dx, dy, dz, we have

$$\frac{x}{a^2}+\lambda l+\mu x=0=\frac{y}{b^2}+\lambda m+\mu y=\frac{z}{c^2}+\lambda n+\mu z. \tag{114}$$

Multiplying these by x, y, z respectively and adding, we have

$$1+\mu r^2=0, \text{ and } \mu=-\frac{1}{r^2}; \tag{115}$$

$$\therefore\quad x=\frac{\lambda l a^2 r^2}{a^2-r^2},\qquad y=\frac{\lambda m b^2 r^2}{b^2-r^2},\qquad z=\frac{\lambda n c^2 r^2}{c^2-r^2}; \tag{116}$$

whence, multiplying these severally by l, m, n, and adding, we have from (110), omitting the factor λr^2,

$$\frac{a^2 l^2}{a^2-r^2}+\frac{b^2 m^2}{b^2-r^2}+\frac{c^2 n^2}{c^2-r^2}=0; \tag{117}$$

which is a quadratic equation in terms of r^2, and accordingly gives two critical values: let these be r_1^2 and r_2^2; then

$$r_1^2+r_2^2=\frac{a^2l^2(b^2+c^2)+b^2m^2(c^2+a^2)+c^2n^2(a^2+b^2)}{a^2l^2+b^2m^2+c^2n^2}, \tag{118}$$

$$r_1^2 r_2^2=\frac{a^2b^2c^2}{a^2l^2+b^2m^2+c^2n^2}: \tag{119}$$

r_1 and r_2 are the principal semi-axes of the elliptic section.

These lines are at right angles to each other. To prove this, let α_1, β_1, γ_1 be the direction-angles of r_1, and α_2, β_2, γ_2 the direction-angles of r_2: then from (116),

$$\cos\alpha_1=\frac{\lambda l a^2 r_1}{a^2-r_1^2},\quad \cos\beta_1=\frac{\lambda m b^2 r_1}{b^2-r_1^2},\quad \cos\gamma_1=\frac{\lambda n c^2 r_1}{c^2-r_1^2}; \tag{120}$$

$$\cos\alpha_2 = \frac{\lambda l a^2 r_2}{a^2 - r_2^2}, \quad \cos\beta_2 = \frac{\lambda m b^2 r_2}{b^2 - r_2^2}, \quad \cos\gamma_2 = \frac{\lambda n c^2 r_2}{c^2 - r_2^2}; \tag{121}$$

$\therefore \quad \cos\alpha_1 \cos\alpha_2 + \cos\beta_1 \cos\beta_2 + \cos\gamma_1 \cos\gamma_2$

$$= \lambda^2 r_1 r_2 \left\{ \frac{a^4 l^2}{(a^2 - r_1^2)(a^2 - r_2^2)} + \frac{b^4 m^2}{(b^2 - r_1^2)(b^2 - r_2^2)} + \frac{c^4 n^2}{(c^2 - r_1^2)(c^2 - r_2^2)} \right\}$$

$$= \lambda^2 \frac{r_1 r_2}{r_1^2 - r_2^2} \left\{ \frac{a^2 l^2 r_1^2}{a^2 - r_1^2} + \frac{b^2 m^2 r_1^2}{b^2 - r_1^2} + \frac{c^2 n^2 r_1^2}{c^2 - r_1^2} - \frac{a^2 l^2 r_2^2}{a^2 - r_2^2} - \frac{b^2 m^2 r_2^2}{b^2 - r_2^2} - \frac{c^2 n^2 r_2^2}{c^2 - r_2^2} \right\}$$

$= 0$, by reason of (117);

therefore r_1 and r_2 are at right angles to each other.

Hence also as the area of the section $= \pi r_1 r_2$,

$$\text{the area} = \frac{\pi abc}{\{a^2 l^2 + b^2 m^2 + c^2 n^2\}^{\frac{1}{2}}} = \frac{\pi abc}{p}, \tag{122}$$

if p is the length of the perpendicular from the centre on either of the tangent planes which are parallel to the plane of section (110).

Hence the area is the same for all sections for which p is constant.

19.] If from the centre of the ellipsoid lines are drawn at right angles to central planes, and lengths are taken on them from the centre equal to the principal semi-axes of the sections made by the planes, the extremities of these lines will form a locus surface, which have been called by Professor Maccullagh the apsidal surface of the ellipsoid. Its equation may be found immediately from the preceding investigation.

Let the central plane be that given in (110), so that the equations to the line passing through the centre and perpendicular to it are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}, \tag{123}$$

this being the line along which the lengths r_1 and r_2, as determined by (117), are to be measured. Let (x, y, z) be the extremity of r measured along the perpendicular to the plane: then eliminating l, m, n by means of (123) and (117), we have

$$\frac{a^2 x^2}{a^2 - r^2} + \frac{b^2 y^2}{b^2 - r^2} + \frac{c^2 z^2}{c^2 - r^2} = 0, \tag{124}$$

which is the equation to the apsidal surface.

This equation may also be put into the following form: subtracting $x^2+y^2+z^2 = r^2$ from both sides of the equation, we have

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2} = 1. \qquad (125)$$

If the equation is put into an integral form, we have

$$r^2(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(c^2+a^2)y^2-c^2(a^2+b^2)z^2 + a^2b^2c^2 = 0, \qquad (126)$$

where $r^2 = x^2+y^2+z^2$.

The surface is of the fourth degree, and evidently from its mode of generation consists of two closed sheets: these sheets cut the axes of x, y, z respectively at distances b and c, c and a, a and b, from the centre; a line drawn through the centre cuts the surface in four points, except when the line is perpendicular to a cyclic plane of the ellipsoid, in which case the two sheets intersect, as all the radii vectores of the plane section are equal; they intersect in a point, and the surface takes the form of a conical cusp. There are evidently four such cuspal points, all lying in the plane of (z, x), on lines perpendicular to the cyclic planes, and consequently making with the x-axis $\pm \tan^{-1}\frac{a}{c}\left(\frac{b^2-c^2}{a^2-b^2}\right)^{\frac{1}{2}}$. Hence on the planes of (y, z) and of (x, y) the traces lie wholly one within the other; whereas on the plane of (z, x) the traces intersect at the cuspal points. The geometrical properties of the cuspal points may be investigated analytically in the ordinary way by means of the equation (124) or of (125).

Also the traces on each coordinate plane consist of a circle and an ellipse. Thus, if $x = 0$, we have

$$(y^2+z^2)(b^2y^2+c^2z^2)-b^2(c^2+a^2)y^2-c^2(a^2+b^2)z^2+a^2b^2c^2 = 0,$$

which is the same as

$$(y^2+z^2-a^2)(b^2y^2+c^2z^2-b^2c^2) = 0,$$

and is satisfied by $y^2+z^2 = a^2$, or by $b^2y^2+c^2z^2-b^2c^2 = 0$, which represent respectively a circle and an ellipse. Similar results are true for the other coordinate planes. It will be observed that in the plane of (y, z) the circle lies wholly outside the ellipse; that in the plane of (z, x) the circle intersects the ellipse at the cuspal points; and in the plane of (x, y) the circle lies wholly inside the ellipse.

The preceding processes are equally applicable to the other central quadrics, viz. the hyperboloid of one sheet and the hyperboloid of two sheets.

20.] If from a given point perpendiculars are drawn to the tangent planes of a given surface, the locus of the point of intersection of the perpendicular with the tangent plane is called the pedal of the surface with respect to that point. Let us investigate the equation to the pedal to the ellipsoid with respect to its centre, the resulting surface being the central pedal of the ellipsoid.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then the equation to the tangent plane is

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}=1,$$

and the equations to the perpendicular passing through the centre are

$$\frac{a^2\xi}{x}=\frac{b^2\eta}{y}=\frac{c^2\zeta}{z}=\xi^2+\eta^2+\zeta^2=(a^2\xi^2+b^2\eta^2+c^2\zeta^2)^{\frac{1}{2}};$$

$$\therefore\quad (\xi^2+\eta^2+\zeta^2)^2=a^2\xi^2+b^2\eta^2+c^2\zeta^2; \tag{127}$$

which is the equation to the central pedal, and represents a closed surface of the fourth degree.

21.] If r_1 and r_2 are the maximum and the minimum values of the radii vectores of the section of this surface made by the plane

$$lx+my+nz=0,$$

then, as may easily be shewn, r_1 and r_2 are the roots of the equation

$$\frac{l^2}{r^2-a^2}+\frac{m^2}{r^2-b^2}+\frac{n^2}{r^2-c^2}=0.$$

Consequently, if $r^2=x^2+y^2+z^2$, the equation to the apsidal of this central pedal surface of the ellipsoid is

$$\frac{x^2}{r^2-a^2}+\frac{y^2}{r^2-b^2}+\frac{z^2}{r^2-c^2}=0. \tag{128}$$

This surface is of the sixth order, of which however the origin is two coincident points. It consists of two sheets, and consequently a line drawn through the origin generally cuts it in four points, but in two particular positions the sheets intersect and the line cuts it in only two points. The traces on each principal plane consist of a circle and a central pedal of an ellipse. In the plane of (y, z), the radius of the circle is a, and the pedal lies wholly within it; in the plane of (z, x) the radius of the circle is b, and the pedal intersects it, so that

at this point, and similarly on the other three quadrants of this principal plane, the line drawn from the origin cuts the surface in only two points; these points are evidently cuspal points; in the plane of (x, y) the radius of the circle is c and the circle lies wholly within the pedal. This surface evidently resembles in general form that described in the preceding article.

22.] We shall also require in the sequel the application of another geometrical principle, viz. that of duality, which arises from the theory of reciprocation, as applied to central quadric surfaces, of which we take the ellipsoid to be the typical form.

Let the equation to an ellipsoid be

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}=1, \tag{129}$$

and from every point of it as a pole let the polar plane be taken relatively to the sphere

$$x^2+y^2+z^2=k^2; \tag{130}$$

the general equation of the polar plane is

$$x\xi+y\eta+z\zeta=k^2: \tag{131}$$

we propose to find the envelope of these planes; differentiating (129) and (131), we have

$$\frac{\xi}{a^2}\,d\xi+\frac{\eta}{b^2}\,d\eta+\frac{\zeta}{c^2}\,d\zeta=0,$$

$$x\,d\xi+y\,d\eta+z\,d\zeta=0;$$

whence we have
$$\frac{a^2x}{\xi}=\frac{b^2y}{\eta}=\frac{c^2z}{\zeta};$$

and eliminating ξ, η, ζ, we have

$$a^2x^2+b^2y^2+c^2z^2=k^4; \tag{132}$$

which is the equation to another ellipsoid, and is called the sphero-polar reciprocal of (129).

Now it is evident that a tangent plane of (132) corresponds to a point of (129); and to the intersection of two tangent planes of (132) corresponds a line passing through the two corresponding points of (129); and to a point on (132) corresponds a tangent plane of (129). Also to a tangent line of (132) corresponds a tangent line of (129). These surfaces therefore have reciprocal properties, and to a plane a line and a point on either, a point a line and a plane on the other severally correspond, so that all properties admit of being doubled. It is manifest that the theory of the reciprocal cone which has been explained in Art. 15 is a particular case of this principle.

I may also observe that the sphero-polar reciprocal of any quadric is also another quadric; but as we shall require only the simple form which has just been discussed, the more general case may be omitted.

23.] It is necessary also for us to investigate the relations which exist between an axis and its conjugate plane relatively to the ellipsoid. We have already investigated the condition which generally exists between a central radius vector and the plane which bisects all chords of the ellipsoid which are parallel to that radius vector; but some further properties of axes in conjugate relations to each other will be required in the sequel.

Let the equations to a radius vector be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}; \tag{133}$$

then, by reason of equation (65), the equation to its conjugate plane is

$$\frac{lx}{a^2} + \frac{my}{b^2} + \frac{nz}{c^2} = 0; \tag{134}$$

and therefore if the equation to a central plane is

$$\text{L}x + \text{M}y + \text{N}z = 0, \tag{135}$$

the equations to the axis conjugate to it are

$$\frac{x}{a^2\text{L}} = \frac{y}{b^2\text{M}} = \frac{z}{c^2\text{N}}. \tag{136}$$

If (x', y', z') is the point where the central radius vector cuts the surface, then the equation to the conjugate plane is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 0, \tag{137}$$

which is evidently the plane parallel to that which touches the ellipsoid at (x', y', z'); so that if a tangent plane be drawn to an ellipsoid at a given point, the central plane parallel to that plane is conjugate to the axis drawn to the point of contact.

Now if three axes of an ellipsoid are such that each is the axis conjugate to the plane which contains the other two, these lines form a system of conjugate axes. And if three planes are such that the line of intersection of any two is the conjugate axis of the third, these planes form a system of conjugate planes. Of such systems we have already had an instance in the principal axes and the principal planes. Let us determine the relations which exist generally between these lines and planes.

Let (x_1, y_1, z_1) (x_2, y_2, z_2) (x_3, y_3, z_3) be the three points on the ellipsoid to which the system of conjugate axes corresponds; so that the equations to the three axes are

$$\left.\begin{aligned} \frac{x}{x_1} &= \frac{y}{y_1} = \frac{z}{z_1}, \\ \frac{x}{x_2} &= \frac{y}{y_2} = \frac{z}{z_2}, \\ \frac{x}{x_3} &= \frac{y}{y_3} = \frac{z}{z_3}; \end{aligned}\right\} \tag{138}$$

and the equations to the conjugate planes are

$$\left.\begin{aligned} \frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} &= 0, \\ \frac{xx_2}{a^2} + \frac{yy_2}{b^2} + \frac{zz_2}{c^2} &= 0, \\ \frac{xx_3}{a^2} + \frac{yy_3}{b^2} + \frac{zz_3}{c^2} &= 0. \end{aligned}\right\} \tag{139}$$

But since the first of (138) coincides with the line of intersection of the second and third of (139), we have

$$\frac{x_1}{a^2(y_2z_3 - z_2y_3)} = \frac{y_1}{b^2(z_2x_3 - x_2z_3)} = \frac{z_1}{c^2(x_2y_3 - y_2x_3)}; \tag{140}$$

which equations are equivalent to the two equations,

$$\left.\begin{aligned} \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} &= 0, \\ \frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} + \frac{z_1z_3}{c^2} &= 0; \end{aligned}\right\} \tag{141}$$

and as the other two lines of (138) must coincide with the lines of intersection of the other planes of (139), we shall, in addition to (141), have also the equation

$$\frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} = 0; \tag{142}$$

these are three relations between the coordinates of the extremities of three conjugate axes.

By a similar process it may be shewn, that if

$$\left.\begin{aligned} \mathrm{L}_1 x + \mathrm{M}_1 y + \mathrm{N}_1 z &= 0, \\ \mathrm{L}_2 x + \mathrm{M}_2 y + \mathrm{N}_2 z &= 0, \\ \mathrm{L}_3 x + \mathrm{M}_3 y + \mathrm{N}_2 z &= 0, \end{aligned}\right\} \tag{143}$$

are the equations to three planes of a conjugate system, then

$$\left.\begin{aligned} a^2\mathrm{L}_2\mathrm{L}_3 + b^2\mathrm{M}_2\mathrm{M}_3 + c^2\mathrm{N}_2\mathrm{N}_3 &= 0, \\ a^2\mathrm{L}_3\mathrm{L}_1 + b^2\mathrm{M}_3\mathrm{M}_1 + c^2\mathrm{N}_3\mathrm{N}_1 &= 0, \\ a^2\mathrm{L}_1\mathrm{L}_2 + b^2\mathrm{M}_1\mathrm{M}_2 + c^2\mathrm{N}_1\mathrm{N}_2 &= 0. \end{aligned}\right\} \tag{144}$$

With respect to these equations I would observe, that if a point $\mathrm{P}_1(x_1, y_1, z_1)$ is given on the ellipsoid, the central plane conjugate

to the axis through P_1 is also given: now any axis may be taken in this plane as the second conjugate axis: let it be that which passes through the point $P_2(x_2, y_2, z_2)$; then the third conjugate axis must be that axis in this plane which is conjugate to the axis passing through P_2: let this cut the surface in the point $P_3(x_3, y_3, z_3)$: so that we thus obtain a complete system of conjugate axes.

Hence if P_1 is given, P_2 and P_3 must be in a determinate plane; that is, in the plane which is conjugate to the axis through P_1: P_2 may be *any* point in this plane, but P_3 must be such that the axis through it is conjugate to the axis through P_2: hence, as P_1 is wholly indeterminate, and P_2 is indeterminate in the conjugate plane, the number of systems of conjugate axes is infinite.

24.] Again, a system of conjugate axes may be defined by the following equations;

$$\left.\begin{array}{l} x_1 = a l_1, \\ y_1 = b m_1, \\ z_1 = c n_1; \end{array}\right\} \quad \left.\begin{array}{l} x_2 = a l_2, \\ y_2 = b m_2, \\ z_2 = c n_2; \end{array}\right\} \quad \left.\begin{array}{l} x_3 = a l_3, \\ y_3 = b m_3, \\ z_3 = c n_3; \end{array}\right\} \qquad (145)$$

in which cases the equation of the ellipsoid gives

$$l_1^2 + m_1^2 + n_1^2 = l_2^2 + m_2^2 + n_2^2 = l_3^2 + m_3^2 + n_3^2 = 1. \qquad (146)$$

and from (141) and (142) we have

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = l_3 l_1 + m_3 m_1 + n_3 n_1 = l_1 l_2 + m_1 m_2 + n_1 n_2 = 0; \quad (1$$

and from these six equations we have the inverse systems

$$l_1^2 + l_2^2 + l_3^2 = m_1^2 + m_2^2 + m_3^2 = n_1^2 + n_2^2 + n_3^2 = 1; \quad (148)$$

$$m_1 n_1 + m_2 n_2 + m_3 n_3 = n_1 l_1 + n_2 l_2 + n_3 l_3 = l_1 m_1 + l_2 m_2 + l_3 m_3 = 0. \quad (1$$

Also we have theorems analogous to (11), (12), and (13) of Art. 2. Now these relations are useful for the proof of many properties of conjugate axes; thus, let r_1, r_2, r_3 be three conjugate axes; then

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 &= a^2(l_1^2 + l_2^2 + l_3^2) + b^2(m_1^2 + m_2^2 + m_3^2) \\ &\qquad + c^2(n_1^2 + n_2^2 + n_3^2) \\ &= a^2 + b^2 + c^2; \end{aligned} \qquad 150$$

that is, the sum of the squares of three conjugate axes is constant.

25.] Again, let there be three central radii vectores of an ellipsoid mutually at right angles to each other; then the sum of the squares of their reciprocals is constant.

Let r_1, r_2, r_3 be the three central radii vectores, of which le

the direction-cosines be (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3); then we have

$$\left.\begin{aligned} \frac{1}{r_1^2} &= \frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}, \\ \frac{1}{r_2^2} &= \frac{l_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2}, \\ \frac{1}{r_3^2} &= \frac{l_3^2}{a^2} + \frac{m_3^2}{b^2} + \frac{n_3^2}{c^2}; \end{aligned}\right\} \tag{151}$$

and therefore by addition

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}. \tag{152}$$

26.] The normal and the tangent plane drawn at any point of a quadric central surface meet each of the principal planes at a point and along a straight line respectively, and these are such that in each principal plane the point is the pole, and the straight line is the corresponding polar, relatively to a certain determinate conic in that principal plane.

Let us take the ellipsoid whose equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \tag{153}$$

to be the typical case; and consider the normal and the tangent plane at the point (x, y, z); and let us also take the principal plane of (x, y), which is that of the greatest and mean principal axes. Then the normal pierces this plane at $\left(\frac{a^2-c^2}{a^2}x, \frac{b^2-c^2}{b^2}y\right)$; and the equation to the line of intersection of the tangent plane and the plane of (x, y) is

$$\frac{x\xi}{a^2} + \frac{y\eta}{b^2} = 1; \tag{154}$$

this is evidently the polar of the pole $\left(\frac{a^2-c^2}{a^2}x, \frac{b^2-c^2}{b^2}\right)$ relatively to the conic

$$\frac{x^2}{a^2-c^2} + \frac{y^2}{b^2-c^2} = 1. \tag{155}$$

By a similar process we may shew that the like conics in the other principal planes are expressed by the equations

$$\frac{x^2}{a^2-b^2} + \frac{z^2}{c^2-b^2} = 1, \tag{156}$$

$$\frac{y^2}{b^2-a^2} + \frac{z^2}{c^2-a^2} = 1; \tag{157}$$

of these equations (155) is that of an ellipse in the plane of (x, y); (156) of an hyperbola in the plane of (x, z); and (157) of a curve

which is wholly imaginary in the plane of (y, z). These curves are called the focal conics of the ellipsoid (153); and for this reason; the vertices of (155) are the foci of the elliptic sections of the ellipsoid by the principal planes of (y, z) and (x, z); and the foci of it are the foci of the elliptic section by the plane of (x, y): also the vertices of (156) are the foci of the elliptic sections of the ellipsoid in the planes of (x, y) and of (z, y); and the foci are the foci of the elliptic section made by the principal plane of (x, z). The third curve is imaginary, although its foci are, as in the other two cases, real. It will be observed that the hyperbola (156) passes through the umbilics of the ellipsoid.

27.] Now we call those surfaces of the second degree confocal the principal sections of which are confocal; hence it appears that all quadric surfaces, which have the same focal conics, are confocal.

Thus the general equation of all surfaces of the second degree confocal with (153) is

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1, \tag{158}$$

where θ, which is called the parameter, is indeterminate, the equations to its focal conics being (155), (156), and (157). And if (158) passes through a given point (x', y', z'), we have from it the cubic equation in θ,

$$(\theta+a^2)(\theta+b^2)(\theta+c^2)-x'^2(\theta+b^2)(\theta+c^2)-y'^2(\theta+c^2)(\theta+a^2)$$
$$-z'^2(\theta+a^2)(\theta+b^2)=0; \tag{159}$$

in which if we substitute for θ successively $+\infty$, $-c^2$, $-b^2$, $-a^2$ the results are severally $+$, $-$, $+$, $-$; so that the roots are real, and lie respectively between $+\infty$ and $-c^2$, $-c^2$ and $-b^2$, $-b^2$ and $-a^2$; in which cases (158) represents respectively an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets. Thus, at the point (x', y', z') these three confocal surfaces intersect. We have also proved (see Vol. I, Art. 411, Ed. 2.) that they intersect at right angles and along their lines of curvature. Thus, at the common points of intersection of these three surfaces, their normals are at right angles to each other. It is also evident that these surfaces intersect in eight points, one in each octant of space about their centre.

Now if $\theta = -c^2$, (158) requires that $z=0$, and we have

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1, \tag{160}$$

which is the equation to the focal conic in the plane of (x, y); similarly if $\theta = -b^2$, and if $\theta = -a^2$, we have the focal conics in the planes of (z, x) and (y, z) respectively; whence it appears that the focal conics are only particular cases of quadrics confocal with (153).

And therefore surfaces which are confocal may also be described as those which have the same focal conics.

28.] If to a system of confocal ellipsoids tangent planes are drawn parallel to each other, the locus of the points of contact is a rectangular hyperbola.

Let the equation to the system of ellipsoids be

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1 \tag{161}$$

where θ is the variable parameter; and let the equation to the tangent plane be

$$lx+my+nz = p \tag{162}$$

where p varies, and l, m, n are constant, being the direction-cosines of the common normals to all these parallel planes. Then, since (162) touches (161),

$$\frac{\frac{x}{a^2+\theta}}{l} = \frac{\frac{y}{b^2+\theta}}{m} = \frac{\frac{z}{c^2+\theta}}{n} = \frac{1}{lx+my+nz} \tag{163}$$

$$= \frac{\frac{x}{l}}{a^2+\theta} = \frac{\frac{y}{m}}{b^2+\theta} = \frac{\frac{z}{n}}{c^2+\theta} = \frac{\frac{y}{m}-\frac{z}{n}}{b^2-c^2} = \frac{\frac{z}{n}-\frac{x}{l}}{c^2-a^2} = \frac{\frac{x}{l}-\frac{y}{m}}{a^2-b^2}; \tag{164}$$

$$\therefore \quad \frac{b^2-c^2}{l}x + \frac{c^2-a^2}{m}y + \frac{a^2-b^2}{n}z = 0, \tag{165}$$

which is the equation to a plane passing through the origin containing the common normal to all the planes and their points of contact; and consequently perpendicular to every one of them; the locus lies in this plane, and consequently is a plane curve.

Also from (163) and (164) we have

$$\left(\frac{y}{m}-\frac{z}{n}\right)(lx+my+nz) = b^2-c^2, \tag{166}$$

and also two other symmetrical equations. Each of these is a central quadric, having its centre at the origin, and is evidently a hyperboloid of two sheets; so that the locus is a plane central section of such a hyperboloid, and is consequently a hyperbola.

Also the planes whose equations are

$$\frac{y}{m}-\frac{z}{n}=0, \quad \text{and} \quad lx+my+nz=0,$$

are asymptotic to the surface; and the asymptotes are the lines of intersection of these planes with the plane (165). If (l_1, m_1, n_1) (l_2, m_2, n_1) are these lines respectively, then

$$\frac{l_1}{l}=\frac{m_1}{m}=\frac{n_1}{n}; \text{ and } ll_2+mm_2+nn_2=0; \qquad (167)$$

$$\therefore \quad l_1 l_2+m_1 m_2+n_1 n_2=0; \qquad (168)$$

and consequently the asymptotes are perpendicular to each other, and the hyperbola is rectangular.

The former of (167) shews that one asymptote is perpendicular to the tangent planes; this theorem is also evident from the fact that the confocal ellipsoid becomes a sphere when $\theta=\infty$, and the point of contact in that case lies in the perpendicular from the origin on the tangent plane.

As the other asymptote is the line of intersection of the plane (165) with the plane $lx+my+nz=0$, we have

$$\frac{b^2-c^2}{l}l_2+\frac{c^2-a^2}{m}m_2+\frac{a^2-b^2}{n}n_2=0,$$

$$ll_2+mm_2+nn_2=0;$$

whence, if $a^2l^2+b^2m^2+c^2n^2=p^2$,

$$\frac{mnl_2}{a^2-p^2}=\frac{nlm_2}{b^2-p^2}=\frac{lmn_2}{c^2-p^2},$$

which determine the position of the second asymptote.

29.] If from any point (ξ, η, ζ) an enveloping cone is drawn to the ellipsoid (153), the principal axes of that cone coincide with the normals of the three confocal surfaces of the second degree which intersect at the vertex of the cone.

By Ex. 2, Art. 355, Vol. I, Ed. 2, the equation to the cone whose vertex is (ξ, η, ζ), and which envelopes the ellipsoid (153), is

$$\left\{\frac{x\xi}{a^2}+\frac{y\eta}{b^2}+\frac{z\zeta}{c^2}-1\right\}^2-\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0. \quad (16$$

For the sake of abbreviation let

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}-1=\kappa; \qquad (170)$$

so that (169) on expansion becomes

$$\left(\frac{\xi^2}{a^2}-\kappa\right)\frac{x^2}{a^2}+\left(\frac{\eta^2}{b^2}-\kappa\right)\frac{y^2}{b^2}+\left(\frac{\zeta^2}{c^2}-\kappa\right)\frac{z^2}{c^2}$$
$$+\frac{2\eta\zeta}{b^2c^2}yz+\frac{2\zeta\xi}{c^2a^2}zx+\frac{2\xi\eta}{a^2b^2}xy+\ldots\ldots=0; \qquad (171)$$

the other terms being omitted because the position of the principal axes of the cone depends on the first six terms only of the expanded equation; see Article 9.

In this case equation (48) becomes

$$\frac{\xi^2}{a^2(a^2\mathrm{X}+\kappa)}+\frac{\eta^2}{b^2(b^2\mathrm{X}+\kappa)}+\frac{\zeta^2}{c^2(c^2\mathrm{X}+\kappa)}=1; \qquad (172)$$

also from (170) we have

$$\frac{\xi^2}{a^2\kappa}+\frac{\eta^2}{b^2\kappa}+\frac{\zeta^2}{c^2\kappa}=1+\frac{1}{\kappa};$$

therefore by subtraction

$$\frac{\xi^2}{a^2+\frac{\kappa}{\mathrm{X}}}+\frac{\eta^2}{b^2+\frac{\kappa}{\mathrm{X}}}+\frac{\zeta^2}{c^2+\frac{\kappa}{\mathrm{X}}}=1. \qquad (173)$$

Now κ and X are functions of the coordinates of the given vertex (ξ, η, ζ) and are therefore known: hence if we describe the quadric surface whose equation is

$$\frac{x^2}{a^2+\frac{\kappa}{\mathrm{X}}}+\frac{y^2}{b^2+\frac{\kappa}{\mathrm{X}}}+\frac{z^2}{c^2+\frac{\kappa}{\mathrm{X}}}=1, \qquad (174)$$

(173) shews that that surface passes through the vertex of the enveloping cone; and this surface is evidently confocal with the original ellipsoid (153); and as X has three values which are the roots of (172), so, as we have shewn in Art. 27, the equation (174) represents three surfaces which are severally an ellipsoid, a hyperboloid of one sheet, and a hyperboloid of two sheets, all of which are confocal with (153); and which intersect orthogonally at (ξ, η, ζ).

For the determination of the principal axes of the cone, let us take the system of direction-cosines of Art. 9, and the forms of them given in (55); in the case of (173), these take the following values

$$\frac{a_1}{\xi}\left(a^2+\frac{\kappa}{\mathrm{X}}\right)=\frac{a_2}{\eta}\left(b^2+\frac{\kappa}{\mathrm{X}}\right)=\frac{a_3}{\zeta}\left(c^2+\frac{\kappa}{\mathrm{X}}\right); \qquad (175)$$

and to fix our thoughts let us suppose X in this equation to be

that root of (172) which corresponds to the ellipsoid. Now the direction-cosines of the normal to the ellipsoid (174) at the point (ξ, η, ζ) are proportional to

$$\frac{\xi}{a^2+\frac{\mathrm{K}}{\mathrm{X}}}, \qquad \frac{\eta}{b^2+\frac{\mathrm{K}}{\mathrm{X}}}, \qquad \frac{\zeta}{c^2+\frac{\mathrm{K}}{\mathrm{X}}}; \tag{176}$$

and a comparison of these values with (175) shews that the principal axis (a_1, a_2, a_3), the position of which is determined by (175), coincides with the normal of the confocal ellipsoid which passes through the point. This axis is generally the internal axis of the cone. By a similar process we may shew that the two external axes of the cone, which correspond to the two other roots of the cubic, are normal to the two confocal hyperboloids which intersect at the given vertex.

The normals to these two hyperboloids are, as we have shewn, tangents to the lines of curvature on the ellipsoid at the point (ξ, η, ζ); and therefore if a cone envelopes an ellipsoid, and if through the vertex of the cone an ellipsoid be described confocal with the former ellipsoid, the normal to the ellipsoid, and the tangents to the two lines of curvature on it, are the principal axes of the enveloping cone.

M. Chasles has also proved that the generating lines of the confocal hyperboloid of one sheet which passes through the vertex are the focal lines of the cone.

Now these same properties are true if instead of the ellipsoid (153) we had taken any other quadric confocal with it; and therefore are true if the focal conics are the directors of the cone, because the focal conics are the limiting forms of the confocal quadrics.

Hence also it follows that if two quadrics have the same focal conic, and if from any point in space as vertex two cones are described enveloping these surfaces, these cones have the same principal axes, and the same focal lines.

For other properties of confocal quadrics and confocal conics, many of which are interesting and important, I must refer the reader to M. Chasles, Memoire de Geomètrie.

CHAPTER II.

THE KINEMATICS OF A RIGID BODY. ANGULAR VELOCITIES; THEIR COMPOSITION AND RESOLUTION, AND RELATION TO LINEAR VELOCITIES.

30.] SEVERAL times in the course of the treatise on Mechanics allusion has been made to a division of the subject into two parts, Kinematics and Dynamics proper. In the former of these the affections of pure motion are investigated; that is, motion and its incidents are discussed apart from all consideration of the action of forces which produce that motion; thus, for instance, it is shewn that motion takes place in time and space; that a particle moves with a certain velocity, and that velocity depends on time and space, and is measured by the ratio of space to time. In the latter, motion is considered as the effect of certain producing causes, and the relations between it as the effect and force as the cause are investigated; thus the laws of motion and the equations of motion belong to Dynamics proper. In the exposition of the first principles of Dynamics, see Chapter VII, Vol. III, this division of the subject is made, and the parts are treated separately; but it was unnecessary to bring the division into more special prominence, because the Kinematics of a moving particle do not present to the mind images difficult of formation. Every one can form a conception, more or less perfect, of the motion of a single particle; it describes a certain line, which is its path, and we can easily imagine that path; it moves with a certain velocity, and if its velocity varies, it is not difficult to conceive the rate of variation. But the motion of a system of particles is more complex; we can indeed follow the path of any one particle of it; it describes a line, just as if it were not connected with the other particles; but what is the motion of all the particles of the system relatively to that particle? Let us however, at present, confine our attention to the motion of a rigid body, which is a system of particles of invariable form, being such that the distance between every pair of particles remains unaltered. The body can, as it were, pirouette about any one particle in all ways, but it is difficult

to imagine and to trace the motion of any other particle. So if a body rotates about an axis absolutely fixed, we can easily picture to ourselves the path described by every particle; it is a circle, the plane of which is perpendicular to the fixed axis, and the centre of which is in that axis. But when the body has the most general motion of which it is capable, our conception is for the most part very obscure. Hence arises the necessity of resolving that motion into its simplest elements; so that when we have an adequate conception of each separate element of motion, we may combine them, and thus obtain an adequate conception of the motion of the whole body. We must therefore first discuss the several motions of which a rigid body is capable, independently of the forces which produce these motions; and subsequently consider the relations which subsist between these effects and their causes. In the present Chapter we shall confine our attention to the former part, viz. the Kinematics of a rigid body; and in the following Chapters we shall consider the Dynamics proper, the fundamental axioms, and the theorems deducible from them.

31.] Let us imagine a rigid body or a system of material particles of invariable form to be in motion. The form of this system will be definite if (1) the distances from each other of three particles which are not on the same straight line are given; (2) the distances of every other particle from each of these three particles are given, the distances being affected with proper signs, as indicating direction; so that, as the system is rigid, if the positions of the first three particles are determined, the position of every other particle is also determined, and that of the whole body is also known. The analytical proof of the sufficiency of this enquiry will be given hereafter. We shall however now presume that it is sufficient for us to consider the motion of the first three particles.

Let the three particles of the system which form a triangle, and relatively to which every other particle is known, be P, Q, R, and let these be joined by straight lines. Now if the motion is such that the sides of this triangle are always parallel to their original positions, it is plain that the line joining any other point to each of these three points is also parallel to its original position; such a motion is said to be a motion of translation of the body or system. In this case the paths of all particles

are equal and parallel lines, whether straight or curved, and are described with equal and parallel velocities; and the motion of the whole body will be easily inferred from that of any one particle. As the incidents of such a motion have been fully discussed in the previous volume of our work it is unnecessary to say more on this part of the subject.

32.] If however the paths described by the several particles are not equal and parallel the body has another motion besides that of translation. Let us consider its nature.

Now let P, Q, R be the positions of the three particles of the body, to which the position of every other particle is referred at a given time; and let P′, Q′, R′ be their positions after a certain motion; suppose moreover that the motion is most general and that the paths described by these three particles in their motion are not equal and parallel; we may analyse the motion by the following process: first let all the three particles be moved so as to describe paths equal and parallel to that described by P; that is, let us suppose a motion of translation of the whole body such that every particle of it moves over a space equal and parallel to PP′; let the positions taken by Q and R after this motion be Q″, R″; through P′, Q″, R″ let a plane be drawn, which is manifestly parallel to the original plane PQR; and also let a plane be drawn through the three final positions P′, Q′, R′; let these two planes intersect along the line P′N; let the body rotate about the line P′N, until the plane Q″R″P′ is brought into the plane Q′R′P′; and, if it is necessary, let the body again rotate about a line passing through P′ and perpendicular to the plane P′Q′R′, until Q″ and R″ coincide respectively with Q′ and R′; by these several motions the body will have passed from its first to its final position. The motions are three; the first is a motion of translation; the other two are motions of turning or of rotation about certain axes; and as the motion has been of the most general kind, so may all motion be resolved into separate motions of the kinds which we have mentioned.

This motion of rotation requires careful consideration. It always takes place about a certain straight line or axis. If a body rotates all points along the axis are at rest so far as the motion of rotation is concerned; they may move by reason of other circumstances, but they do not move by reason of the rotation of the body about that axis; and the straight

line along which the quiescent points are is called the axis of rotation.

Also this axis may or may not meet the body. If it meets the body, the particles of the body along the axis are at rest; if it does not meet the body, all the particles of the body move by reason of the rotation.

Many rotations about different axes may co-exist; we must consider how this is, and investigate laws by which these may be combined into one or more resultants.

33.] Now the most simple rotation is that of a body rotating about an axis fixed absolutely; that is, relatively to it and to space. In this case every particle of the body describes a circle in a plane perpendicular to the axis; and the body being rigid, the times in which the circles are described are the same for all the particles; and their relative position is not changed by or during the motion.

Imagine a particle m at a distance r from the axis of a rotating body; and through the fixed axis and containing it let a plane be drawn fixed in space; then the position of the particle may be determined at any instant by means of r and the angle at which r is inclined to this fixed plane. Thus in Fig. 3, let P be the place of m at the time t; let Oz be the rotation-axis, fixed relatively to the body and to space; through it let the plane zOx be drawn, and let it be fixed in space, so that when the body rotates, this plane as well as the axis remains fixed; let OP be drawn at right angles to Oz; OP $= r$, PO$x = \theta$; then r and θ are sufficient to determine the place of m.

Firstly, let us suppose the body to rotate uniformly about the axis; that is, let us suppose θ to receive equal increments in equal times; let ω be the angle by which θ is increased, that is through which r revolves, in an unit of time; then if θ is the angle through which r has revolved in t units of time,

$$\theta = \omega t; \tag{1}$$

so that if OP coincides with Ox when $t = 0$, PO$x = \theta = \omega t$. And from (1) we have

$$\omega = \frac{\theta}{t}. \tag{2}$$

We must enlarge our language; let us take our nomenclature from that of motion of translation. Since the linear velocity of a particle moving uniformly is the linear space described

by it in an unit of time; so let the angle through which an uniformly rotating body rotates in an unit of time be called the angular velocity of the body. Thus ω is the angular velocity of the body, and is defined mathematically by (2). It is to be observed that the angular velocity is independent of r and is the same for all points of the body.

Secondly, suppose the body to rotate about the axis but not uniformly, so that the radius vector of any particle does not describe equal angles in equal times; then the angular velocity varies and is a function of the time. Let the time be resolved into infinitesimal elements; and let us suppose the angular velocity at the time t to be ω, and to be $\omega+d\omega$ at the time $t+dt$; and let $d\theta$ be the angle through which the body has rotated in the time dt. Then since ω is the angular velocity at the beginning of dt, and $\omega+d\omega$ is the angular velocity at the end of dt, the *mean* angular velocity with which $d\theta$ has been described in $\omega+\phi d\omega$ if ϕ is a proper fraction; and ϕ is positive or negative according as the angular velocity is increasing or decreasing; so that by reason of (1)

$$d\theta = (\omega+\phi d\omega)\,dt;$$

and omitting the infinitesimal of the second order,

$$d\theta = \omega\,dt; \tag{3}$$

thus $d\theta$ is the angle described in dt units of time by the body rotating with the angular velocity ω at the beginning of dt; and therefore dividing both sides by dt, we have

$$\omega = \frac{d\theta}{dt}; \tag{4}$$

and therefore ω or $\frac{d\theta}{dt}$ is the angle described in an unit of time, and is, according to our definition, the angular velocity of the body.

Thus in both cases, of uniform and of continuously varying angular velocity, angular velocity is the angle described by the radius vector of any particle in an unit of time; and is the ratio of the angle described in a given time to the time in which it is described; in the case of varying velocity this ratio is the ratio of two infinitesimals.

The unit angular velocity is that of a body which rotates through an unit angle in an unit of time; and if the angular velocity of a body is ω, ω is a number designating the number of unit angles through which the body rotates in an unit of time.

34.] Again let a rigid body rotate about a fixed axis; at a given instant the angular velocity is the same for all particles of the body; but the linear velocity is evidently not the same for all; the linear velocity of those at a greater distance from the axis is greater than of those at a less distance; the relation between the angular and the linear velocities of a particle is thus found.

Let us take a particle m at a distance r from the axis. Let ω be the angular velocity, $d\theta$ be the angle described by r in the time dt, and let ds be the space described by m; then $ds = r\,d\theta$; and

$$\frac{ds}{dt} = r\frac{d\theta}{dt}, \tag{5}$$

$$= r\omega; \tag{6}$$

so that the linear velocity of m is the product of the angular velocity and the radius of m, and therefore varies directly as the distance of m from the rotation-axis. If therefore $r = 1$, the angular velocity is identical with the linear velocity.

35.] Hence is derived the principle on which angular velocities are measured; if two bodies rotate with angular velocities such that the particles in each at an unit distance from the axis describe equal spaces in equal times, the angular velocities of the bodies being uniform during that time, these angular velocities are said to be equal. And this mode of determining equal angular velocities being adopted, it is evident that one angular velocity may be double, or treble, or n times another. If the equal spaces are described by each particle in the same direction, the angular velocities are equal and in the same direction; but if the equal spaces are described in opposite directions, the angular velocities are equal and opposite. Angular velocities may therefore be affected with signs. Thus if ω represents the angular velocity with which a body rotates in a given direction, $-\omega$ will represent the equal angular velocity of a body rotating in the opposite direction. As angular velocities have rotation-axes, intensities or magnitudes, and directions, it is evidently desirable to have some geometrical and graphical representative of them, as of linear velocities. This is supplied by a straight line on a principle similar to that by which the line-representatives of couples are determined in Statics. Along the rotation-axis let a length be taken containing the same number of linear

units as ω contains angle-units; then this line by its position and its length represents the axis of rotation and the intensity of the angular velocity. Let a point on this rotation-axis be taken as a fixed pole; as the body may rotate about this axis in either of two directions, so may the line-representative of the angular velocity be measured from this pole in either of two opposite directions, and therefore we must choose a principle by which direction of rotation may be determined. Let it be this; if, as we look along the axis from the pole, the body rotates from left to right, like the hands of a watch when we face it, let that rotation be called positive, and let its line-representative be measured from the pole in the direction in which we look; but if the body rotates from right to left, that is in the direction opposite to that of the motion of the hands of a watch, let that rotation be negative, and let the line-representative be measured from the pole in a direction opposite to that along which we look. Thus in Fig. 4; let O be the pole and Ox the rotation-axis; as we look from O towards x let the body rotate as the hands of a watch which we face; that is, in the direction of the letters PQRS, then that rotation is positive, and its line-representative is to be measured from O towards the right; let OA be that line, then OA is as to direction and length a representative of the angular velocity. If, on the contrary, when we look from O towards x, the rotation of the body is in the opposite direction, then the line OA is to be measured along Ox produced backwards; that is, OA′ is the line-representative of the angular velocity. I may observe that if we look from O towards A′, that is towards the left instead of the right, OA′ is the representative of the rotation in the second case according to the principle we have adopted, for as we look from O towards A′ the body rotates in the same direction as the hands of a watch; thus the line-representative is independent of the direction in which we look from O. These line-representatives are called vectors, and we shall hereafter use them as adequate representatives of angular velocities. This mode of representation is the foundation of Graphical Kinematics.

36.] Thus much as to single angular velocities, and their line-representatives. We will now investigate the circumstances of a body which rotates with many simultaneous angular velocities; that is, we suppose a body to rotate about a determinate axis

with a given angular velocity, and another angular velocity to be communicated to it; what change of motion is due to the addition of this new angular velocity, and what is the combined resultant of the new and the original angular velocities? And again, we shall suppose other angular velocities to be communicated, and we shall have to determine the resultant of all of them. At present we say nothing about the source of these velocities or the mode of communication; we shall consider only the combined result of them as expressed angular velocities. The problem therefore is the composition and resolution of angular velocities; and we shall in order consider (1) those which have the same rotation-axis; (2) those whose rotation-axes meet in a point; (3) those whose rotation-axes do not meet; and, as a special case of (2), those whose rotation-axes are parallel, that is, meet at an infinite distance.

We consider first then the composition of angular velocities which have the same rotation-axis. Let a body rotate about the axis Ox, Fig. 5, with two positive angular velocities ω_a, ω_b, whose representatives are OA and OB; let P be the place of a particle of the body, which we will take to be in the plane of the paper. Let PM be drawn from P at right angles to Ox, and let PM $= r$; as both the angular velocities are positive, all particles of the body lying in the plane of the paper and above Ox move from below to above the plane of the paper; and all particles lying below Ox move from above to below; therefore the spaces through which P passes from below to above the plane of the paper in the time dt are $\omega_a r\, dt$ due to ω_a and $\omega_b r\, dt$ due to ω_b; so that the whole space over which P passes in dt

$$= (\omega_a + \omega_b) r\, dt.$$

Now suppose ω to be the resultant of ω_a and ω_b; so that if the body rotates with this velocity, P will pass in the time dt over a space due to ω which is equal to the sum of the spaces due to ω_a and to ω_b; but the space passed over by P due to the angular velocity ω in the time dt is $\omega r\, dt$; therefore

$$\omega r\, dt = (\omega_a + \omega_b) r\, dt;$$

$$\therefore \quad \omega = \omega_a + \omega_b; \tag{7}$$

that is, the resultant angular velocity is the sum of the two component angular velocities.

If one of the components, say ω_b, is negative, the positive

space passed over by P in the time dt will manifestly be $(\omega_a - \omega_b) r\, dt$; and we shall have

$$\omega = \omega_a - \omega_b; \tag{8}$$

and if $\omega_b = -\omega_a$, then from (7) $\omega = 0$, and two equal and opposite angular velocities neutralise each other.

Similarly if a body rotates about a given axis with coaxial angular velocities $\omega_1, \omega_2, \ldots\ldots, \omega_n$, and if Ω is the resultant angular velocity,

$$\begin{aligned} \Omega &= \omega_1 + \omega_2 + \ldots\ldots + \omega_n, \\ &= \Sigma.\omega; \end{aligned} \tag{9}$$

and as this holds true whether the angular velocities are positive or negative, $\Sigma.\omega$ expresses the algebraical sum of the several component angular velocities.

In all these cases the line-representative of the resultant is the algebraical sum of the line-representatives of the components.

37.] Next let us suppose the body to move with two simultaneous angular velocities ω_a and ω_b, whose axes OA and OB intersect each other in the point O.

Let us take O to be the pole, and OA, OB, Fig. 6, to be the line-representatives of the angular velocities ω_a, ω_b respectively; so that due to ω_a all particles of the body in the plane of the paper which are above the line OA pass from below to above the plane of the paper, and all those below OA pass from above to below; similarly by means of ω_b all particles to the right of OB pass from above to below and all those to the left from below to above. Let us take a point P within the angle BOA, and from P let PL and PK be drawn perpendicular to OA and OB respectively; let OM $= x$, MP $= y$, BOA $= \gamma$; then PK $= x \sin\gamma$, PL $= y \sin\gamma$. Let us investigate the paths described by P in the time dt, which are due to these two angular velocities; the upward path of P due to $\omega_a = \omega_a y \sin\gamma\, dt$; and the downward path of P due to $\omega_b = \omega_b x \sin\gamma\, dt$: so that the resultant upward path of P $= (y\omega_a - x\omega_b) \sin\gamma\, dt$. Now let us suppose P to be at rest under the effects of the two angular velocities, then

$$\frac{x}{\omega_a} = \frac{y}{\omega_b}; \tag{10}$$

and replacing ω_a and ω_b by their representatives OA and OB,

$$\frac{x}{\text{OA}} = \frac{y}{\text{OB}}; \tag{11}$$

but either of these is the equation to a straight line passing through O; and as all particles along it are at rest, it is the axis of the resultant angular velocity. From (11) it appears that it is the diagonal of the parallelogram of which OA and OB are the containing sides; so that the axis of the resultant angular velocity lies along the diagonal of the parallelogram of which the line-representatives of the component angular velocities are the containing sides.

The intensity of the resultant may be found as follows; let us suppose it to be ω_c; then as the path which A, or indeed any particle on the line OA, describes in dt in the case of the component angular velocities is that due to ω_b only, and is $\omega_b \, \text{OA} \sin \gamma \, dt$, and as the path described by A in dt in the case of the resultant angular velocity is $\omega_c \, \text{OA} \sin \text{COA} \, dt$, these paths are to be equal; and therefore

$$\omega_b \sin \gamma = \omega_c \sin \text{COA}; \tag{12}$$

similarly if we equate to each other the two paths described by B in the cases of the component and of the resultant angular velocities, we shall have

$$\omega_a \sin \gamma = \omega_c \sin \text{COB}; \tag{13}$$

from either of which equations it appears that ω_c is represented in length by the diagonal OC. Hence it follows that if a body rotates with two simultaneous velocities, whose line-representatives meet in a point and are the adjacent sides of a parallelogram, the resultant angular velocity is equivalently expressed in all respects by that diagonal of the parallelogram which abuts at the point of intersection of the line-representatives of the component angular velocities.

Hence if, as in Fig. 7, CO is produced to C′, so that C′O = CO, then if OA, OB, OC′ are the line-representatives of three simultaneous angular velocities, the body is at rest; because OC, which represents the resultant of ω_a and ω_b, represents an angular velocity which is neutralised by that of which OC′ is the line-representative; and therefore if $\text{AOC}' = \beta$, $\text{BOC}' = \alpha$, from (12) and (13) we have

$$\frac{\omega_a}{\sin \alpha} = \frac{\omega_b}{\sin \beta} = \frac{\omega_c}{\sin \gamma}; \tag{14}$$

hence also if a body rotates with three simultaneous angular velocities whose axes meet in a point, and whose line-representatives are parallel and equal to the three sides of a triangle, the

angular velocities neutralise each other, and the body remains at rest.

Hence also it follows that if a body has two simultaneous angular velocities ω_a and ω_b about two axes which intersect at an angle γ, these are equivalent to a single angular velocity ω_c, which is given by the equation

$$\omega_c^2 = \omega_a^2 + 2\omega_a\omega_b \cos\gamma + \omega_b^2; \tag{15}$$

the rotation-axis of which makes angles α and β with the rotation-axes of ω_b and ω_a, which are such that

$$\frac{\omega_a}{\sin\alpha} = \frac{\omega_b}{\sin\beta} = \frac{\omega_c}{\sin\gamma}. \tag{16}$$

38.] Now suppose a body to have two simultaneous angular velocities ω_x and ω_y about two axes intersecting each other at right angles; then, if ω is the resultant angular velocity,

$$\omega^2 = \omega_x^2 + \omega_y^2; \tag{17}$$

and, if α is the angle between the axes of ω and ω_x,

$$\left.\begin{aligned} \omega_x &= \omega \cos\alpha, \\ \omega_y &= \omega \sin\alpha; \end{aligned}\right\} \tag{18}$$

so that angular velocities may by means of their line-representatives be resolved and compounded according to the projective laws of pure geometry, the laws of resolution and composition of statical pressures, and of dynamical linear velocities.

Similarly if a body has three simultaneous angular velocities $\omega_x, \omega_y, \omega_z$ about three axes which intersect at right angles; then, if ω is the resultant angular velocity,

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2; \tag{19}$$

and if α, β, γ are the angles which the axis of the resultant angular velocity makes with those of the component angular velocities,

$$\frac{\omega_x}{\cos\alpha} = \frac{\omega_y}{\cos\beta} = \frac{\omega_z}{\cos\gamma} = \omega. \tag{20}$$

39.] Hence we can deduce the single resultant angular velocity of many angular velocities, whose rotation-axes pass through a given point: let that point be the origin, and at it let a system of coordinate axes originate; let the several angular velocities be $\omega_1, \omega_2, \ldots\ldots \omega_n$; and let their rotation-axes be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $\ldots\ldots (\alpha_n, \beta_n, \gamma_n)$; let each angular velocity be resolved into three components along the three coordinate axes; so that those, whose rotation-axis is the axis of x,

are $\omega_1 \cos a_1, \omega_2 \cos a_2, \ldots\ldots \omega_n \cos a_n$; and if $\Sigma.\omega \cos a$ is the sum of all these,

$$\left.\begin{array}{l} \Sigma.\omega \cos a = \omega_1 \cos a_1 + \omega_2 \cos a_2 + \ldots\ldots + \omega_n \cos a_n; \\ \text{similarly} \\ \Sigma.\omega \cos \beta = \omega_1 \cos \beta_1 + \omega_2 \cos \beta_2 + \ldots\ldots + \omega_n \cos \beta_n, \\ \Sigma.\omega \cos \gamma = \omega_1 \cos \gamma_1 + \omega_2 \cos \gamma_2 + \ldots\ldots + \omega_n \cos \gamma_n. \end{array}\right\} \quad (21)$$

Let Ω be the resultant angular velocity; and let a, b, c be the direction-angles of its rotation-axis; then

$$\Omega \cos a = \Sigma.\omega \cos a, \quad \Omega \cos b = \Sigma.\omega \cos \beta, \quad \Omega \cos c = \Sigma.\omega \cos \gamma; \quad (22)$$

$$\therefore \quad \Omega^2 = (\Sigma.\omega \cos a)^2 + (\Sigma.\omega \cos \beta)^2 + (\Sigma.\omega \cos \gamma)^2; \quad (23)$$

and
$$\frac{\Sigma.\omega \cos a}{\cos a} = \frac{\Sigma.\omega \cos \beta}{\cos b} = \frac{\Sigma.\omega \cos \gamma}{\cos c} = \Omega; \quad (24)$$

which equations give the intensity of the resultant angular velocity, and the direction-cosines of its rotation-axis.

40.] A machine devised by Bohnenberger*, which is represented in Fig. 8, may probably facilitate the conception of simultaneous angular velocities, and their combined effects; the machine is delineated in its primary state of rest. O is the centre of a sphere, through which a horizontal axis AA′ passes, the ends of which are pivots acting in a horizontal circular ring ABA′B′, so that the sphere can rotate about this axis. To this horizontal ring two pivots are attached at B and B′, the line joining which is perpendicular to AA′; these pivots work in a vertical circular ring CBC′B′, so that the ring containing the sphere can rotate about the axis BB′; the ring CC′ has also two pivots at C and C′, which are fixed points, the line joining which is vertical and is at right angles to both AA′ and BB′; and the vertical ring can rotate about CC′ as an axis. The three lines AA′, BB′, CC′ intersect in O the centre of the sphere, and thus form a system of rectangular axes in space, the origin of which is at the centre of the sphere. Now this is the state of the machine at rest, and the problem is this; let an angular velocity ω_x be given to the sphere about the line AA′; to the ring ABA′B′ carrying the sphere let an angular velocity ω_y be given about the axis BB′; and to the ring CBC′B′ carrying the former ring and the sphere let an angular velocity ω_z be given about the axis CC′; then the sphere being connected with the rings has all these angular velocities simultaneously, and the

* See Bohnenberger, Gilbert's "Annalen der Physik," Band LX, 1819.

question is, what line of particles is at rest? What is the resultant axis of rotation, and what is the angular velocity of the sphere about it? Try to follow in your mind the path described by any particle of the sphere, when it moves with all these simultaneous angular velocities; and try thence to determine the line of quiescent particles. Probably in the difficulty of doing so, you will perceive the necessity of such composition and resolution of angular velocities as we have just explained. Let us assume the three angular velocities to be positive; then the resultant angular velocity will be given by equation (19) and the direction-cosines of its axis by (20). This may be thus exhibited; let us suppose $\omega_x = \omega_y = \omega_z$; then $\alpha = \beta = \gamma$; let a diameter of the sphere be drawn making equal angles with AA′, BB′, CC′ in its original position; and at the poles where this diameter meets the surface, let the surface be divided into three equal lunes, and let them be coloured respectively red, yellow, and blue; then it will be found that the sphere will rotate about this axis which is equally inclined to the three lines AA′, BB′, CC′, and the rotating sphere will, if its angular velocity is great enough, appear white; whereas if the resultant rotation-axis does not pass through the point where the differently coloured lunes meet, the colour of the rotating sphere near its pole will be that of the lune in which the rotation-axis pierces the sphere.

41.] The experiment with the pendulum, devised by Foucault to exhibit to the eye the rotation of the earth about its axis, is a simple application of the laws of resolution and composition of angular velocities which have been investigated. Let us suppose the earth to be a perfect sphere, of which a plane section through the poles is drawn in Fig. 9; P and P′ being the north and south poles, C being the centre, and WCB being the intersection of the plane of the paper and the plane of the equator. Let a pendulum be suspended at the north pole, so that it may vibrate freely in all directions. Now if the pendulum is at rest, and suspended from a point not rotating with the earth, but fixed absolutely in space, the earth would rotate under the pendulum from west to east in 24 hours; and the apparent effect to a person on the earth would be a complete rotation of the pendulum through 360° from east to west in the same time. A point of suspension however not fixed to the earth cannot be obtained, and of course if the point of suspension

is joined to the earth, all moves together, and the pendulum has no apparent rotatory motion. Let however the pendulum be suspended from a point fixed to the earth; and let it vibrate in a plane; then as the earth turns, the point of suspension turns, and the pendulum turns, but the *plane of vibration* is not affected by these rotations; it is as stationary as if the point of suspension were absolutely fixed; so that in the course of 24 hours, if the pendulum vibrates so long, the plane of vibration will apparently pass in succession over all the meridians from east to west, because the earth in that time performs a complete revolution from west to east under the pendulum; and the angular velocity of the earth will be the apparent angular velocity of the plane of vibration of the pendulum. A similar phænomenon will be presented by a pendulum at the south pole, but the direction of the apparent rotation of the plane of vibration will be from west to east. At the equator no such effect takes place. For suppose a pendulum to be suspended at the equator, and its plane of vibration to be, say, north and south; as the earth rotates about its axis, it is evident that neither the point of suspension of the pendulum nor the plane of vibration has any rotation; the point of suspension of the pendulum is carried round in a circle, and the plane of vibration continues north and south. At the equator therefore no effects of the earth's rotation such as we have described will be exhibited by a pendulum. The full effect is exhibited at either pole; and no effect at the equator. Now let us take any place A, whose latitude is λ; and let AC be drawn to the centre of the earth. Let ω be the angular velocity of the earth about its axis; then $\omega \sin \lambda$ is the angular velocity about the line AC, which is the normal to the earth's surface at the point A; so that the plane of vibration of the pendulum at A will undergo a displacement from east to west similar to that which takes place at P, but more slowly; for whereas the time of a complete revolution at $\text{P} = \frac{2\pi}{\omega} = 24$ hours, the time of a complete revolution at $\text{A} = \frac{2\pi}{\omega \sin \lambda}$; so that

$$\text{The time of revolution at A} = \frac{24 \text{ hours}}{\sin \lambda}. \qquad (25)$$

This law has been verified by numerous observations made at various places on the earth; for although the vibration of the

pendulum has not been continued through 24 hours, yet the arcs described by the plane of vibration in a given time have been found to vary in different latitudes as the sines of these latitudes.

42.] We proceed now to consider the resolution and composition of angular velocities, the rotation-axes of which do not meet; and we will first consider the particular case of angular velocities whose axes are parallel, and about which separately the body rotates in the same direction. Let the angular velocities be ω_a and ω_b; and let their poles be O and O′, and their axes OA, O′B; OO′ being perpendicular to each of these lines; see Fig. 10; let P be the place of any particle in the line OO′, and to fix our thoughts let us take it between O and O′; let $OO' = c$, $OP = x$, $PO' = y$; then the downward path of P in the time dt which is due to ω_a is $x\omega_a dt$, and the upward path in the same time due to ω_b is $y\omega_b dt$; so that the downward path of P in the time dt $= (x\omega_a - y\omega_b)\,dt.$

Now suppose Q to be a point in OO′ which under the effects of the two angular velocities ω_a and ω_b remains at rest; then if x and y are the respective distances of Q from OA and from O′B, $x\omega_a - y\omega_b = 0$; whence

$$\frac{x}{\omega_b} = \frac{y}{\omega_a}; \tag{26}$$

whereby Q is determined; and as every point in the line through Q perpendicular to OO′ is at rest, so QC is the axis of the resultant angular velocity; and (26) shews that it divides the distance between the axes of the two component angular velocities into two parts which are to each other inversely as the angular velocities.

Let ω be the resultant angular velocity; then in the case of the component angular velocities, the downward path of O′ in the time $dt = \omega_a c\,dt$; and in the case of the resultant angular velocity, the downward path of $O' = \omega y\,dt$; these are of course equal; whence we have

$$\omega_a c = \omega y; \tag{27}$$

similarly, if we equate the two paths of O in the two cases, we have

$$\omega_b c = \omega x; \tag{28}$$

whence

$$\frac{\omega}{c} = \frac{\omega_a}{y} = \frac{\omega_b}{x} = \frac{\omega_a + \omega_b}{y + x}, \tag{29}$$

$$\frac{\omega}{c} = \frac{\omega_a + \omega_b}{c}; \tag{30}$$

$$\therefore \quad \omega = \omega_a + \omega_b; \tag{31}$$

that is, the resultant angular velocity is the sum of the component angular velocities.

A similar theorem is true, whatever is the number of the component angular velocities which have parallel axes.

If one of the component angular velocities, say ω_b, is negative, the paths of all particles between O and O′ due to both ω_a and ω_b will be downward; let us then, see Fig. 11, consider the path taken by a point P in the line OO′ produced; now the line-representative of ω_b is O′B, which is drawn from O′ in a direction opposite to that in which OA is drawn from O. Let OP $= x$, O′P $= y$; then the downward path described by P in dt due to ω_a is $\omega_a x\, dt$; and the upward path due to ω_b is $\omega_b y\, dt$; therefore the whole downward path in the time dt

$$= (\omega_a x - \omega_b y)\, d t.$$

Let Q be a point in OO′ which remains at rest; then, if OQ $= x$, O′Q $= y$,

$$\omega_a x = \omega_b y; \tag{32}$$

so that the line OO′ is divided externally into two parts which are inversely proportional to the component angular velocities.

Also let ω be the resultant angular velocity whose rotation-axis is QC; then equating the downward paths of O which are due to ω_b and to ω respectively, we have

$$\omega_b c = \omega x; \tag{33}$$

and equating the downward paths of O′ which are due to ω_a and to ω respectively,

$$\omega_a c = \omega y; \tag{34}$$

$$\therefore \quad \frac{\omega}{c} = \frac{\omega_b}{x} = \frac{\omega_a}{y} = \frac{\omega_b - \omega_a}{x - y},$$

$$= \frac{\omega_b - \omega_a}{c};$$

$$\therefore \quad \omega = \omega_b - \omega_a; \tag{35}$$

that is, the resultant angular velocity is the excess of the greater component over the less; and has therefore the same sign as the greater.

Hence if a body moves with many angular velocities $\omega_1, \omega_2, \ldots \omega_n$, all of which have parallel rotation-axes, and if Ω is the resultant angular velocity,

$$\Omega = \omega_1 + \omega_2 + \ldots\ldots + \omega_n$$

$$= \Sigma \,.\, \omega; \tag{36}$$

where $\Sigma . \omega$ is the algebraical sum of the several components; but more will be said hereafter on this subject.

43.] If however the difference between ω_a and ω_b is infinitesimal, then ω is also infinitesimal; and if $\omega_a = \omega_b$, $\omega = 0$, and the resultant angular velocity vanishes. In this case however $x = y$; which can be only if $x = y = \infty$. Here then a paradox presents itself; when two component angular velocities with parallel axes are equal and have opposite signs, the resultant angular velocity is zero, and its axis is at an infinite distance. We must return to first principles.

Consider Fig. 12, wherein OA and O′B are the line-representatives of two equal angular velocities which have opposite directions: let OO′ = C, and take any particle P in the line OO′: let $\text{OP} = x$, $\text{O}'\text{P} = y$; then the downward path of P in the time dt due to ω_a and to $-\omega_a$

$$\begin{aligned} &= \omega_a x\, dt + \omega_a y\, dt, \\ &= \omega_a (x+y)\, dt, \\ &= \omega_a c\, dt; \end{aligned} \tag{37}$$

and therefore is the same, whatever is the place of P. Thus all particles of the body are advanced in the time dt along a distance equal to $\omega_a c dt$ and perpendicular to the plane containing the two parallel axes of the component angular velocities. The effect therefore of a body moving with such a pair of equal and opposite angular velocities is a displacement of translation of the body over a distance proportional to the product of either angular velocity and the perpendicular distance between the two axes. M. Poinsot, to whom we are indebted for the laws of composition and resolution of angular velocities, calls such a pair of equal and opposite angular velocities a couple of angular velocities, and the product $\omega_a c$ he calls the moment of the couple. The analogy is evident between these theorems and those of statical couples.

Hence a couple of angular velocities gives a body a displacement of translation equal to $\omega_a c dt$ in the time dt, and along a line perpendicular to the plane of the axes of the couple.

Hence also it is evident that a couple may be equivalently replaced by any other equimomental couple provided that the planes containing the axes of the couples are either parallel or identical. And the geometrical representation of a couple is a straight line whose length is proportional to the moment of the

couple, and which is perpendicular to the plane of the axes of the couple.

44.] Lastly, let us consider the most general case; that in which a body moves with many simultaneous angular velocities, the rotation-axes of which do not pass through one and the same point, and are not parallel.

As the signs of angular velocity are arbitrary, it is convenient for our present purpose to affect them with those which are best suited to a system of coordinate axes in space. Let then those angular velocities be considered positive with which, having for their rotation-axes severally the axes of x, y, z, the body turns from the y-axis to the z-axis, from the z-axis to the x-axis, from the x-axis to the y-axis respectively; and let those be negative with which the body rotates in opposite directions. This system is evidently cyclical, and is easily remembered.

Let the angular velocities be $\omega_1, \omega_2, \dots \omega_n$; and let a point O rigidly connected with the body be the origin; at it let a system of rectangular coordinates fixed in space originate; and let the direction-angles of the rotation-axes be $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2, \dots \alpha_n, \beta_n, \gamma_n$; let (x_1, y_1, z_1), (x_2, y_2, z_2), $\dots (x_n, y_n, z_n)$ be points severally on the rotation-axis of each, and let $p_1, p_2, \dots p_n$ be the perpendicular distances from O on the several rotation-axes. And of all these quantities let ω, (α, β, γ), (x, y, z), p be the types. Let us consider the type velocity ω. At the origin let a pair of equal and opposite angular velocities be introduced, each of which is equal to ω, and the rotation-axis of which is parallel to that of ω; and from O let the perpendicular distance p be drawn to the rotation-axis of ω; so that instead of the original ω, we have ω at O equal to the original ω and with a rotation-axis parallel to that of the original ω, and a couple of angular velocities, each of which is ω, and the distance between whose axes is p; so that $p\omega$ is the moment of the couple; and the effect of which is a displacement of the body in the time dt over a distance equal to $\omega p\, dt$ in a line perpendicular to the plane which contains the origin and the rotation-axis of ω. Let a similar process be performed on all the angular velocities; then we have a system of angular velocities the rotation-axes of which pass through O, and also a system of couples of angular velocities, the effects of which are severally a displacement of translation of the body.

Let Ω be the resultant angular velocity of all those which act at O; let $\Omega_x, \Omega_y, \Omega_z$ be its axial components; and let a, b, c be the direction-angles of its rotation-axis; then

$$\Omega_x = \omega_1 \cos a_1 + \omega_2 \cos a_2 + \ldots\ldots + \omega_n \cos a_n$$
$$= \Sigma . \omega \cos a; \tag{38}$$
$$\Omega_y = \Sigma . \omega \cos \beta; \tag{39}$$
$$\Omega_z = \Sigma . \omega \cos \gamma; \tag{40}$$
$$\therefore \quad \Omega^2 = \Omega_x^2 + \Omega_y^2 + \Omega_z^2,$$
$$= (\Sigma . \omega \cos a)^2 + (\Sigma . \omega \cos \beta)^2 + (\Sigma . \omega \cos \gamma)^2; \tag{41}$$

and
$$\frac{\Sigma . \omega \cos a}{\cos a} = \frac{\Sigma . \omega \cos \beta}{\cos b} = \frac{\Sigma . \omega \cos \gamma}{\cos c} = \Omega; \tag{42}$$

whereby the intensity and the direction-cosines of the rotation-axis of the resultant angular velocity through O are known.

45.] As to the couple of angular velocities which arises from ω, the moment of the couple is $p\omega$; and as p is the perpendicular from the origin on a line passing through (x, y, z), whose direction-angles are (a, β, γ), we have

$$p^2 = (z\cos\beta - y\cos\gamma)^2 + (x\cos\gamma - z\cos a)^2 + (y\cos a - x\cos\beta)^2. \tag{43}$$

Now the displacement of translation which the body undergoes by virtue of this couple of angular velocities is along a line perpendicular to the rotation-axis of ω and to p; so that its direction-cosines are

$$\frac{z\cos\beta - y\cos\gamma}{p}, \quad \frac{x\cos\gamma - z\cos a}{p}, \quad \frac{y\cos a - x\cos\beta}{p}. \tag{44}$$

Let $\Delta\sigma$ be the space through which the body is displaced in the time dt by reason of this couple of angular velocities; then

$$\Delta\sigma = \omega p\, dt; \tag{45}$$

and the direction-cosines of $\Delta\sigma$ are given by (44); so that if $\Delta\xi, \Delta\eta, \Delta\zeta$ are the axial projections of $\Delta\sigma$,

$$\left.\begin{aligned} \Delta\xi &= \omega\,(z\cos\beta - y\cos\gamma)\,dt, \\ \Delta\eta &= \omega\,(x\cos\gamma - z\cos a)\,dt, \\ \Delta\zeta &= \omega\,(y\cos a - x\cos\beta)\,dt. \end{aligned}\right\} \tag{46}$$

A result similar to this is true for each component angular velocity; and therefore if σ is the whole space through which the origin is transferred, and if ξ, η, ζ are the axial projections of σ,

$$\left.\begin{aligned} \xi &= \Sigma . \omega\,(z\cos\beta - y\cos\gamma)\,dt, \\ \eta &= \Sigma . \omega\,(x\cos\gamma - z\cos a)\,dt, \\ \zeta &= \Sigma . \omega\,(y\cos a - x\cos\beta)\,dt; \end{aligned}\right\} \tag{47}$$

and
$$\sigma^2 = \xi^2 + \eta^2 + \zeta^2; \tag{48}$$

and the direction-cosines of σ are

$$\frac{\Sigma.\omega(z\cos\beta-y\cos\gamma)\,dt}{\sigma}, \quad \frac{\Sigma.\omega(x\cos\gamma-z\cos\alpha)\,dt}{\sigma}, \quad \frac{\Sigma.\omega(y\cos\alpha-x\cos\beta)\,dt}{\sigma}; \quad (4$$

whereby we have the resultant motions, both of translation and of rotation about an axis through the origin, of a body moving with many simultaneous angular velocities.

46.] If the simultaneous angular velocities $\omega_1, \omega_2, \ldots \omega_n$ are such that the body is at rest, then $\Omega = 0$, and $\sigma = 0$; so that we have the six conditions

$$\left.\begin{aligned} \Sigma.\omega\cos\alpha &= 0,\\ \Sigma.\omega\cos\beta &= 0,\\ \Sigma.\omega\cos\gamma &= 0; \end{aligned}\right\} \quad (50) \qquad \left.\begin{aligned} \Sigma.\omega(z\cos\beta-y\cos\gamma) &= 0,\\ \Sigma.\omega(x\cos\gamma-z\cos\alpha) &= 0,\\ \Sigma.\omega(y\cos\alpha-x\cos\beta) &= 0; \end{aligned}\right\} \quad (51)$$

from these equations theorems can be deduced similar to those of Arts. 71–75, Vol. III.

47.] If the angular velocities are capable of composition into a single angular velocity, let Ω be the single angular velocity, $\Omega_x, \Omega_y, \Omega_z$ its axial components, and (x', y', z') a point on its axis; then as the effect of Ω is to be identical in all respects with the combined effects of the component angular velocities,

$$\Omega_x = \Sigma.\omega\cos\alpha, \quad \Omega_y = \Sigma.\omega\cos\beta, \quad \Omega_z = \Sigma.\omega\cos\gamma; \qquad (52)$$

$$\left.\begin{aligned} \Omega_y z' - \Omega_z y' &= \xi,\\ \Omega_z x' - \Omega_x z' &= \eta,\\ \Omega_x y' - \Omega_y x' &= \zeta; \end{aligned}\right\} \qquad (53)$$

from the last of which we have the condition

$$\xi\Omega_x + \eta\Omega_y + \zeta\Omega_z = 0; \qquad (54)$$

which equation shews that the resultant line of displacement is perpendicular to the rotation-axis of the resultant angular velocity; and as the resultant displacement may be replaced by two equal and opposite angular velocities whose rotation-axes are perpendicular to the line of displacement, we may take these rotation-axes to be parallel to that of Ω, whereby we shall have three angular velocities with parallel rotation-axes which may be compounded into a single angular velocity.

48.] If the axes of all the simultaneous couples are parallel, then

$$\Omega\cos a = \cos\alpha\,\Sigma.\omega, \quad \Omega\cos b = \cos\beta\,\Sigma.\omega, \quad \Omega\cos c = \cos\gamma\,\Sigma.\omega;$$

$$\left.\begin{aligned} \therefore\quad a = \alpha, \qquad b = \beta, \qquad c &= \gamma;\\ \Omega = \Sigma.\omega; \end{aligned}\right\} \qquad (55)$$

that is, the resultant angular velocity is equal to the sum of all the component angular velocities, and its rotation-axis is parallel to the rotation-axes of the components. Also

$$\left.\begin{aligned}\xi &= (\cos\beta\,\Sigma.\omega z - \cos\gamma\,\Sigma.\omega y)\,dt,\\ \eta &= (\cos\gamma\,\Sigma.\omega x - \cos\alpha\,\Sigma.\omega z)\,dt,\\ \zeta &= (\cos\alpha\,\Sigma.\omega y - \cos\beta\,\Sigma.\omega x)\,dt.\end{aligned}\right\}\tag{56}$$

In this case (54) is satisfied; and the angular velocities are capable of reduction to a single angular velocity; let Ω be the resultant, and let $(\bar{x}, \bar{y}, \bar{z})$ be a point on its rotation-axis; then as it produces the same effect as all the components taken in combination, and as the direction-angles of its axis are α, β, γ,

$$\left.\begin{aligned}\xi &= (\cos\beta\,\Omega\bar{z} - \cos\gamma\,\Omega\bar{y})\,dt,\\ \eta &= (\cos\gamma\,\Omega\bar{x} - \cos\alpha\,\Omega\bar{z})\,dt,\\ \zeta &= (\cos\alpha\,\Omega\bar{y} - \cos\beta\,\Omega\bar{x})\,dt;\end{aligned}\right\}\tag{57}$$

which are severally equal to the values given in (56). But as these results are true, whatever be the system of coordinate axes, α, β, γ are manifestly indeterminate; so that we have

$$\bar{x} = \frac{\Sigma.\omega x}{\Sigma.\omega}, \qquad \bar{y} = \frac{\Sigma.\omega y}{\Sigma.\omega}, \qquad \bar{z} = \frac{\Sigma.\omega z}{\Sigma.\omega}. \tag{58}$$

$(\bar{x}, \bar{y}, \bar{z})$ is called the *centre* of the parallel angular velocities.

49.] From the preceding Articles it appears that if a body is moving with many simultaneous angular velocities, the resultant motion consists (1) of a determinate angular velocity Ω, the rotation-axis of which is determinate in direction but not in position; and (2) of a displacement of translation of an arbitrarily chosen point on the axis of the resultant angular velocity Ω, the line of motion and amount of which are given by equations (47) and (48). Now whatever is the position of this arbitrarily chosen point, the resultant angular velocity is the same in intensity and direction, but the amount and the line of the displacement of translation of the point varies with the point; for although in the infinitesimal time dt the displacement is infinitesimal, yet equations (47) shew that it varies with the origin. Let us therefore inquire what is the position of the origin, when that displacement is the least; the form of (47) indicates that it does not admit of a maximum, but it may be a minimum.

Let the directions of the coordinate axes be unchanged; and let (x_0, y_0, z_0) be the origin for which the displacement of translation is the least; so that for x, y, z in (47) we must substitute

$x-x_0, y-y_0, z-z_0$; and let ξ_0, η_0, ζ_0 be the axial components of the displacement σ_0 of the new origin; then

$$\xi_0 = \Sigma.\omega\,\{(z-z_0)\cos\beta-(y-y_0)\cos\gamma\}\,dt;$$
$$= \Sigma.\omega\,(z\cos\beta-y\cos\gamma)\,dt+y_0\Sigma.\omega\cos\gamma\,dt-z_0\Sigma.\omega\cos\beta\,dt;$$
$$\left.\begin{aligned} &= \xi-(z_0\,\Omega_y-y_0\,\Omega_z)\,dt,\\ \eta_0 &= \eta-(x_0\,\Omega_z-z_0\,\Omega_x)\,dt,\\ \zeta_0 &= \zeta-(y_0\,\Omega_x-x_0\,\Omega_y)\,dt;\end{aligned}\right\} \qquad (59)$$

$$\therefore\quad \sigma_0^2 = \{\xi-(z_0\,\Omega_y-y_0\,\Omega_z)\,dt\}^2+\{\eta-(x_0\Omega_z-z_0\Omega_x)\,dt\}^2$$
$$+\{\zeta-(y_0\Omega_x-x_0\Omega_y)\,dt\}^2. \qquad (60)$$

Thus σ_0^2 is a function of x_0, y_0, z_0 which are three independent variables; whence equating to zero the three partial differentials of σ_0, as in Art. 82, Vol. III, we have

$$\frac{x_0-\dfrac{\zeta\,\Omega_y-\eta\,\Omega_z}{\Omega^2\,dt}}{\Omega_x} = \frac{y_0-\dfrac{\xi\,\Omega_z-\zeta\,\Omega_x}{\Omega^2\,dt}}{\Omega_y} = \frac{z_0-\dfrac{\eta\,\Omega_x-\xi\,\Omega_y}{\Omega^2\,dt}}{\Omega_z}; \qquad (61)$$

which are the equations to a straight line; this line therefore is the locus of points for which the displacement of translation due to the simultaneous angular velocities is a minimum. The line is evidently parallel to the rotation-axis of Ω; and passes through a point whose coordinates are

$$\frac{\zeta\,\Omega_y-\eta\,\Omega_z}{\Omega^2\,dt},\qquad \frac{\xi\,\Omega_z-\zeta\,\Omega_x}{\Omega^2\,dt},\qquad \frac{\eta\,\Omega_x-\xi\,\Omega_y}{\Omega^2\,dt}; \qquad (62)$$

which give the following geometrical construction. At any point O taken arbitrarily, let the displacement of translation σ and its axial components ξ, η, ζ be drawn; also let the rotation-axis of Ω be drawn. Let ϕ be the angle between these two lines; through O draw a line perpendicular to both of them; and along that perpendicular from O take a length OD $= p$, such that

$$p = \frac{\sigma\sin\phi}{\Omega\,dt}; \qquad (63)$$

then a line drawn through D parallel to the rotation-axis of Ω is that whose equations are (61). This line is called the central axis of the system, and the equations to it can be found directly from this conception of it.

50.] If the origin is taken on the line, $p = 0$; and therefore $\sin\phi = 0$; so that the line of displacement lies along the rotation-axis of Ω. In the case therefore of the central axis the system of simultaneous angular velocities is reduced to (1) an angular velocity Ω about a determinate axis, and (2) a displacement of

translation σ along this axis. In the infinitesimal time dt the body rotates through an angle $\Omega\, dt$ about the axis, and moves along the axis over a distance σ which is given by equation (48). Thus it has a helical motion: while it rotates with a given angular velocity, it also advances along the rotation-axis with a determinate linear velocity. Hence it is, that such a motion is called a screw, and the central axis is called the axis of the screw. The ratio of the velocity of translation along the central axis to the angular velocity about it is called the pitch of the screw; thus if v is the velocity of translation along the central axis, and Ω is the angular velocity,

$$\text{the pitch} = \frac{\text{v}}{\Omega}\,*.$$

This is one of the most simple images which the motion of a rigid body admits of.

Hence as the motion of any particle of the body is made up of the motion of rotation about the central axis, and of the motion of translation along it, and as the projection of the former on the central axis is zero, it follows that the projections on the central axis, and consequently on any axis of rotation, since all rotation-axes are parallel, of the displacements of all particles of a body are equal.

I may observe that the equations (61) of the central axis may be found by investigating the locus of points at which the displacement of translation lies along the rotation-axis of Ω.

51.] Now when a body has a continuous motion the system of the central axes forms a ruled surface in space, and another ruled surface in the body; all the generators of the second successively coincide with those of the first. In imagining such a motion let us suppose that of translation or the sliding of the axis to take place before the rotation about the axis; let two generators be placed on each other in their corresponding positions, and let the sliding along them take place over the distance σ; and then let the body turn about the common generator of the two surfaces through the angle $\Omega\, dt$; by this means the next two generators will be brought into coincidence; and the corresponding sliding and rotation will again take place; and so on; whereby the two surfaces will be successively brought into

* For a further discussion on Screws, see "The Theory of Screws," by R. S. Ball. Dublin. 1876.

contact with each other along their generators. If one of the surfaces is developable, the other is; and if $\sigma = 0$ throughout the motion, in which case there is no sliding, the two surfaces are evidently cones. This also is the case when one point in the body is fixed, so that all the axes pass through that point. The line of contact of the two surfaces is called the instantaneous sliding axis.

The case however in which the body has a fixed point, and in which the ruled surfaces become cones, deserves further illustration. Let O, see Fig. 13, be the fixed point of the body; and from it let a sphere be described of any radius; and let the cones which are respectively fixed in space and fixed in the body be cut by the surface of the sphere in the curves IS and IS′; let I be the point where the instantaneous axis meets the spherical surface; let s and s' be the arcs of the two curves IS and IS′ respectively; of which let the length-elements be ds and ds'; let OI be the instantaneous line of contact of the two cones, and let the time be divided into equal infinitesimal elements dt; also let the curves s and s' be divided into elements corresponding to successive dt's, and so that the corresponding elements are equal. Let OS be the cone fixed in space, and OS′ that fixed in the body; then the motion of the body will be represented by the rolling, without sliding, of the latter cone on the former; the line of contact of the two cones being always the instantaneous axis of the body. Hereafter we shall have a case in which we shall find the equations to these cones, and thus have their relative magnitudes and position. In the mean time I may observe that the rolling may take place in many ways. One cone may roll outside the other, as in Fig. 13; or the moveable cone may roll inside, as in Fig. 14; or again as in Fig. 15, where the moveable cone is larger than the fixed cone and rolls on the outside of it. Or again either of the cones may degenerate into a plane, as in Fig. 16; and either of the cones may become a straight line, in which case the axis of rotation is fixed. Or the cones may become identical, in which case the position of the rotation-axis is indeterminate.

Also the vertex of either cone may be at an infinite distance, in which case the cone becomes a cylinder; and as the instantaneous axes are all parallel, the path of every particle is in a plane perpendicular to the axis of the cylinder.

52.] Let us consider other properties of the central axis. From (59) we have

$$\frac{\xi_0 \Omega_x + \eta_0 \Omega_y + \zeta_0 \Omega_z}{\Omega} = \frac{\xi \Omega_x + \eta \Omega_y + \zeta \Omega_z}{\Omega}; \tag{64}$$

and by a process similar to that of Art. 77, Vol. III, it may be shewn that this quantity is invariant, through whatever angle the system of coordinate axes is turned. But if ϕ is the angle between σ corresponding to any origin and the rotation-axis of ω,

$$\cos\phi = \frac{\xi \Omega_x + \eta \Omega_y + \zeta \Omega_z}{\sigma \Omega};$$

$$\therefore \quad \sigma \cos\phi = \text{a constant} = \sigma_0; \tag{65}$$

where σ_0 is the displacement of translation along the central axis; that is, the projections of the displacements of translation of all points of the body on the central axis are equal; in other words, the body has a motion of translation along the central axis, and passes through the distance σ_0 in the time dt. If therefore a plane is drawn perpendicular to the central axis, the distances of all points of the body from this fixed plane are increased or diminished by the same quantity in the time dt. And the line of this quantity is, as shewn in Art. 49, that along which the displacement of translation is the least. Hereby we have the following construction for the direction of the central axis; through any point in space draw straight lines equal and parallel to the actual displacements of all the points of a body due to the time dt; then the ends of these lines will be all in one plane which is perpendicular to the central axis.

Again, all the properties of reciprocal lines which are demonstrated in Vol. III, Arts. 89–93, are true, mutatis mutandis, of angular velocities; indeed the principle of duality is completely applicable to these theorems. X, Y, Z, R, L, M, N, G are to be changed into Ω_x, Ω_y, Ω_z, Ω, ξ, η, ζ, σ respectively, and the enunciations of the theorems are to be changed accordingly; that is, pressures are to be changed into angular velocities, and moments are to be changed into displacements of translation. Thus, it is there proved that any system of pressures may be reduced into two pressures acting along lines at right angles to each other: hence it follows that every system of angular velocities may be reduced into two angular velocities whose axes are perpendicular to each other.

Again, as all moment-centres of equal moment lie on a cylin-

drical surface whose axis lies along the central axis; so all points for which the displacement of translation is the same lie on a cylindrical surface; also all the lines of equal displacement corresponding to points on a circle whose plane is perpendicular to the central axis lie on the surface of a hyperboloid of revolution.

So for all points on a line perpendicular to the central axis, the lines of displacement are in the surface of a hyperbolic paraboloid.

From the theorems of Art. 91, Vol. III, we have the following deductions. If a body moves with many simultaneous angular velocities, they may be reduced to two others whose rotation-axes are such that if one is given, the other is the reciprocal line; and the perpendicular line to these two axes passes through and is perpendicular to the central axis of the system.

And from Art. 93, it follows that every system of angular velocities may be replaced by two equal angular velocities, whose rotation-axes are perpendicular to each other, and each of which is inclined at 45° to the central axis, and the axes are perpendicular to a line which is bisected at right angles by the central axis; each of the angular velocities $= \frac{\Omega}{2^{\frac{1}{2}}}$; and the length of the perpendicular distance between the axes $= \frac{2\sigma_0}{\Omega}$.

53.] We have in the next place to determine the velocity of any particle of a body which has the most general motion; that is a motion of rotation as well as a motion of translation.

I will first take the case where one point is fixed, through which of course the rotation-axis passes, so that the displacement of a particle may be due to the rotation only; and I shall consider the displacements which take place during the infinitesimal time dt.

Let the fixed point of the body be taken as the origin, and let a system of coordinate axes fixed in space originate at it; let ω be the angular velocity with which the body rotates about the axis, of which let the direction-angles be (α, β, γ); let (x, y, z) be the place of the particle P, the displacement of which in the time dt is to be calculated; let ds be the displacement, of which let dx, dy, dz be the axial projections; let $\omega_x, \omega_y, \omega_z$ be the axial components of ω; so that

$$\frac{\omega_x}{\cos\alpha} = \frac{\omega_y}{\cos\beta} = \frac{\omega_z}{\cos\gamma} = \omega. \tag{66}$$

Let p be the perpendicular distance from (x, y, z) on the rotation-axis (α, β, γ); then as $\omega\, dt$ is the infinitesimal angle through which the body rotates in the time dt,

$$ds = p\,\omega\, dt; \tag{67}$$

and

$$p^2 = (z\cos\beta - y\cos\gamma)^2 + (x\cos\gamma - z\cos\alpha)^2 + (y\cos\alpha - x\cos\beta)^2. \tag{68}$$

Now ds is perpendicular to the rotation-axis, and to the line drawn from the origin to (x, y, z); also dx, dy, dz are proportional to its direction-cosines; therefore

$$\left.\begin{aligned} \cos\alpha\, dx + \cos\beta\, dy + \cos\gamma\, dz &= 0, \\ x\,dx + \quad y\,dy + \quad z\,dz &= 0; \end{aligned}\right\} \tag{69}$$

$$\therefore \quad \frac{dx}{z\cos\beta - y\cos\gamma} = \frac{dy}{x\cos\gamma - z\cos\alpha} = \frac{dz}{y\cos\alpha - x\cos\beta} \tag{70}$$

$$= \frac{ds}{p}; \tag{71}$$

$$\therefore \quad \left.\begin{aligned} dx &= (z\cos\beta - y\cos\gamma)\,\omega\, dt = (z\,\omega_y - y\,\omega_z)\,dt, \\ dy &= (x\cos\gamma - z\cos\alpha)\,\omega\, dt = (x\,\omega_z - z\,\omega_x)\,dt, \\ dz &= (y\cos\alpha - x\cos\beta)\,\omega\, dt = (y\,\omega_x - x\,\omega_y)\,dt; \end{aligned}\right\} \tag{72}$$

$$\therefore \quad \frac{ds^2}{dt^2} = (z\,\omega_y - y\,\omega_z)^2 + (x\,\omega_z - z\,\omega_x)^2 + (y\,\omega_x - x\,\omega_y)^2. \tag{73}$$

In all these expressions there has been an ambiguity of sign, which I have omitted; that sign has been taken which is in accordance with the principle of signs of angular velocities assumed in Article 44; for suppose the body to rotate about the axis of x only; so that $\omega_y = \omega_z = 0$; then for a particle in the first octant of space, dz, the increment of z, will be positive, and dy, the increment of y, will be negative; similarly for single rotations about the other coordinate axes; and equations (72) are in accordance with these conditions.

Hence the equations to the tangent line of the path which the particle m at (x, y, z) is taking at the time t are

$$\frac{\xi - x}{z\,\omega_y - y\,\omega_z} = \frac{\eta - y}{x\,\omega_z - z\,\omega_x} = \frac{\zeta - z}{y\,\omega_x - x\,\omega_y}. \tag{74}$$

Hence also if a body has a fixed point at the origin, and rotates with angular velocities $\omega_x, \omega_y, \omega_z$ about its three coordinate axes respectively, the rotation-axis of the body is the locus of the points which are at rest, and its equations consequently are

$$\frac{\xi}{\omega_x} = \frac{\eta}{\omega_y} = \frac{\zeta}{\omega_z}. \tag{75}$$

If $\frac{ds}{dt} =$ a constant, (73) is the equation to a circular cylinder: hence the locus of particles in a body moving about a fixed point which at any given instant have the same velocity is a circular cylinder.

54.] As an exact comprehension of (72) is of great importance, let us investigate these expressions by another and a more elementary process.

From (x, y, z), the place of m, let perpendiculars r_x, r_y, r_z be drawn to the three rectangular axes of x, y, z respectively; and let θ be the angle between r_x and the plane of (x, y), ϕ the angle between r_y and the plane of (y, z), ψ the angle between r_z and the plane of (z, x); so that $d\theta, d\phi, d\psi$ are positive according to our hypothesis of signs for small rotations about the axes of x, y, z respectively. Let rotations through infinitesimal angles take place successively about these axes, and let the changes of the variables due to these rotations be calculated. For a rotation about the axis of x through $d\theta$ we have

$$\begin{aligned} y &= r_x \cos\theta, & z &= r_x \sin\theta, \\ dy &= -r_x \sin\theta\, d\theta, & dz &= r_x \cos\theta\, d\theta, \\ &= -z\, d\theta; & &= y\, d\theta; \end{aligned}$$

so that the infinitesimal variations of y and z are respectively $-z\,d\theta$, and $y\,d\theta$; similarly for a rotation through an angle $d\phi$ about the axis of y, the variations of z and x are respectively $-x\,d\phi$ and $z\,d\phi$; and for a rotation through $d\psi$ about the axis of z, the variations of x and y are respectively $-y\,d\psi$ and $x\,d\psi$; so that if dx, dy, dz are the total variations of x, y, and z due to these combined rotations,

$$\left.\begin{aligned} dx &= z\,d\phi - y\,d\psi, \\ dy &= x\,d\psi - z\,d\theta, \\ dz &= y\,d\theta - x\,d\phi. \end{aligned}\right\} \qquad (76)$$

55.] We shall hereafter find it convenient to refer a body and its motion to two sets of coordinate axes at the same time; one of which is assumed to be fixed in space, and the other to be fixed in the body and to be moving with it. At first I will assume these two systems to originate at the same fixed point of the body so that the body is capable of only a motion of rotation about an axis passing through this fixed point; and I will suppose a point P to be (x, y, z) and (ξ, η, ζ) in reference re-

spectively to the systems fixed in space and fixed in the body; and I shall suppose the two systems to be related by the scheme of direction-cosines given in Art. 2. As the body moves about an axis, which passes through the origin, x, y, z and the nine direction-cosines vary, but ξ, η, ζ do not vary; so that from (2), Art. 2, we have for the variations in the time dt

$$\left.\begin{aligned} \frac{dx}{dt} &= \xi\frac{da_1}{dt} + \eta\frac{db_1}{dt} + \zeta\frac{dc_1}{dt}, \\ \frac{dy}{dt} &= \xi\frac{da_2}{dt} + \eta\frac{db_2}{dt} + \zeta\frac{dc_2}{dt}, \\ \frac{dz}{dt} &= \xi\frac{da_3}{dt} + \eta\frac{db_3}{dt} + \zeta\frac{dc_3}{dt}; \end{aligned}\right\} \tag{77}$$

which values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ express at the time t the axial components of the velocity of the particle which is at (x, y, z).

Let $\omega_x, \omega_y, \omega_z$ be the axial components of the angular velocity ω of the body along the fixed axes of x, y, z respectively; then from (72), Art. 53,

$$\begin{aligned} \frac{dx}{dt} &= z\,\omega_y - y\,\omega_z \\ &= \omega_y(a_3\xi + b_3\eta + c_3\zeta) - \omega_z(a_2\xi + b_2\eta + c_2\zeta); \end{aligned}$$

$$\left.\begin{aligned} \frac{dx}{dt} &= \xi(a_3\omega_y - a_2\omega_z) + \eta(b_3\omega_y - b_2\omega_z) + \zeta(c_3\omega_y - c_2\omega_z), \\ \frac{dy}{dt} &= \xi(a_1\omega_z - a_3\omega_x) + \eta(b_1\omega_z - b_3\omega_x) + \zeta(c_1\omega_z - c_3\omega_x), \\ \frac{dz}{dt} &= \xi(a_2\omega_x - a_1\omega_y) + \eta(b_2\omega_x - b_1\omega_y) + \zeta(c_2\omega_x - c_1\omega_y). \end{aligned}\right\} \tag{78}$$

In (77) and (78) (ξ, η, ζ) is the place of any particle, and therefore ξ, η, ζ are indeterminate; so that the systems of equations are identical; hence we may equate coefficients, and we have

$$\left.\begin{aligned} \frac{da_1}{dt} &= a_3\omega_y - a_2\omega_z, & \frac{db_1}{dt} &= b_3\omega_y - b_2\omega_z, & \frac{dc_1}{dt} &= c_3\omega_y - c_2\omega_z; \\ \frac{da_2}{dt} &= a_1\omega_z - a_3\omega_x, & \frac{db_2}{dt} &= b_1\omega_z - b_3\omega_x, & \frac{dc_2}{dt} &= c_1\omega_z - c_3\omega_x; \\ \frac{da_3}{dt} &= a_2\omega_x - a_1\omega_y, & \frac{db_3}{dt} &= b_2\omega_x - b_1\omega_y, & \frac{dc_3}{dt} &= c_2\omega_x - c_1\omega_y. \end{aligned}\right\} \tag{79}$$

56.] These formulæ are important, and it is necessary to understand their meaning; we have arrived at them indirectly, and therefore let us prove them by another process.

The circumstances of the body are these. A point in it is

fixed and is the origin; at it originate (1) a system of coordinate axes (x, y, z) fixed absolutely in space; (2) a system of coordinate axes (ξ, η, ζ) fixed in the moving body: the body rotates with an angular velocity ω about an axis such that the axial components of ω along the axes of x, y, z are $\omega_x, \omega_y, \omega_z$; at the time t the direction-cosines of the axis of ξ are a_1, a_2, a_3: What are the changes of these quantities due to the angular velocity ω? Take a point (x, y, z) on the axis of ξ at a distance ρ from the origin; then

$$\frac{x}{a_1} = \frac{y}{a_2} = \frac{z}{a_3} = \rho;$$

$$\therefore \quad \frac{dx}{dt} = \rho\frac{da_1}{dt}, \qquad \frac{dy}{dt} = \rho\frac{da_2}{dt}, \qquad \frac{dz}{dt} = \rho\frac{da_3}{dt};$$

and replacing $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ by their values given in (72), we have

$$\left.\begin{aligned} \frac{da_1}{dt} &= \frac{1}{\rho}(z\omega_y - y\omega_z) \\ &= a_3\omega_y - a_2\omega_z, \\ \frac{da_2}{dt} &= a_1\omega_z - a_3\omega_x, \\ \frac{da_3}{dt} &= a_2\omega_x - a_1\omega_y; \end{aligned}\right\} \tag{80}$$

and the other six equivalents of (79) may be found by similar

$= 1, \frac{da_1}{dt}, \frac{da_2}{dt}, \frac{da_3}{dt}$ are severally the axial com-

velocity of the particle on the axis of ξ at an unit

the origin at the time t.

by also the axial components $\omega_x, \omega_y, \omega_z$ may be de-

terms of the t-differentials of the direction-cosines.

multiply the three equations in the last horizontal row

y a_2, b_2, c_2 respectively and add, then

$$\omega_x = a_2\frac{da_3}{dt} + b_2\frac{db_3}{dt} + c_2\frac{dc_3}{dt}; \tag{81}$$

t us multiply the three equations in the middle row of

y a_3, b_3, c_3 respectively and add; then

$$-\omega_x = a_3\frac{da_2}{dt} + b_3\frac{db_2}{dt} + c_3\frac{dc_2}{dt}; \tag{82}$$

these two values of ω_x are in accordance with the first equation of (6), Art. 2.

Hence, and from similar processes, we have

$$\omega_x = a_2\frac{da_3}{dt} + b_2\frac{db_3}{dt} + c_2\frac{dc_3}{dt} = -\left\{a_3\frac{da_2}{dt} + b_3\frac{db_2}{dt} + c_3\frac{dc_2}{dt}\right\}; \quad (83)$$

$$\omega_y = a_3\frac{da_1}{dt} + b_3\frac{db_1}{dt} + c_3\frac{dc_1}{dt} = -\left\{a_1\frac{da_3}{dt} + b_1\frac{db_3}{dt} + c_1\frac{dc_3}{dt}\right\}; \quad (84)$$

$$\omega_z = a_1\frac{da_2}{dt} + b_1\frac{db_2}{dt} + c_1\frac{dc_2}{dt} = -\left\{a_2\frac{da_1}{dt} + b_2\frac{db_1}{dt} + c_2\frac{dc_1}{dt}\right\}. \quad (85)$$

Hence also the equations (75) to the rotation-axis may be expressed in terms of the t-differentials of the direction-cosines of the axes fixed in the body.

58.] Also, if ω_ξ, ω_η, ω_ζ are the axial components along the axes of ξ, η, ζ of the angular velocity of the body at the time t,

$$\left.\begin{aligned} \omega_\xi &= a_1\omega_x + a_2\omega_y + a_3\omega_z, \\ \omega_\eta &= b_1\omega_x + b_2\omega_y + b_3\omega_z, \\ \omega_\zeta &= c_1\omega_x + c_2\omega_y + c_3\omega_z; \end{aligned}\right\} \quad (86)$$

and conversely,

$$\left.\begin{aligned} \omega_x &= a_1\omega_\xi + b_1\omega_\eta + c_1\omega_\zeta, \\ \omega_y &= a_2\omega_\xi + b_2\omega_\eta + c_2\omega_\zeta, \\ \omega_z &= a_3\omega_\xi + b_3\omega_\eta + c_3\omega_\zeta. \end{aligned}\right\} \quad (87)$$

Also, from the second vertical row of (79), we have

$$\begin{aligned} c_1\frac{db_1}{dt} + c_2\frac{db_2}{dt} + c_3\frac{db_3}{dt} &= (b_2c_3 - b_3c_2)\,\omega_x + (b_3c_1 - b_1c_3)\,\omega_y + (b_1c_2 - b_2c_1)\,\omega_z \\ &= a_1\omega_x + a_2\omega_y + a_3\omega_z \\ &= \omega_\xi, \end{aligned}$$

using the equalities contained in (11), (12), and (13) of Art. 2. By a similar process, equivalents of ω_η and ω_ζ in terms of the t-differentials of the direction-cosines are determined, and we have

$$\left.\begin{aligned} \omega_\xi &= c_1\frac{db_1}{dt} + c_2\frac{db_2}{dt} + c_3\frac{db_3}{dt} = -\left\{b_1\frac{dc_1}{dt} + b_2\frac{dc_2}{dt} + b_3\frac{dc_3}{dt}\right\}, \\ \omega_\eta &= a_1\frac{dc_1}{dt} + a_2\frac{dc_2}{dt} + a_3\frac{dc_3}{dt} = -\left\{c_1\frac{da_1}{dt} + c_2\frac{da_2}{dt} + c_3\frac{da_3}{dt}\right\}, \\ \omega_\zeta &= b_1\frac{da_1}{dt} + b_2\frac{da_2}{dt} + b_3\frac{da_3}{dt} = -\left\{a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right\}. \end{aligned}\right\} \quad (88)$$

The t-differentials of the direction-cosines may be determined as follows in terms of ω_ξ, ω_η, and ω_ζ. From (79) and (87) we have

$$\begin{aligned} \frac{da_1}{dt} &= a_3\omega_y - a_2\omega_z \\ &= a_3(a_2\omega_\xi + b_2\omega_\eta + c_2\omega_\zeta) - a_2(a_3\omega_\xi + b_3\omega_\eta + c_3\omega_\zeta) \\ &= (c_2a_3 - c_3a_2)\,\omega_\zeta - (a_2b_3 - a_3b_2)\,\omega_\eta \\ &= b_1\omega_\zeta - c_1\omega_\eta. \end{aligned}$$

So that hence, and by similar processes, we have

$$\left.\begin{aligned}
\frac{da_1}{dt} &= b_1\omega_\zeta - c_1\omega_\eta, & \frac{da_2}{dt} &= b_2\omega_\zeta - c_2\omega_\eta, & \frac{da_3}{dt} &= b_3\omega_\zeta - c_3\omega_\eta,\\
\frac{db_1}{dt} &= c_1\omega_\xi - a_1\omega_\zeta, & \frac{db_2}{dt} &= c_2\omega_\xi - a_2\omega_\zeta, & \frac{db_3}{dt} &= c_3\omega_\xi - a_3\omega_\zeta,\\
\frac{dc_1}{dt} &= a_1\omega_\eta - b_1\omega_\xi, & \frac{dc_2}{dt} &= a_2\omega_\eta - b_2\omega_\xi, & \frac{dc_3}{dt} &= a_3\omega_\eta - b_3\omega_\xi.
\end{aligned}\right\} \quad (89)$$

59.] The groups (79) and (89) involve some remarkable results which will be useful in the sequel.

From (79) we have

$$\left.\begin{aligned}
\left(\frac{da_1}{dt}\right)^2 + \left(\frac{db_1}{dt}\right)^2 + \left(\frac{dc_1}{dt}\right)^2 &= \omega_y^2 + \omega_z^2,\\
\left(\frac{da_2}{dt}\right)^2 + \left(\frac{db_2}{dt}\right)^2 + \left(\frac{dc_2}{dt}\right)^2 &= \omega_z^2 + \omega_x^2,\\
\left(\frac{da_3}{dt}\right)^2 + \left(\frac{db_3}{dt}\right)^2 + \left(\frac{dc_3}{dt}\right)^2 &= \omega_x^2 + \omega_y^2.
\end{aligned}\right\} \quad (90)$$

And similarly from (89) we have

$$\left.\begin{aligned}
\left(\frac{da_1}{dt}\right)^2 + \left(\frac{da_2}{dt}\right)^2 + \left(\frac{da_3}{dt}\right)^2 &= \omega_\eta^2 + \omega_\zeta^2,\\
\left(\frac{db_1}{dt}\right)^2 + \left(\frac{db_2}{dt}\right)^2 + \left(\frac{db_3}{dt}\right)^2 &= \omega_\zeta^2 + \omega_\xi^2,\\
\left(\frac{dc_1}{dt}\right)^2 + \left(\frac{dc_2}{dt}\right)^2 + \left(\frac{dc_3}{dt}\right)^2 &= \omega_\xi^2 + \omega_\eta^2.
\end{aligned}\right\} \quad (91)$$

Also we have from (79)

$$\left.\begin{aligned}
\frac{da_2}{dt}\frac{da_3}{dt} + \frac{db_2}{dt}\frac{db_3}{dt} + \frac{dc_2}{dt}\frac{dc_3}{dt} &= -\omega_y\omega_z,\\
\frac{da_3}{dt}\frac{da_1}{dt} + \frac{db_3}{dt}\frac{db_1}{dt} + \frac{dc_3}{dt}\frac{dc_1}{dt} &= -\omega_z\omega_x,\\
\frac{da_1}{dt}\frac{da_2}{dt} + \frac{db_1}{dt}\frac{db_2}{dt} + \frac{dc_1}{dt}\frac{dc_2}{dt} &= -\omega_x\omega_y.
\end{aligned}\right\} \quad (92)$$

And from (89)

$$\left.\begin{aligned}
\frac{db_1}{dt}\frac{dc_1}{dt} + \frac{db_2}{dt}\frac{dc_2}{dt} + \frac{db_3}{dt}\frac{dc_3}{dt} &= -\omega_\eta\omega_\zeta,\\
\frac{dc_1}{dt}\frac{da_1}{dt} + \frac{dc_2}{dt}\frac{da_2}{dt} + \frac{dc_3}{dt}\frac{da_3}{dt} &= -\omega_\zeta\omega_\xi,\\
\frac{da_1}{dt}\frac{db_1}{dt} + \frac{da_2}{dt}\frac{db_2}{dt} + \frac{da_3}{dt}\frac{db_3}{dt} &= -\omega_\xi\omega_\eta.
\end{aligned}\right\} \quad (93)$$

Also from (79) we have

$$\left.\begin{aligned}
a_2\frac{da_3}{dt}-a_3\frac{da_2}{dt} &= b_1\omega_\eta+c_1\omega_\zeta,\\
a_3\frac{da_1}{dt}-a_1\frac{da_3}{dt} &= b_2\omega_\eta+c_2\omega_\zeta,\\
a_1\frac{da_2}{dt}-a_2\frac{da_1}{dt} &= b_3\omega_\eta+c_3\omega_\zeta;
\end{aligned}\right\} \qquad (94)$$

$$\left.\begin{aligned}
b_2\frac{db_3}{dt}-b_3\frac{db_2}{dt} &= c_1\omega_\zeta+a_1\omega_\xi,\\
b_3\frac{db_1}{dt}-b_1\frac{db_3}{dt} &= c_2\omega_\zeta+a_2\omega_\xi,\\
b_1\frac{db_2}{dt}-b_2\frac{db_1}{dt} &= c_3\omega_\zeta+a_3\omega_\xi;
\end{aligned}\right\} \qquad (95)$$

$$\left.\begin{aligned}
c_2\frac{dc_3}{dt}-c_3\frac{dc_2}{dt} &= a_1\omega_\xi+b_1\omega_\eta,\\
c_3\frac{dc_1}{dt}-c_1\frac{dc_3}{dt} &= a_2\omega_\xi+b_2\omega_\eta,\\
c_1\frac{dc_2}{dt}-c_2\frac{dc_1}{dt} &= a_3\omega_\xi+b_3\omega_\eta.
\end{aligned}\right\} \qquad (96)$$

Similarly from (89) we have

$$\left.\begin{aligned}
c_1\frac{db_1}{dt}-b_1\frac{dc_1}{dt} &= a_2\omega_y+a_3\omega_z,\\
a_1\frac{dc_1}{dt}-c_1\frac{da_1}{dt} &= b_2\omega_y+b_3\omega_z,\\
b_1\frac{da_1}{dt}-a_1\frac{db_1}{dt} &= c_2\omega_y+c_3\omega_z;
\end{aligned}\right\} \qquad (97)$$

$$\left.\begin{aligned}
c_2\frac{db_2}{dt}-b_2\frac{dc_2}{dt} &= a_3\omega_z+a_1\omega_x,\\
a_2\frac{dc_2}{dt}-c_2\frac{da_2}{dt} &= b_3\omega_z+b_1\omega_x,\\
b_2\frac{da_2}{dt}-a_2\frac{db_2}{dt} &= c_3\omega_z+c_1\omega_x;
\end{aligned}\right\} \qquad (98)$$

$$\left.\begin{aligned}
c_3\frac{db_3}{dt}-b_3\frac{dc_3}{dt} &= a_1\omega_x+a_2\omega_y,\\
a_3\frac{dc_3}{dt}-c_3\frac{da_3}{dt} &= b_1\omega_x+b_2\omega_y,\\
b_3\frac{da_3}{dt}-a_3\frac{db_3}{dt} &= c_1\omega_x+c_2\omega_y.
\end{aligned}\right\} \qquad (99)$$

60.] The components of the absolute velocity of the particle m at the time t along the moving axes which are fixed in the body may thus be found; let them be v_ξ, v_η, v_ζ; then

$$v_\xi = a_1\frac{dx}{dt} + a_2\frac{dy}{dt} + a_3\frac{dz}{dt}$$
$$= \xi\left(a_1\frac{da_1}{dt} + a_2\frac{da_2}{dt} + a_3\frac{da_3}{dt}\right) + \eta\left(a_1\frac{db_1}{dt} + a_2\frac{db_2}{dt} + a_3\frac{db_3}{dt}\right)$$
$$+ \zeta\left(a_1\frac{dc_1}{dt} + a_2\frac{dc_2}{dt} + a_3\frac{dc_3}{dt}\right);$$

whence, and by similar processes for the other components, we have

$$\left.\begin{aligned} v_\xi &= \zeta\omega_\eta - \eta\,\omega_\zeta, \\ v_\eta &= \xi\omega_\zeta - \zeta\,\omega_\xi, \\ v_\zeta &= \eta\,\omega_\xi - \xi\,\omega_\eta. \end{aligned}\right\} \qquad (100)$$

These equations enable us to determine the position of the rotation-axis at the time t relatively to the axes fixed in the moving body; for if $v_\xi = v_\eta = v_\zeta = 0$,

$$\frac{\xi}{\omega_\xi} = \frac{\eta}{\omega_\eta} = \frac{\zeta}{\omega_\zeta}; \qquad (101)$$

and these are the equations to the line of quiescent particles.

Now, in the general motion of a body, the axial components of the angular velocities are functions of the time, and may be expressed in terms of t; and therefore the position of the rotation-axis will vary from time to time relatively to both systems of axes, and will describe a conical surface the vertex of which is the fixed point. If then we eliminate t from (75), the resulting equation will be that of the conical surface fixed in space; and if we eliminate t from (101), the resultant equation will be that of the conical surface fixed in the body; and these two conical surfaces will always have a generating line common; which will be the rotation-axis at the time t. These are the cones referred to in Art. 51.

61.] Let us now suppose the body to be free from all constraint: let us as heretofore take a point in or rigidly connected with the body to be the origin of a system of rectangular coordinate axes fixed in it and moving with it; relatively to a system of axes fixed in space, let, at the time t, (x, y, z) be the place of a type-particle m of the body, and let (x_0, y_0, z_0) be the place of the moving origin; and let (ξ, η, ζ) be the place of m relatively to the origin and to the axes which are fixed in and move with the body; then, taking the scheme of direction-cosines of Art. 2, we have

$$\left.\begin{aligned} x &= x_0 + a_1\xi + b_1\eta + c_1\zeta, \\ y &= y_0 + a_2\xi + b_2\eta + c_2\zeta, \\ z &= z_0 + a_3\xi + b_3\eta + c_3\zeta; \end{aligned}\right\} \qquad (102)$$

we have also the inverse system,

$$\left.\begin{aligned}\xi &= a_1(x-x_0)+a_2(y-y_0)+a_3(z-z_0),\\ \eta &= b_1(x-x_0)+b_2(y-y_0)+b_3(z-z_0),\\ \zeta &= c_1(x-x_0)+c_2(y-y_0)+c_3(z-z_0).\end{aligned}\right\} \tag{103}$$

As the body moves x_0, y_0, z_0 and the nine direction-cosines evidently vary; but ξ, η, ζ are constant; therefore

$$\left.\begin{aligned}\frac{dx}{dt} &= \frac{dx_0}{dt}+\xi\frac{da_1}{dt}+\eta\frac{db_1}{dt}+\zeta\frac{dc_1}{dt},\\ \frac{dy}{dt} &= \frac{dy_0}{dt}+\xi\frac{da_2}{dt}+\eta\frac{db_2}{dt}+\zeta\frac{dc_2}{dt},\\ \frac{dz}{dt} &= \frac{dz_0}{dt}+\xi\frac{da_3}{dt}+\eta\frac{db_3}{dt}+\zeta\frac{dc_3}{dt};\end{aligned}\right\} \tag{104}$$

which are the components along the fixed axes of the velocity of any particle m, and admit also of expression in the following forms;

$$\begin{aligned}\frac{dx}{dt} = \frac{dx_0}{dt} &+ (x-x_0)\left(a_1\frac{da_1}{dt}+b_1\frac{db_1}{dt}+c_1\frac{dc_1}{dt}\right)\\ &+ (y-y_0)\left(a_2\frac{da_1}{dt}+b_2\frac{db_1}{dt}+c_2\frac{dc_1}{dt}\right)\\ &+ (z-z_0)\left(a_3\frac{da_1}{dt}+b_3\frac{db_1}{dt}+c_3\frac{dc_1}{dt}\right);\end{aligned}$$

$$\therefore \quad \frac{dx}{dt} = \frac{dx_0}{dt}+(z-z_0)\,\omega_y-(y-y_0)\,\omega_z.$$

Also similarly,

$$\left.\begin{aligned}\frac{dx}{dt} &= \frac{dx_0}{dt}+(z-z_0)\,\omega_y-(y-y_0)\,\omega_z,\\ \frac{dy}{dt} &= \frac{dy_0}{dt}+(x-x_0)\,\omega_z-(z-z_0)\,\omega_x,\\ \frac{dz}{dt} &= \frac{dz_0}{dt}+(y-y_0)\,\omega_x-(x-x_0)\,\omega_y.\end{aligned}\right\} \tag{105}$$

Let v_ξ, v_η, v_ζ be the components of the velocity of m, along the moving axes of ξ, η, ζ; so that

$$\begin{aligned}v_\xi &= a_1\frac{dx}{dt}+a_2\frac{dy}{dt}+a_3\frac{dz}{dt}\\ &= a_1\frac{dx_0}{dt}+a_2\frac{dy_0}{dt}+a_3\frac{dz_0}{dt}+\xi\left(a_1\frac{da_1}{dt}+a_2\frac{da_2}{dt}+a_3\frac{da_3}{dt}\right)\\ &\quad+\eta\left(a_1\frac{db_1}{dt}+a_2\frac{db_2}{dt}+a_3\frac{db_3}{dt}\right)+\zeta\left(a_1\frac{dc_1}{dt}+a_2\frac{dc_2}{dt}+a_3\frac{dc_3}{dt}\right);\end{aligned}$$

$$\therefore \quad \left.\begin{aligned}v_\xi &= a_1\frac{dx_0}{dt}+a_2\frac{dy_0}{dt}+a_3\frac{dz_0}{dt}+\zeta\,\omega_\eta-\eta\,\omega_\zeta,\\ v_\eta &= b_1\frac{dx_0}{dt}+b_2\frac{dy_0}{dt}+b_3\frac{dz_0}{dt}+\xi\,\omega_\zeta-\zeta\,\omega_\xi,\\ v_\zeta &= c_1\frac{dx_0}{dt}+c_2\frac{dy_0}{dt}+c_3\frac{dz_0}{dt}+\eta\,\omega_\xi-\xi\,\omega_\eta.\end{aligned}\right\} \tag{106}$$

In these expressions the first three terms of each are the resolved parts along the axes of ξ, η, ζ of the velocity of the moving origin; and the last two terms are the velocities due to the rotation about an axis passing through the moving origin. But the point (x_0, y_0, z_0) is arbitrary; so that we have the following theorem;

An infinitesimal motion of a body in a time dt is compounded of a motion of translation of any particle of it at (x_0, y_0, z_0), and of a motion of rotation of the body about an axis passing through that point.

Now a question arises whether, when a body has the general motion described in the preceding paragraph, there is a point, or a locus of points in it, which is at the instant under consideration at rest. To shorten the expressions (106) let u, v, w, be the components along the axes of ξ, η, ζ of the absolute velocity of the moving origin; so that if the point (ξ, η, ζ) is at rest

$$u + \zeta\omega_\eta - \eta\,\omega_\zeta = 0, \quad v + \xi\omega_\zeta - \zeta\omega_\xi = 0, \quad w + \eta\,\omega_\xi - \xi\,\omega_\eta = 0; \quad (107)$$

$$\therefore \frac{\xi\omega^2 + v\,\omega_\zeta - w\,\omega_\eta}{\omega_\xi} = \frac{\eta\,\omega^2 + w\,\omega_\xi - u\,\omega_\zeta}{\omega_\eta} = \frac{\zeta\omega^2 + u\,\omega_\eta - v\,\omega_\xi}{\omega_\zeta}, \quad (108)$$

which are the equations to a straight line, all points of which are for the instant at rest. This straight line is called the instantaneous axis: it is evidently parallel to the rotation-axis which passes through the moving origin. It describes in its successive positions a ruled surface both in the moving body and in space, its position being at any instant the line of contact of these surfaces, as one rolls on the other: as its equations are the same as (61), Art. 49, this line is the central axis of the system; but since from the preceding equations $u\omega_\xi + v\,\omega_\eta + w\,\omega_\zeta = 0$, there is no motion of translation along it; and the axis of the resultant screw is the instantaneous axis, when the pitch of the screw vanishes. All this is it once apparent on kinematical principles.

If all the particles move in one and the same plane, or if every particle moves in one of a series of parallel planes, to which the plane of (x, y) is parallel, the instantaneous axis pierces the plane at a point which is called the instantaneous centre: and the loci of this centre in the body and in space are called centrodes. These curves are determined by means of the equations

$$u - \eta\,\omega = 0, \qquad v + \xi\omega = 0;$$

and in reference to the fixed origin

$$u-(y-y_0)\omega = 0, \qquad v+(x-x_0)\omega = 0.$$

62.] If the displacement of translation lies along the rotation-axis, the motion is compounded of a sliding along a line and of a rotation about that line: this is indeed the case which we have considered in Art. 50, the sliding axis of rotation being the central axis of the body; we may thus find the equations of the central axis.

Let (x_0, y_0, z_0) be any point chosen arbitrarily; of the velocity of which let u, v, w be the components. Let (x, y, z) be a point on the central axis; this point has therefore only the sliding displacement of translation, and the line of its motion is along the rotation-axis of the angular velocity; therefore dx, dy, dz are proportional to ω_x, ω_y, ω_z respectively. Hence from (105),

$$\frac{\frac{dx_0}{dt}+(z-z_0)\,\omega_y-(y-y_0)\,\omega_z}{\omega_x} = \frac{\frac{dy_0}{dt}+(x-x_0)\,\omega_z-(z-z_0)\,\omega_x}{\omega_y}$$
$$= \frac{\frac{dz_0}{dt}+(y-y_0)\,\omega_x-(x-x_0)\,\omega_y}{\omega_z}; \quad (109)$$

from which we have

$$\frac{(x-x_0)\,\omega^2+v\,\omega_z-w\,\omega_y}{\omega_x} = \frac{(y-y_0)\,\omega^2+w\,\omega_x-u\,\omega_z}{\omega_y}$$
$$= \frac{(z-z_0)\,\omega^2+u\,\omega_y-v\,\omega_x}{\omega_z}; \quad (110)$$

which are the equations to the central axis.

63.] Any system of rectangular axes may be transferred into any other rectangular system having the same origin by a single rotation through a definite angle about an axis passing through the common origin.

Let P be any point on this axis; and let P be (x, y, z) and (ξ, η, ζ) respectively in reference to the systems of coordinate axes given in Art. 2. And let these systems be connected by the scheme of direction-cosines given in that article. Then, as the position of P is not changed by the rotation, ξ, η, ζ are respectively equal to x, y, z; and therefore

$$\left.\begin{aligned} (a_1-1)x+a_2y+a_3z &= 0, \\ b_1x+(b_2-1)y+b_3z &= 0, \\ c_1x+c_2y+(c_3-1)z &= 0. \end{aligned}\right\} \quad (111)$$

Let λ, μ, ν be the direction-angles of the axes about which the

rotation takes place, and let θ be the angle through which the system is turned; then

$$\left.\begin{array}{l}(a_1-1)\cos\lambda + a_2\cos\mu + a_3\cos\nu = 0,\\ b_1\cos\lambda+(b_2-1)\cos\mu + b_2\cos\nu = 0,\\ c_1\cos\lambda + c_2\cos\mu+(c_3-1)\cos\nu = 0;\end{array}\right\} \quad (112)$$

whence, having regard to (11), (12), (13), Art. 2,

$$\left.\begin{array}{l}\dfrac{\cos\lambda}{a_1-b_2-c_3+1} = \dfrac{\cos\mu}{a_2+b_1} = \dfrac{\cos\nu}{a_3+c_1};\\ \dfrac{\cos\lambda}{a_2+b_1} = \dfrac{\cos\mu}{b_2-a_1-c_3+1} = \dfrac{\cos\nu}{b_3+c_2};\\ \dfrac{\cos\lambda}{a_3+c_1} = \dfrac{\cos\mu}{b_3+c_2} = \dfrac{\cos\nu}{c_3-a_1-b_2+1};\end{array}\right\} \quad (113)$$

and therefore

$$\frac{(\cos\lambda)^2}{a_1-b_2-c_3+1} = \frac{(\cos\mu)^2}{b_2-a_1-c_3+1} = \frac{(\cos\nu)^2}{c_3-a_1-b_2+1}; \quad (114)$$

which assign the position of the axis about which the rotation takes place.

If we make the following substitutions,

$$1-a_1 = \alpha, \quad 1-b_2 = \beta, \quad 1-c_3 = \gamma, \quad (115)$$

then $$\frac{(\cos\lambda)^2}{\beta+\gamma-\alpha} = \frac{(\cos\mu)^2}{\gamma+\alpha-\beta} = \frac{(\cos\nu)^2}{\alpha+\beta-\gamma} = \frac{1}{\alpha+\beta+\gamma}. \quad (116)$$

To determine θ. Imagine a sphere of radius unity described about the origin; and let the axes of x, y, z, of ξ, η, ζ, of rotation, respectively intersect the surface of the sphere in the points A, B, C, A$'$, B$'$, C$'$, and G; and let GA, GA$'$, &c. be joined by arcs of great circles. Then GA = GA$'$ = λ, GB = GB$'$ = μ, GC = GC$'$ = ν; also $\cos$ AA$'$ = a_1, $\cos$ BB$'$ = b_2, $\cos$ CC$'$ = c_3: also

AGA$'$ = BGB$'$ = CGC$'$ = θ.

Hence in the isosceles triangle AGA$'$

$$\cos\theta(\sin\lambda)^2 = a_1-(\cos\lambda)^2, \quad (117)$$

$$\therefore \quad 4\left(\sin\frac{\theta}{2}\right)^2 = \alpha+\beta+\gamma, \quad (118)$$

which gives the angle through which the system is turned about OG.

From (115)

$$\alpha = 2\left(\sin\frac{AA'}{2}\right)^2, \quad \beta = 2\left(\sin\frac{BB'}{2}\right)^2, \quad \gamma = 2\left(\sin\frac{CC'}{2}\right)^2. \quad (119)$$

It is evident from the preceding construction that A and A$'$ lie in a small circle of the sphere whose angular distance from G, the pole, is λ, and that the axis OA describes about OG a right circular cone whose semi-vertical angle is λ, as OA moves from its original position to OA$'$; similarly, OB and OC describe right cir-

cular cones about OG whose semi-vertical angles are respectively μ and ν.

Equations (111) are those of planes, each of which passes through the axis of rotation. The first of these evidently bisects the angle between the planes of (y, z) and of (η, ζ); and consequently bisects the arc AA′; similarly the other planes bisect the arcs BB′ and CC′ respectively.

64.] The two systems to which the rotation of a rigid body has been referred in the preceding Articles are related to each other by means of nine direction-cosines, but as six equations of condition are given, only three of the direction-cosines are arbitrary; that is, three independent variables are sufficient for passing from one system of rectangular coordinates to another, both of which originate at the same point. I proceed to explain the mode of expressing angular velocities referred to one system in terms of angular velocities referred to another system by means of the formulæ which are investigated in Art. 3; and to give clearness to our thoughts let us consider the systems of axes which are delineated in Fig. 1. O is the common point at which the two systems originate, and from O as a centre let a sphere of radius unity be described, on the surface of which the great circles delineated in the figure are supposed to be. Let ω_x, ω_y, ω_z, ω_ξ, ω_η, ω_ζ be the angular velocities about the axes of $x, y, z, \xi, \eta, \zeta$ respectively. Let the planes of (x,y) and of (ξ,η) intersect in the line ON, and be inclined to each other at the angle θ; so that θ is also the angle between the axes of z and ζ; ON is technically called the line of nodes. Let, as in Art. 3, xON $= \psi$, ξON $= \phi$; then, as the body moves, θ, ψ, and ϕ vary. The angle θ is technically called the obliquity, and $\frac{d\theta}{dt}$ is the angular velocity of the body about the line ON; $\frac{d\phi}{dt}$ is the angular velocity about the axis of ζ; $\frac{d\psi}{dt}$ is the angular velocity about the axis of z, and indicates the velocity with which ON moves along the plane of (x,y); it is called the velocity of precession, the precession being the angle xON; and the precession is direct or retrograde according as the angle ψ is increasing or diminishing. The angular velocity $\frac{d\theta}{dt}$ is sometimes called the nutation; of these terms however and of their origin more will be said hereafter.

Let us express the angular velocities ω_ξ, ω_η, ω_ζ in terms of $\frac{d\theta}{dt}$, $\frac{d\phi}{dt}$, and $\frac{d\psi}{dt}$; that is, let us resolve the latter along the rotation axes of the former;

$$\omega_\xi = \frac{d\theta}{dt}\cos\xi\text{ON} + \frac{d\phi}{dt}\cos\xi\text{O}\zeta + \frac{d\psi}{dt}\cos\xi\text{O}z$$

$$= \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi; \tag{120}$$

$$\omega_\eta = \frac{d\theta}{dt}\cos\eta\text{ON} + \frac{d\phi}{dt}\cos\eta\text{O}\zeta + \frac{d\psi}{dt}\cos\eta\text{O}z$$

$$= -\frac{d\theta}{dt}\sin\phi + \frac{d\psi}{dt}\sin\theta\cos\phi; \tag{121}$$

$$\omega_\zeta = \frac{d\theta}{dt}\cos\zeta\text{ON} + \frac{d\phi}{dt}\cos\zeta\text{O}\zeta + \frac{d\psi}{dt}\cos\zeta\text{O}z$$

$$= \frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta. \tag{122}$$

Therefore by elimination

$$\frac{d\theta}{dt} = \omega_\xi\cos\phi - \omega_\eta\sin\phi. \tag{123}$$

$$\frac{d\psi}{dt} = \frac{\omega_\xi\sin\phi + \omega_\eta\cos\phi}{\sin\theta}. \tag{124}$$

$$\frac{d\phi}{dt} = \omega_\zeta - \frac{\cos\theta}{\sin\theta}(\omega_\xi\sin\phi + \omega_\eta\cos\phi); \tag{125}$$

whereby $\frac{d\theta}{dt}$, $\frac{d\psi}{dt}$, $\frac{d\phi}{dt}$ are given in terms of the angular velocities of the body about three axes fixed in and moving with it; and if by integration or otherwise θ, ψ, ϕ can be found in terms of t, these quantities will determine the place of the body at a given time.

If, to fix our thoughts, the moving body is the earth, of which O is the centre, Oζ is the polar axis; then the plane of (ξ, η) is the equator; and, if the plane of (x, y) is the ecliptic, the axis of z passes through the pole of the ecliptic, which is approximately fixed in the heavens. In this case θ is the obliquity of the ecliptic, ON passes through the vernal and the autumnal equinoxes and is the line of equinoxes; and ϕ is the longitude of a certain meridian plane, viz. ξOζ, measured from the line of equinoxes.

CHAPTER III.

THE DYNAMICS PROPER OF A MATERIAL SYSTEM.

SECTION I.—*D'Alembert's Principle; the equations of motion of a material system.*

65.] WE now come to the dynamics proper of a material system.

A material system is an assemblage of particles dependent on each other by the action of certain forces which have their origin in and pervade the system. These are called internal forces and are generally different in different systems. In consequence of this dependence, if one particle or body of the system moves by the action of a force external to the system, one or more of the other particles will also move. Thus a material system is always subject to the action of internal forces; and may also be acted on by external forces, in which case the system will move. Let us first consider the former forces; and the constitution of the system of the particles as it depends on the nature of these internal forces.

(1) The system may be a rigid body; then the internal forces are molecular, and of such an intensity that all the particles of the body are at relative rest during the whole motion; and the external forces, whatever are the particles they act on, do not effect any separation of the particles; so that the molecular forces are infinitely great in comparison of them. Such a system is not probably found in nature: all bodies are more or less compressible and elastic, and the particles have a relative motion under the action of external forces. Nevertheless as such a system is imaginable, and is the limit towards which rigid bodies in nature tend, it is necessary to consider it and to disscuss its properties.

(2) The system may consist of particles invariably connected by rigid and inextensible rods, which are capable of bearing force of either compression or extension; so that during the motion the particles are at relative rest. This system has dynamically the same properties as the former, and differs only in the nature of the internal forces. As in this and the former cases the

particles have always the same relative places, these systems are called geometrical.

(3) The several particles of the system may be connected by flexible and extensible strings or rods; in which case the internal forces acting on the particles, by the strings or rods, may vary from time to time as the system moves, and according to the nature of the external forces. We have instances of these systems, when bodies move about pulleys by means of flexible strings, either extensible or inextensible; when a perfect pendulum vibrates with an extensible rod.

(4) The system may consist of particles and of bodies which act on each other by mutual attractions or repulsions; and these may be functions of the several distances of the particles or bodies; so that during the motion of the system its form may not be invariable. The solar system is of this nature, in which we have a series of planets, secondaries, &c. subject to mutual attractions. We have also other systems of the same kind in the motion of liquids; of oscillating flexible cords; of air; of the ethereal medium, &c.; in all which cases the form of the system is continually changing.

These latter two are generally called dynamical systems.

66.] Such are some of the material systems, the motion of which we have now to examine. They consist of material particles, each of which is acted on by certain forces both internal and external. Now if all the forces which act on any one particle are given, the motion of that particle may be determined by the processes of Vol. III. The internal forces however are generally not known, and the determination of them is beyond our power. Let us then consider whether the equations of motion of the aggregate system cannot be constructed without a knowledge of these internal forces, and thus without a knowledge of the motion of the several constituent particles. With this object in view, we must examine the circumstances of motion more exactly, and this we shall do by contrasting it with that of a single particle.

When a single particle moves under the action of a certain external force, the expressed momentum is equal to that impressed on it by this force, whether these are respectively infinitesimal, which is the case when a finite force acts for an infinitesimal time-element; or whether the momentum-increase is

finite, as is the case when an instantaneous force acts. These results are consequent on that property of matter which we call Inertia.

In the motion of a material system, the momentum expressed in any particle is not necessarily equal to that impressed on it from an external source, because each particle is influenced or constrained by one or more of the other particles; that is, is acted on by internal forces, as we have just now explained, and thus it is not free for the development of the momentum communicated to it. The expressed momentum may be either greater or less than that impressed by the external forces according to the internal forces which act on it from the other particles. Some other principle therefore is necessary for the construction of equations by which the motion of the system may be determined. This has been supplied by D'Alembert. It was first enunciated in a Memoir read before the Academy of Sciences in Paris at the end of the year 1742; and it is now always known as "D'Alembert's Principle." I propose to consider the circumstances which require it in one or two particular cases; because by these it will be better understood. And although I shall take continuously acting finite forces by which momentum-increments are impressed, yet, as what is true of them will also be true of momenta impressed by instantaneous forces, the explanation will be applicable to both kinds of force.

67.] In the first place, a difference usually exists between the momentum-increment impressed on, and that expressed in the motion of, a particle of a material system. To shew that this is the case, let us suppose a heavy rigid body to be composed, say, one half of pith, and the other half of gold; and suppose it to fall towards the earth through the air which is a resisting medium. Now gravity acts as an accelerating force equally on the pith and the gold; that is, gravity impresses equal velocity on both; and in an exhausted receiver of an air pump, as we well know, both fall through equal vertical spaces in equal times, and both in equal times acquire equal velocities continuously generated with equal accelerations. The air however by its resistance is a retarding force, and acts with greater effect, cæteris paribus, in diminishing velocity on bodies whose density is more nearly equal to its own density than it does on those

whose density is greater. Hence the velocity of the pith would be diminished by the action of the air more than the velocity of the gold, if both were of equal size and shape and were separate. If the two substances were separated the gold would fall faster than the pith; that is, in other words, in a given time greater velocity would be impressed upon and expressed in the gold than in the pith. Let the two be connected so that the retarding action of the air is equal on both; by reason of the connection all falls together; and has therefore a common velocity; and in neither the pith nor the gold is the momentum expressed equal to that impressed, for the expressed momentum of the gold is less than it would be if it were not attached to the pith; and the expressed momentum of the pith is greater than it would be if it were separate from the gold. Some of the momentum which is impressed on the gold is not expressed in its motion; and that expressed in the pith is in excess of tha which is impressed on it. The gold therefore loses momentun as exhibited in that expressed, and the pith gains.

Again, let us take an instance which is similar to that for th solution of which the principle was first devised. Let us sup pose a circular horizontal plate to rotate about a vertical ax passing through its centre; and let us suppose it to rotate i the exhausted receiver of an air pump, so that no diminution velocity takes place by reason of the resistance of the air. A the ends of the vertical axis let pivots be placed in fixed centr so that the plate continues to rotate about the fixed vertic axis; and let the friction of the pivots be the sole cause of t diminution of the velocity of the plate. Now to the plate le certain angular velocity be imparted; then if the plate th rotating were divided into two equal concentric parts, the m mentum of the exterior part would be greater than that of interior, in the ratio indeed of $2^{\frac{3}{2}}-1$ to 1; and therefore if interior part alone rotated it would be brought to rest by friction of the pivots much sooner than the exterior part wo if it were connected with the axis, the interior part having b removed. However if the whole forms one rigid body a simultaneously brought to rest; the exterior in less time tha would be, if it were separated from the interior; and the terior in longer time than it would be if it were alone. In withdrawal therefore of the momentum of the plate by the

tion of the pivots, the diminution of the exterior part is greater, and that of the interior part is less than it would be if each were separate from the other. Thus the exterior loses and the interior gains momentum. In neither one part nor the other is the momentum expressed equal to that impressed. Similarly if the plate is divided into concentric rings of infinitesimal breadth, generally the expressed momentum-increment of any ring (I use the term increment algebraically) is not equal to the impressed. It is so doubtless in a certain ring, but all rings external to that lose momentum; and all rings internal to it gain momentum; and, as we shall presently shew, the aggregate of the momentum lost throughout the plate is equal to the aggregate of that which is gained.

68.] What has here been said of rigid bodies, is also generally true of material systems. In the motion of each particle a difference will exist between the momentum impressed by a given external force and that expressed in the motion of the particle; and this difference too exists not only in the intensity of these momenta but also as to their lines of action; the particle m (say) will not move along the line of action of the force which impresses momentum, as it would do if it were free, but it will generally move along some other line; thus the momentum due to the acting force is not expressed in the particle's motion either as to intensity or as to line of action. And the connection of the particle with other particles of the system is the cause of this difference.

On what however is this difference spent? the momentum which is impressed is indestructible; matter is inert and cannot absorb it; it is not expressed in the motion of the particle on which the force acts; it must therefore be expressed elsewhere; and must in the first place produce an action between that particle and one or more of the other particles of the system, that is, it puts the system into a state of strain, and produces a stress. And what is the result of this? It must be that these other particles will gain exactly as much momentum as the original particle m has lost. A similar result will also be true for every other particle of the system; so that the sum of the momenta expressed in all the particles will be equal to the sum of those impressed. Thus, if the momenta are calculated throughout the system, the sum of those which are lost is exactly equal to the

sum of those which are gained. Hence the differences between the impressed and the expressed momenta, taken throughout the system, are in equilibrium, and satisfy the equations (109) and (110), Art. 70, Vol. III. This theorem of the equality of the impressed and expressed momenta, taken through the whole system in motion, is what was devised first by D'Alembert, and is known as D'Alembert's Principle. The statement of it is as follows:

When a material system is acted on by forces which impress momenta, the momenta lost by all the particles of the system are in equilibrium.

In this enunciation the term "momentum lost" is equivalent to the excess of the momentum impressed over that expressed in any particle of the system. The term is employed algebraically, and includes cases in which the expressed momentum is in excess of that impressed. It will be perceived that in the explanation above, we have fixed our thoughts on a particle in which the impressed momentum is greater than that expressed.

If the system is acted on by finite accelerating forces, so that infinitesimal momentum-increments are impressed in infinitesimal time-elements, the term momentum in the preceding enunciation must be replaced by "momentum-increment."

69.] To shew an application of the principle let us take the particular case in which finite accelerating forces act; and let us consider the equations of translation of such a system leaving the general case to be treated hereafter.

Let $m_1, m_2, m_3, \ldots m_n$ be the particles of which a system is composed; of which let m be the type, and let (x, y, z) be its place at the time t; let X, Y, Z be the axial components of the momentum-increment impressed by the external forces on m; and let $\text{I}\cos\alpha$, $\text{I}\cos\beta$, $\text{I}\cos\gamma$ be the axial components of the momentum-increment arising from the strain; then the equations of motion of translation of m are

$$\left.\begin{aligned} \text{X} - m\frac{d^2x}{dt^2} &= \text{I}\cos\alpha, \\ \text{Y} - m\frac{d^2y}{dt^2} &= \text{I}\cos\beta, \\ \text{Z} - m\frac{d^2z}{dt^2} &= \text{I}\cos\gamma; \end{aligned}\right\} \qquad (1)$$

let the equations of which these are the types be written for

every particle of the system; and let them be added; then we have

$$\left.\begin{aligned}\Sigma\left(\mathrm{X}-m\frac{d^2x}{dt^2}\right) &= \Sigma.\mathrm{I}\cos\alpha,\\ \Sigma\left(\mathrm{Y}-m\frac{d^2y}{dt^2}\right) &= \Sigma.\mathrm{I}\cos\beta,\\ \Sigma\left(\mathrm{Z}-m\frac{d^2z}{dt^2}\right) &= \Sigma.\mathrm{I}\cos\gamma;\end{aligned}\right\} \qquad (2)$$

if however the system is such that all the momentum-increments arising from the strain neutralise each other, the right-hand members of the preceding equations vanish; and we have

$$\left.\begin{aligned}\Sigma.\left(\mathrm{X}-m\frac{d^2x}{dt^2}\right) &= 0,\\ \Sigma.\left(\mathrm{Y}-m\frac{d^2y}{dt^2}\right) &= 0,\\ \Sigma.\left(\mathrm{Z}-m\frac{d^2z}{dt^2}\right) &= 0.\end{aligned}\right\} \qquad (3)$$

This is one form of an equation of motion which is derived from D'Alembert's Principle.

What I have called the expressed momentum or momentum-increment is often called the effective force, because it is the (so called) force required for the actual motion of the particle, on the supposition that the particle is free from all constraint. It seems to me however that the nomenclature which I have chosen more adequately and appropriately expresses the actual state of the problem which we have to solve.

70.] In the general case then of a body or a system of particles in motion, when there is a difference between the expressed and the impressed momentum or momentum-increment in respect of one or more particles, the system is said to be in a state of strain, and the forces due to the strain are called stresses; it is of these stresses that I have taken I to be the type in the preceding article, and of which the system is in equilibrium. A stress may have simply a pulling or tearing effect, tending to produce what is technically called a parting; or it may have a transversal effect so that a small plate of particles may slide on a contiguous parallel plate with or without rotation, in which case there is what is called a shear, or a twist; or it may have a bending effect, and produce what is commonly called a tendency to break. Hence the intensity of these stresses may

exceed the cohesion or other molecular forces of the system, and disruption or disintegration may ensue. All these circumstances we shall consider at length hereafter when we shall obtain analytical expressions for the stresses. It may be that a body may move without the production of any such stress; thus if a heavy body falls freely in vacuo, each particle moves with a momentum exactly equal to that impressed on it, and there is no strain in the body. Or again the body may be strained when it moves, although no force acts on it. Thus if ring of mass M, transverse section κ, and radius a revolves about an axis through its centre and perpendicular to its plane with an angular velocity Ω, it produces a longitudinal tension T (referred to unit area) along the ring, such that

$$\text{T} = \frac{\text{M}\, a\, \Omega^2}{2\pi\kappa}.$$

We shall have many examples of this kind in our subsequent investigations.

71.] Before we proceed to the complete and purely mathematical expression of the principle, and to the investigation of the general equations which arise out of it, I will explain two or three simple problems, so that the mode of application may be more exactly apprehended. Certain circumstances will be omitted, because we have not yet deduced from the principle theorems which they require. Hereafter the problems will be treated completely.

Ex. 1. Let m and m', Fig. 17, be two heavy particles attached to the ends of a perfectly flexible and inextensible string, which we will suppose to be without weight. The string with the weights at its ends is suspended over a small pulley which we will assume to be without inertia and to be perfectly smooth. It is required to determine the motion of the particles and the tension of the string.

Let A and A$'$ be the places of m and m' at the beginning of the time, P and P$'$ at the time t; $\text{OA} = a$, $\text{O}'\text{A}' = a'$; $\text{OP} = x$, $\text{O}'\text{P}' = x'$;

$$\therefore \quad x + x' = a + a'. \tag{4}$$

Let us consider the circumstances at the beginning of the motion. Let us suppose impulsive forces to act on m and m' downwards, and to impress on them velocities u and u'; that is, m and m' would move with velocities u and u' by the action of the impulsive forces, if they were free. Let v and v' be the ex-

pressed velocities of each in its
τ' be the tensions of the string
as the impressed and express
mu and mv, $\tau =$

and similarly $\tau' = m$

But since from (4) $\frac{dx}{dt} +$

As the pulley has no
$\tau = \tau'$; $\therefore$

which gives the velocit
when $t = 0$. Also

$\tau =$

Let us next consider the
when the bodies are no lo
forces, but of the continuo
Then the impressed and th

of m are respectively mg and m

also let the tension of the string o

$$\therefore \quad T = mg -$$

$$T' = m'g -$$

but from (4) $$\frac{d^2x}{dt^2} + \frac{d^2x'}{dt^2} =$$

And as the pulley has no inertia and i
friction, by D'Alembert's principle, $T = T'$,

$$\therefore \quad mg - m\frac{d^2x}{dt^2} = m'g + m'$$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{m-m'}{m+m'}g = -$$

$$\therefore \quad \frac{dx}{dt} - v = \frac{m-m'}{m+m'}gt.$$

$$\frac{dx'}{dt'} + v = -\frac{m-}{m+}$$

$$\left.\begin{aligned} x &= a + vt + \frac{m-m'}{m+m'}\frac{1}{2}gt^2, \\ x' &= a' - vt - \frac{m-m'}{m+m'}\frac{1}{2}gt^2; \end{aligned}\right\} \tag{10}$$

$$\text{T} = \text{T}' = \frac{2mm'g}{m+m'}; \tag{11}$$

so that all the circumstances of motion are determined.

Ex. 2. Let m and m', Fig. 18, be two heavy particles attached by means of flexible and inextensible strings without weight to a wheel and axle respectively, which are supposed to be without inertia; the initial circumstances being given, it is required to determine the subsequent circumstances of motion.

In Fig. 18, C is the common centre of the wheel and axle; CO $= c$, CO$' = c'$ are the radii; A and A$'$ are the places of m and m' when $t = 0$, P and P$'$ when $t = t$; OA $= a$, O$'$A$' = a'$, OP $= x$, O$'$P$' = x'$.

As x increases by dx, and as x' decreases by dx', let the wheel and axle rotate through an angle $d\theta$; so that

$$dx = c\,d\theta, \qquad dx' = -c'\,d\theta,$$

$$\frac{dx}{dt} = c\frac{d\theta}{dt}, \qquad \frac{dx'}{dt} = -c'\frac{d\theta}{dt}; \tag{12}$$

$$\frac{d^2x}{dt^2} = c\frac{d^2\theta}{dt^2}; \qquad \frac{d^2x'}{dt^2} = -c'\frac{d^2\theta}{dt^2}. \tag{13}$$

Let us consider the circumstances at the beginning of the motion; and let the symbols be the same as those of the preceding example; then

$$\left.\begin{aligned} \tau &= mu - mv, \\ \tau' &= m'u' - m'v'; \end{aligned}\right\} \tag{14}$$

By D'Alembert's principle these tensions are in equilibrium; therefore

$$c\tau = c'\tau', \tag{15}$$

and from (12)

$$\frac{v}{c} = -\frac{v'}{c'};$$

$$\therefore \quad \frac{v}{c} = -\frac{v'}{c'} = \frac{cmu - c'm'u'}{mc^2 + m'c'^2}; \tag{16}$$

$$\frac{\tau}{c'} = \frac{\tau}{c} = \frac{mm'(c'u + cu')}{mc^2 + m'c'^2}; \tag{17}$$

whereby the initial velocities of the particles and the initial tensions of the strings are known.

Let us now consider the circumstances at the time t; and let

the symbols be the same as those of the preceding example; then

$$T = mg - m\frac{d^2x}{dt^2}$$

$$= mg - mc\frac{d^2\theta}{dt^2}; \tag{18}$$

$$T' = m'g - m'\frac{d^2x'}{dt^2}$$

$$= m'g + m'c'\frac{d^2\theta}{dt^2}; \tag{19}$$

and as these tensions are in equilibrium by D'Alembert's principle, we have

$$cT = c'T'; \tag{20}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{mc - m'c'}{mc^2 + m'c'^2} g; \tag{21}$$

$$\frac{1}{c}\frac{d^2x}{dt^2} = -\frac{1}{c'}\frac{d^2x'}{dt^2} = \frac{mc - m'c'}{mc^2 + m'c'^2} g;$$

$$\therefore \quad \frac{1}{c}\left(\frac{dx}{dt} - v\right) = -\frac{1}{c'}\left(\frac{dx'}{dt} - v'\right) = \frac{mc - m'c'}{mc^2 + m'c'^2} gt;$$

$$\therefore \quad x = a + vt + c\frac{mc - m'c'}{mc^2 + m'c'^2}\frac{gt^2}{2}; \tag{22}$$

$$x' = a' + v't - c'\frac{mc - m'c'}{mc^2 + m'c'^2}\frac{gt^2}{2}; \tag{23}$$

$$cT = c'T' = \frac{cc'mm'g(c + c')}{mc^2 + m'c'^2}; \tag{24}$$

whereby all the circumstances of motion in both the initial and the general states are known.

Ex. 3. A heavy chain, flexible and inextensible, homogeneous and smooth, hangs over a small pulley at the common vertex of two smooth inclined planes; it is required to determine the motion of the chain.

Let the two inclined planes, the chain and the pulley, be represented in Fig. 19, each of the inclined planes being supposed to be longer than the length of the chain; so that the chain, as we consider its motion, is on one or the other of the planes. Let O be the common vertex of the two planes; A and A$'$ the ends of the chain when $t = 0$, P and P$'$ the ends when $t = t$; $OA = a$, $OA' = a'$, $OP = x$, $OP' = x'$; and let α and α' be the angles of inclination of the planes to the horizon; $l =$ the length of the chain; therefore

$$x + x' = a + a' = l. \tag{25}$$

We will suppose the chain to be initially at rest. Let ω be the

area of a transverse section, $\rho =$ the density; T = the tension at the time t;

$$\therefore \text{T} = \omega\rho x\left\{g\sin\alpha - \frac{d^2x}{dt^2}\right\} = \omega\rho x'\left\{g\sin\alpha' - \frac{d^2x'}{dt^2}\right\}; \qquad (26)$$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{\sin\alpha + \sin\alpha'}{l}gx - g\sin\alpha'; \qquad (27)$$

$$\frac{d^2x'}{dt^2} = \frac{\sin\alpha + \sin\alpha'}{l}gx' - g\sin\alpha; \qquad (28)$$

$$\therefore \quad \frac{dx^2}{dt^2} = \frac{g(\sin\alpha + \sin\alpha')}{l}(x^2 - a^2) - 2g\sin\alpha'(x - a); \qquad (29)$$

$$\frac{dx'^2}{dt} = \frac{g(\sin\alpha + \sin\alpha')}{l}(x'^2 - a'^2) - 2g\sin\alpha(x' - a'); \qquad (30)$$

whence the relations between x and t, and between x' and t, may be found; but the form of the equations is too complicated to be of any use. Also

$$\text{T} = \frac{\omega\rho g x x'}{l}(\sin\alpha + \sin\alpha'). \qquad (31)$$

If the chain, instead of resting on two inclined planes, hangs over a small pulley without inertia, then, all the other circumstances being the same, $\alpha = \alpha' = 90°$; and the equations of motion are

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \frac{2g}{l}x - g, \\ \frac{d^2x'}{dt^2} &= \frac{2g}{l}x' - g. \end{aligned}\right\} \qquad (32)$$

72.] The following explanation of D'Alembert's principle is much the same as that which he first gave in the Traité de Dynamique; and as it will thus be stated in a mathematical form, the general equations of motion will be most conveniently deduced from it.

Let P, Fig. 20, be the place of a particle m of a material system. During the infinitesimal time dt let a force act on m which would impress on it, if it were free, a velocity whose line-representative is PA; let the impressed velocity be V; so that mV is the impressed momentum along, and proportional to PA; let v be the velocity of m; that is, let mv be the expressed momentum; and let its line of action be PB; let PC be the line which would complete the parallelogram of which PA is the diagonal, and PB one of the containing sides: then resolving V into the velocity v along PB, and v' along PC, v', which is repre-

sented by PC, is the velocity lost; and mv', which is proportional to and acts along PC, is the momentum lost. D'Alembert's principle asserts that all the lost momenta taken throughout the system are in equilibrium. His words are:

Décomposez les mouvements a, b, c, ..., imprimeés à chaque corps, chacun en deux autres a, α; b, β; c, γ; ...... qui soient tels que si l'on n'eût imprimeé aux corps que les mouvements a, b, c, ... ils eussent pu conserver ces mouvements sans se nuire réciproquement; et que si on ne leur eût imprimé que les mouvements α, β, γ, ..., le systême fût demeuré en repos.

Again, produce BP to B′, so that PB′ = PB; then the momentum represented by PC is evidently the resultant of those represented by PA and PB′; hence we have D'Alembert's principle in the following form:

If the expressed momenta of the several particles of a material system are estimated in a direction the contrary of that in which they act, they, together with the impressed momenta when taken through the whole system, will satisfy the conditions of statical equilibrium.

73.] Such is D'Alembert's principle, as to its origin and as to its form of expression; it reduces all the theorems of motion of material systems to those of statical equilibrium; and so it is commonly said that D'Alembert reduced dynamics to statics. The principle does not indeed directly furnish the equations necessary for the solution of the different problems of dynamics; but it teaches the mode by which they are to be deduced from the equations of equilibrium; and thus, if we apply to the "momenta lost," the conditions of statical equilibrium, the dynamical equations will be formed. It is evident too that we may introduce them as pressures into the equation of virtual velocities, and this will hereafter be done. The equations of equilibrium of a system of pressures acting on the several points of a rigid body are investigated in Vol. III. The number of them is six; of which three are of translation and three are of rotation: the momenta lost must satisfy these six conditions.

Firstly, let us suppose the acting forces on the system of particles to be impulsive and instantaneous, so that finite momenta are impressed instantaneously, and the expressed momenta are also instantaneously developed.

Let m be the symbol of a type-particle; (x, y, z) its place at

the time t; v_x, v_y, v_z the axial components of its expressed velocity due to the acting instantaneous forces; x, y, z the axial components of the velocity impressed on m; so that the differences between the axial components of the impressed and expressed momenta are

$$m(\text{x}-\text{v}_x), \qquad m(\text{y}-\text{v}_y), \qquad m(\text{z}-\text{v}_z). \tag{33}$$

By D'Alembert's principle these together with similar quantities for all the other particles of the system are in equilibrium; therefore the six following equations must be satisfied by them;

$$\left.\begin{aligned} \Sigma.m(\text{x}-\text{v}_x) &= 0, \\ \Sigma.m(\text{y}-\text{v}_y) &= 0, \\ \Sigma.m(\text{z}-\text{v}_z) &= 0; \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} \Sigma.m\{y(\text{z}-\text{v}_z)-z(\text{y}-\text{v}_y)\} &= 0, \\ \Sigma.m\{z(\text{x}-\text{v}_x)-x(\text{z}-\text{v}_z)\} &= 0, \\ \Sigma.m\{x(\text{y}-\text{v}_y)-y(\text{x}-\text{v}_x)\} &= 0; \end{aligned}\right\} \tag{35}$$

whereof the first three are the equations of translation, and the last three are the equations of the moments of the couples which arise from the excess of the expressed over the impressed momenta about the three coordinate axes. The sign of summation extends to, and includes, all the particles of the system; and the expressed velocities are those due to the action of the impressed forces.

If the impressed momentum is due to a single blow, of which the momentum is Q, and of the action-line of which λ, μ, ν are the direction-angles, and the axial components of the couple of which are L, M, N, (ξ, η, ζ) being a point on the action-line, the equations take the following form

$$\Sigma.m\text{v}_x = \text{Q}\cos\lambda, \qquad \Sigma.m\text{v}_y = \text{Q}\cos\mu, \qquad \Sigma.m\text{v}_z = \text{Q}\cos\nu:$$

$$\begin{aligned} \Sigma.m(y\text{v}_z-z\text{v}_y) &= \text{Q}(\eta\cos\nu-\zeta\cos\mu) = \text{L}, \\ \Sigma.m(z\text{v}_x-x\text{v}_z) &= \text{Q}(\zeta\cos\lambda-\xi\cos\nu) = \text{M}, \\ \Sigma.m(x\text{v}_y-y\text{v}_x) &= \text{Q}(\xi\cos\mu-\eta\cos\lambda) = \text{N}. \end{aligned}$$

Also if u, v, w are the axial components of the velocity of m before the action of the instantaneous force or forces, and u', v', w' are the axial components of the velocity after such action, then in the preceding equations v_x, v_y, v_z will be replaced by $u'-u$, $v'-v$, $w'-w$ respectively, these last being the axial velocities due to the action of the forces.

Secondly, let us suppose the system of particles to be under the action of finite accelerating forces, so that in infinitesimal

time-elements infinitesimal momentum-increments are impressed upon and expressed in the type-particle m.

Let X, Y, Z be the axial components of the impressed velocity-increment on m, which is supposed to be at (x, y, z) at the time t; and $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ are the axial components of the expressed velocity-increment; so that the differences between the axial components of the impressed and the expressed momentum-increments of m are

$$m\left(\mathrm{X}-\frac{d^2x}{dt^2}\right), \quad m\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right), \quad m\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right). \tag{36}$$

By D'Alembert's principle these and similar quantities for all the other particles of the system are in equilibrium; therefore the six following equations must be satisfied by them;

$$\left.\begin{aligned} \Sigma.m\left(\mathrm{X}-\frac{d^2x}{dt^2}\right) &= 0,\\ \Sigma.m\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right) &= 0,\\ \Sigma.m\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right) &= 0; \end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned} \Sigma.m\left\{y\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)-z\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)\right\} &= 0,\\ \Sigma.m\left\{z\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)-x\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)\right\} &= 0,\\ \Sigma.m\left\{x\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)-y\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)\right\} &= 0; \end{aligned}\right\} \tag{38}$$

the sign of summation includes all the particles of the system.

Also the power of this sign should be carefully observed; it includes all the particles of the moving system, whether that system be continuous or discontinuous. Thus, if the several particles are $m_1, m_2, \ldots m_n$; and their places at the time t are $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots (x_n, y_n, z_n)$, and the impressed velocity-increments are $(\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1), (\mathrm{X}_2, \mathrm{Y}_2, \mathrm{Z}_2), \ldots (\mathrm{X}_n, \mathrm{Y}_n, \mathrm{Z}_n)$; then the first of (37) is the abbreviated form of

$$m_1\left(\mathrm{X}_1-\frac{d^2x_1}{dt^2}\right)+m_2\left(\mathrm{X}_2-\frac{d^2x_2}{dt^2}\right)+\ldots+m_n\left(\mathrm{X}_n-\frac{d^2x_n}{dt^2}\right)=0.$$

Similarly the other five equations are abbreviated forms of analogous expressions.

In the case of a continuous body the process of summation becomes integration.

74.] As m is independent of the time, the equations (37) and (38) may also be written in the following forms; and these are important, because they immediately exhibit the genesis of the expressions;

$$\left.\begin{aligned}\frac{d}{dt}\Sigma.m\frac{dx}{dt} &= \Sigma.m\mathrm{X},\\ \frac{d}{dt}\Sigma.m\frac{dy}{dt} &= \Sigma.m\mathrm{Y},\\ \frac{d}{dt}\Sigma.m\frac{dz}{dt} &= \Sigma.m\mathrm{Z};\end{aligned}\right\}\qquad(39)$$

$$\left.\begin{aligned}\frac{d}{dt}\Sigma.m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right) &= \Sigma.m(y\mathrm{Z}-z\mathrm{Y}) = \mathrm{L},\\ \frac{d}{dt}\Sigma.m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right) &= \Sigma.m(z\mathrm{X}-x\mathrm{Z}) = \mathrm{M},\\ \frac{d}{dt}\Sigma.m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right) &= \Sigma.m(x\mathrm{Y}-y\mathrm{X}) = \mathrm{N};\end{aligned}\right\}\qquad(40)$$

where L, M, N are the moments of the couples of the impressed momentum-increments about the axes of x, y, z respectively, and

$$\Sigma.m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right),\quad \Sigma.m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right),\quad \Sigma.m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$$

are the moments of the couples of the momenta of all the particles of the system about the axes of x, y, z respectively.

Now the position of the coordinate axes is entirely arbitrary; consequently for any line in space either of the equations (39) and its corresponding equation in (40) are true. Hence if V_s expresses the sum of the components of the expressed momenta of all the particles of the system along the line s, and H_s expresses the sum of the moments of the couples of the momenta of all the particles about the line s, then

$$\frac{d\mathrm{V}_s}{dt} = \mathrm{S}, \qquad \frac{d\mathrm{H}_s}{dt} = \mathrm{K}, \qquad (41)$$

where S is the sum of the components of all the impressed momentum-increments along the line s, and K is the sum of the moments of the couples of all the impressed momentum-increments about the same line s. (39) and (40) are particular forms of (41) when s becomes x, y, z respectively.

75.] The equations of motion of a material system may also be expressed in another form. For since D'Alembert's principle enables us to deduce them from the equilibrium which subsists among the "lost momenta," that equilibrium will be obtained

not only from the six equations which correspond to (37) and (38) of the preceding Article, but also from the equation of virtual velocities. The truth of this last equation has been demonstrated in Articles 108 and 467 of Vol. III, but since we shall now apply it somewhat extensively, and very generally, as it will include all dynamics, it is necessary to say a few words on its form and the conditions of its exactness. We imagine a material system to be at rest under the action of many forces, which may be external as well as internal to the system: of these forces we take P to be the type, and we suppose it to act on m, which we take to be the type-particle: so that $\Sigma . \text{P}$ will be the sum of all the forces which act on all the particles; many of which may act on one and the same particle; and others of which will enter in pairs of equal and opposite forces, when there are mutual tensions or reactions or constraints among the particles of the system. We imagine the system to receive an arbitrary infinitesimal displacement, consistently with its geometrical relations, whereby the points of application of the forces are changed, but neither the intensities nor the directions of the lines of action are altered. Let the displacements of the points of application of the forces be estimated along the lines of action of the forces; and let δp be the infinitesimal displacement of (x, y, z), the point of application of P, thus estimated; then the equation of virtual velocities is

$$\Sigma . \text{P} \delta p = 0;$$

and this expresses the condition that the forces are in equilibrium.

Let us put into an equation of this form the several quantities which are active in the motion of a material system. As $\frac{d^2 x}{dt^2}, \frac{d^2 y}{dt^2}, \frac{d^2 z}{dt^2}$ are the axial components of the expressed velocity-increments of m, which is at (x, y, z) at the time t, it is evident that the impressed momentum-increments along these axes, which would have their full effect in producing pressure if the system were at rest, must be diminished by $m\frac{d^2 x}{dt^2}, m\frac{d^2 y}{dt^2}, m\frac{d^2 z}{dt^2}$, now that the system moves; and the actual effects will be the excesses of the former over the latter: in the equation therefore of virtual velocities these latter quantities must be affected with negative signs. We shall use the symbol δ to express the varia-

tions of the points of application of the forces which are due to the arbitrary geometrical displacement of the system; and we shall still indicate by the symbol d the time-variation of the coordinates and velocities. Let mP be the type momentum-increment acting on the type particle m; and let us suppose δp to be the infinitesimal displacement of the point of application of P estimated along its line of action, P tending to remove m from the origin, and δp being positive when the point of application of P is moved in the direction along which P acts; then, if the line of action of a force is along a coordinate axis, say that of x, the variation of the point of application is δx. Now, estimating forces according to these conditions, the equation of virtual velocities is

$$\Sigma . m\text{P}\,\delta p - \Sigma . m\left\{\frac{d^2 x}{dt^2}\delta x + \frac{d^2 y}{dt^2}\delta y + \frac{d^2 z}{dt^2}\delta z\right\} = 0; \qquad (42)$$

which is the most general equation of motion of a material system.

$\Sigma . m\text{P}\,\delta p$ includes all the forces which act on the several particles of the system, both internal and external; if however two particles are acted on by a force along the line which joins them, and if the distance between these particles is unchanged in the geometrical displacement, this force will disappear in the aggregate; because the geometrical displacements of the two particles estimated along the line of force will be equal and opposite, and therefore the two effects, as they are measured in the preceding equation, will neutralise each other, and will disappear.

76.] Let all the impressed momentum-increments, as they are applied to each particle, be resolved into components parallel to the three coordinate axes; and let X, Y, Z be the axial components of P as it acts on m at (x, y, z); then the infinitesimal displacements of m along the three axes will be $\delta x, \delta y, \delta z$; which are the same as the displacements of the point of application of the expressed momentum-increments: so that (42) becomes in this case

$$\Sigma . m\left\{\left(\text{X} - \frac{d^2 x}{dt^2}\right)\delta x + \left(\text{Y} - \frac{d^2 y}{dt^2}\right)\delta y + \left(\text{Z} - \frac{d^2 z}{dt^2}\right)\delta z\right\} = 0; \quad (43)$$

which is another form of the equation of virtual velocities.

Now no restriction has been made as to the kind of displacement of (x, y, z), of which the axial projections are represented by $\delta x, \delta y, \delta z$; it is only to be consistent with the geometrical

relations of the system: let us suppose it therefore to be most general, and to be compounded of motions of translation and of rotation of the whole system. Let the system receive a displacement of translation, so that every particle moves over an equal and parallel space in the direction of the coordinate axes, which we will represent severally by $\delta x_0, \delta y_0, \delta z_0$; and also let the system receive three successive displacements of rotation through the angles $\delta\theta, \delta\phi, \delta\psi$ about the three coordinate axes: then, see (76), Art. 54, the total variations of the coordinates of the point (x, y, z) are

$$\left.\begin{aligned}\delta x &= \delta x_0 + z\,\delta\phi - y\,\delta\psi,\\ \delta y &= \delta y_0 + x\,\delta\psi - z\,\delta\theta,\\ \delta z &= \delta z_0 + y\,\delta\theta - x\,\delta\phi;\end{aligned}\right\}$$

and substituting these in (43) we have

$$\begin{aligned}\delta x_0\,\Sigma.m\Big(\mathrm{X}-\frac{d^2x}{dt^2}\Big)+\delta y_0\,\Sigma.m\Big(\mathrm{Y}-\frac{d^2y}{dt^2}\Big)+\delta z_0\,\Sigma.m\Big(\mathrm{Z}-\frac{d^2z}{dt^2}\Big)&\\ +\,\delta\theta\,\Sigma.m\Big\{y\Big(\mathrm{Z}-\frac{d^2z}{dt^2}\Big)-z\Big(\mathrm{Y}-\frac{d^2y}{dt^2}\Big)\Big\}&\\ +\,\delta\phi\,\Sigma.m\Big\{z\Big(\mathrm{X}-\frac{d^2x}{dt^2}\Big)-x\Big(\mathrm{Z}-\frac{d^2z}{dt^2}\Big)\Big\}&\\ +\,\delta\psi\,\Sigma.m\Big\{x\Big(\mathrm{Y}-\frac{d^2y}{dt^2}\Big)-y\Big(\mathrm{X}-\frac{d^2x}{dt^2}\Big)\Big\}&=0;\end{aligned}$$

as the several variations on which the displacement depends are independent of each other, their coefficients must separately vanish; and hence we have the six equations of motion, viz. (37) and (38) of Art. 73.

In the case of impulsive forces it will be more convenient to state the equation in a slightly different form, because the impulsive forces do not generally act directly on every particle and are not generally proportional to the particles on which they act.

Let u, v, w be the axial components of the velocity of m before the action of the impulsive forces, and u', v', w' the like quantities after the impulse, so that $u'-u, v'-v, w'-w$ are the axial components of velocity of m due to the forces; and let $\Sigma.\mathrm{X}, \Sigma.\mathrm{Y}, \Sigma.\mathrm{Z}$ be the axial components of the momenta impressed by the forces; then the equation of virtual velocities takes the form

$$\Sigma.[\{\mathrm{X}-m(u'-u)\}\,\delta x+\{\mathrm{Y}-m(v'-v)\}\,\delta y+\{\mathrm{Z}-m(w'-w)\}\,\delta z]=0,$$

or

$$\Sigma.m\{(u'-u)\,\delta x+(v'-v)\,\delta y+(w'-w)\,\delta z\}=\Sigma.(\mathrm{X}\,\delta x+\mathrm{Y}\,\delta y+\mathrm{Z}\,\delta z). \quad (44)$$

Giving to $\delta x, \delta y, \delta z$ the values (41), and equating to zero the

coefficients of the six arbitrary quantities, we obtain the six equations (34) and (35).

77.] When the system of particles on which the forces act is rigid or invariable in form, that invariability of form will be secured by means of certain equations which the coordinates of the several particles must satisfy. Thus, if the number of particles is n, and if all the distances of these particles from each other are invariable during the motion, $(3n-6)$ equations are satisfied thereby; for if the distances between three are given, and the distances of every one of the remaining $(n-3)$ from each of these three are given, we have $3n-6$ given distances, and the system of particles is invariable in form. Now the position of every particle of the system at any time is determined, when the coordinates of every particle are expressed in terms of t; and as each particle has three coordinates, $3n$ quantities must be expressed in terms of t: these however are subjected to $3n-6$ conditions of relative position: six other conditions therefore are necessary, and are sufficient for the complete solution of the problem; and these are given by the equations of Art. 73.

Also that six independent conditions are required and are sufficient for the complete determination of a rigid material system is apparent from the fact that the position of such a system is given, when (1) the position of any one point is given, for which three coordinates are required, and (2) the position of any two lines in the body passing through that point is also given. For this latter purpose three more quantities are required, as for instance the values of Euler's three angles (see Art. 64) which determine the position of the axes of ξ and ζ in that system of reference. Thus altogether six conditions are needed and are enough. These may belong to any system of coordinates, provided that they are independent of each other.

When the motion is in two dimensions only, three conditions are enough, viz. the two coordinates of any given point, and the angle made with a fixed line in space by some line about which the body rotates.

It has been customary of late to speak of the degrees of freedom which a body or a system of material particles admits of; this mode of speaking is equivalent to stating the number of independent coordinates which are required for the complete determination of the position of a system. Hence there cannot

be more than six degrees of freedom, and the number may be greater or less, according as the system is subject to fewer or more constraints; thus if one point of a body is fixed, three degrees of freedom are satisfied, and three only remain; these may be determined by three coordinates, such as Euler's three angles, or by the nine direction-cosines which are given in Art. 2, and which are subject to six geometrical conditions, so that only three are independents. If a body moves so that a given plane section of it moves parallel to itself, there are only four degrees of freedom corresponding to four quantities, three of which determine a point in the plane, and the fourth of which gives the angle contained between a given straight line in the plane and a fixed plane perpendicular to the given plane.

78.] And now let us take a more general case; and let us suppose certain particles of a material system to be constrained to move on certain curves, so that certain relations will be given which the coordinates of the particles must satisfy. Suppose the number of these relations to be k; and let them be $F_1 = 0$, $F_2 = 0$, ... $F_k = 0$; then, taking the most general case, and supposing each of these functions to contain all the coordinates of all the points, their total variations are

$$\left.\begin{array}{l} \left(\frac{dF_1}{dx_1}\right)\delta x_1 + \left(\frac{dF_1}{dy_1}\right)\delta y_1 + \ldots + \left(\frac{dF_1}{dy_n}\right)\delta y_n + \left(\frac{dF_1}{dz_n}\right)\delta z_n = 0, \\ \left(\frac{dF_2}{dx_1}\right)\delta x_1 + \left(\frac{dF_2}{dy_1}\right)\delta y_1 + \ldots + \left(\frac{dF_2}{dy_n}\right)\delta y_n + \left(\frac{dF_2}{dz_n}\right)\delta z_n = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \left(\frac{dF_k}{dx_1}\right)\delta x_1 + \left(\frac{dF_k}{dy_1}\right)\delta y_1 + \ldots + \left(\frac{dF_k}{dy_n}\right)\delta y_n + \left(\frac{dF_k}{dz_n}\right)\delta z_n = 0; \end{array}\right\} \quad (45)$$

and from (43) we have an equation of the form

$$A_1\delta x_1 + B_1\delta y_1 + C_1\delta z_1 + \ldots\ldots + A_n\delta x_n + B_n\delta y_n + C_n\delta z_n = 0. \quad (46)$$

If then we multiply (45) severally by indeterminate multipliers $\lambda_1, \lambda_2, \ldots \lambda_k$, and add all and (46); and if we equate to zero the coefficients of $\delta x_1, \delta y_1, \ldots \delta z_n$, we have $3n$ equations, viz.

$$\left.\begin{array}{l} \lambda_1\left(\frac{dF_1}{dx_1}\right) + \lambda_2\left(\frac{dF_2}{dx_1}\right) + \ldots + \lambda_k\left(\frac{dF_k}{dx_1}\right) + A_1 = 0, \\ \lambda_1\left(\frac{dF_1}{dy_1}\right) + \lambda_2\left(\frac{dF_2}{dy_1}\right) + \ldots + \lambda_k\left(\frac{dF_k}{dy_1}\right) + B_1 = 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \lambda_1\left(\frac{dF_1}{dz_n}\right) + \lambda_2\left(\frac{dF_2}{dz_n}\right) + \ldots + \lambda_k\left(\frac{dF_k}{dz_n}\right) + C_n = 0; \end{array}\right\}$$

which together with the k equations of constraint are sufficient to determine the $3n$ coordinates and the k multipliers.

79.] We will now apply the process of the preceding Article to the motion of a flexible and inextensible string, which has its two ends fixed, under the action of given forces.

Let ds be the length-element of the string at the point (x, y, z); let ω be the area of the transverse section, $\rho =$ the density; so that the mass-element of the string is $\rho\,\omega\,ds$. Let the points (x_1, y_1, z_1), (x_2, y_2, z_2) be the two ends of the string, and let $\rho_1, \omega_1, \rho_2, \omega_2$ be the values of ρ, ω at them respectively. Let $\mathrm{X}, \mathrm{Y}, \mathrm{Z}$ be the axial components of the impressed velocity-increment at the point (x, y, z); and let $\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1, \mathrm{X}_2, \mathrm{Y}_2, \mathrm{Z}_2$ be the values of these quantities at the two ends of the string which we suppose to be dynamically fixed, but to admit of geometrical variation. Then (43) becomes

$$\int_1^2 \rho\,\omega\,ds\Big\{\Big(\mathrm{X}-\frac{d^2x}{dt^2}\Big)\delta x + \Big(\mathrm{Y}-\frac{d^2y}{dt^2}\Big)\delta y + \Big(\mathrm{Z}-\frac{d^2z}{dt^2}\Big)\delta z\Big\}$$
$$+\mathrm{X}_1\delta x_1+\mathrm{Y}_1\delta y_1+\mathrm{Z}_1\delta z_1+\mathrm{X}_2\delta x_2+\mathrm{Y}_2\delta y_2+\mathrm{Z}_2\delta z_2 = 0; \quad (47)$$

and since the string is of constant length

$$\int_1^2 ds = \text{a constant.}$$

$$\therefore \quad \int_1^2 \Big\{\frac{dx}{ds}\delta.dx+\frac{dy}{ds}\delta.dy+\frac{dz}{ds}\delta.dz\Big\} = \delta.\int_1^2 ds$$
$$= 0; \quad (48)$$

and multiplying the quantity under the sign of integration by λ, and adding it to (47), we have

$$\int_1^2 \Big\{\rho\,\omega\,ds\Big(\mathrm{X}-\frac{d^2x}{dt^2}\Big)\delta x+\lambda\,\frac{dx}{ds}d.\delta x\Big\} + \dots$$
$$\dots + \mathrm{X}_1\delta x_1 + \mathrm{Y}_1\delta y_1 + \dots\dots + \mathrm{Z}_2\delta z_2 = 0; \quad (49)$$

and integrating by parts the second terms of the several members of the upper line, as we have explained in the Calculus of Variations, Vol. II, we have

$$\int_1^2 \Big\{\rho\,\omega\,ds\Big(\mathrm{X}-\frac{d^2x}{dt^2}\Big)-d.\lambda\frac{dx}{ds}\Big\}\delta x + \dots\dots$$
$$+\Big[\lambda\frac{dx}{ds}\delta x+\lambda\frac{dy}{ds}\delta y+\lambda\frac{dz}{ds}\delta z\Big]_1^2$$
$$+\mathrm{X}_1\delta x_1+\mathrm{Y}_1\delta y_1+\dots\dots+\mathrm{Z}_2\delta z_2 = 0. \quad (50)$$

As no other relation is given between δx, δy, and δz, we have

$$\left.\begin{aligned} \rho\,\omega\,ds\left(X-\frac{d^2x}{dt^2}\right)-d.\lambda\frac{dx}{ds}&=0,\\ \rho\,\omega\,ds\left(Y-\frac{d^2y}{dt^2}\right)-d.\lambda\frac{dy}{ds}&=0,\\ \rho\,\omega\,ds\left(Z-\frac{d^2z}{dt^2}\right)-d.\lambda\frac{dz}{ds}&=0;\end{aligned}\right\} \qquad (51)$$

which are the equations of motion of the cord. The latter terms in (50) give values at the ends of the string; if the ends are independent of each other,

$$\left.\begin{aligned} \left(-\lambda_1\frac{dx_1}{ds_1}+X_1\right)\delta x_1+\left(-\lambda_1\frac{dy_1}{ds_1}+Y_1\right)\delta y_1+\left(-\lambda_1\frac{dz_1}{ds_1}+Z_1\right)\delta z_1=0,\\ \left(\lambda_2\frac{dx_2}{ds_2}+X_2\right)\delta x_2+\left(\lambda_2\frac{dy_2}{ds_2}+Y_2\right)\delta y_2+\left(\lambda_2\frac{dz_2}{ds_2}+Z_2\right)\delta z_2=0;\end{aligned}\right\} (52)$$

and if the ends are fixed

$$\left.\begin{aligned} -\lambda_1\frac{dx_1}{ds_1}+X_1=0, \qquad -\lambda_1\frac{dy_1}{ds_1}+Y_1=0, \qquad -\lambda_1\frac{dz_1}{ds_1}+Z_1=0;\\ \lambda_2\frac{dx_2}{ds_2}+X_2=0, \qquad \lambda_2\frac{dy_2}{ds_2}+Y_2=0, \qquad \lambda_2\frac{dz_2}{ds_2}+Z_2=0.\end{aligned}\right\}$$

The form of the last terms of (51) shews that λ is the tension of the string at the point (x, y, z), and acts along the length-element ds: indeed the equations (51) are only the particular form of (37) when the forces arising from the tension are introduced into them. If we eliminate λ from (51), we shall have two equations in terms of x, y, z, t, which will give the position of the string at any time. I may observe too that λ is evidently the tension, because the mode in which it is introduced shews that it is an internal force acting along ds; and that the variations of its points of action are the same as the variations of the ends of the length-element.

Thus much must at present suffice for this problem; we proceed to the demonstration of various general theorems which arise out of the equations of motion.

SECTION 2.—*Independence of the motions of translation of the mass-centre, and of rotation about an axis passing through it.*

80.] In the preceding chapter it has been proved that the motion of a body in its most general form may be resolved into a motion of translation of any one point of it chosen arbitrarily, and a motion of rotation about an axis passing through that

arbitrarily chosen point. One particular form of the resolution was pointed out on principles purely kinematical; viz. that if the point was taken on the central axis, the motion of the point would lie along that axis, and that we should have what is technically known as a screw; and this is of course true when all the expressed velocities of the system due to the acting forces are given, whereby the position of the central axis at the time t may be determined. The question however of choice of the moving particle about which rotation is to be estimated now offers itself from another standpoint, viz. from the equations of motion which have been arrived at from D'Alembert's Principle; and it presents itself in this form: Is there any point such that, if it be taken as the moving point of translation, the equations become simplified? or, in other words, is there any reason why a certain point should be chosen as the moving point of translation rather than any other? Further on we shall have another question, viz. Is there any reason why one particular system of coordinate axes should be chosen in preference to any other? This latter question we will leave for the present. As to the former, a simple inspection of the equations (34) and (37) suggests that the point which is commonly known as the centre of gravity should be taken as the moving point of translation, because the coordinates of that point involve quantities analogous to those of these equations, and will lead to certain simplifications and theorems, as is shewn in the following Articles.

81.] I must observe on the name "centre of gravity": it has been given to the so-called point of a heavy body or heavy system of particles, which is the centre of the several parallel forces of weight acting on the particles. This meaning, though precise, is too narrow for our present purpose; we require a term of wider signification, because the systems of particles which we have to consider may never have weight at all, and yet must have a certain point in reference to the configuration of the particles which, in the case of weight, is the centre of gravity. This point has been called the centre of masses, or the mass-centre, and is defined as the point whose coordinates $\bar{x}, \bar{y}, \bar{z}$ are given by the following expressions, viz.

$$\left.\begin{aligned} \bar{x}\,\Sigma.m &= \Sigma.mx, \\ \bar{y}\,\Sigma.m &= \Sigma.my, \\ \bar{z}\,\Sigma.m &= \Sigma.mz. \end{aligned}\right\} \qquad (53)$$

I shall generally call this point the mass-centre, as being a short and expressive term; but in cases where gravitation is the acting force shall not hesitate to call it the centre of gravity. In geometry, as it is well known, there is a similar point which is called the centre of mean distances, and is the point given by (53) when all the m's are equal to each other.

Since m does not vary with the time, from (53) we have

$$\left.\begin{aligned} \frac{d\bar{x}}{dt}\Sigma.m &= \Sigma.m\frac{dx}{dt}, \\ \frac{d\bar{y}}{dt}\Sigma.m &= \Sigma.m\frac{dy}{dt}, \\ \frac{d\bar{z}}{dt}\Sigma.m &= \Sigma.m\frac{dz}{dt}; \end{aligned}\right\} (54) \qquad \left.\begin{aligned} \frac{d^2\bar{x}}{dt^2}\Sigma.m &= \Sigma.m\frac{d^2x}{dt^2}, \\ \frac{d^2\bar{y}}{dt^2}\Sigma.m &= \Sigma.m\frac{d^2y}{dt^2}, \\ \frac{d^2\bar{z}}{dt^2}\Sigma.m &= \Sigma.m\frac{d^2z}{dt^2}. \end{aligned}\right\} (55)$$

Hence in respect of the mass-centre as origin,

$$\Sigma.mx = \Sigma.my = \Sigma.mz = 0; \tag{56}$$

$$\Sigma.m\frac{dx}{dt} = \Sigma.m\frac{dy}{dt} = \Sigma.m\frac{dz}{dt} = 0; \tag{57}$$

$$\Sigma.m\frac{d^2x}{dt^2} = \Sigma.m\frac{d^2y}{dt^2} = \Sigma.m\frac{d^2z}{dt^2} = 0. \tag{58}$$

These are the expressions which lead to the simplification of the equations of motion.

82.] Let $(\bar{x}, \bar{y}, \bar{z})$ be the place of the mass-centre at the time t; (x, y, z) the place of the type-particle m; and let us suppose a system of coordinate axes to originate at the mass-centre and be parallel to the original system of reference; and let the place of m relatively to these axes be (x', y', z'); so that

$$x = \bar{x}+x', \qquad y = \bar{y}+y', \qquad z = \bar{z}+z',$$

and

$$\Sigma.mx' = \Sigma.my' = \Sigma.mz' = 0;$$

$$\therefore \quad \Sigma.m\frac{dx'}{dt} = \Sigma.m\frac{dy'}{dt} = \Sigma.m\frac{dz'}{dt} = 0;$$

$$\Sigma.m\frac{d^2x'}{dt^2} = \Sigma.m\frac{d^2y'}{dt^2} = \Sigma.m\frac{d^2z'}{dt^2} = 0.$$

Let $\text{M} =$ the mass of all the particles of the moving system; then

$$\Sigma.mx = \text{M}\bar{x}, \qquad \Sigma.my = \text{M}\bar{y}, \qquad \Sigma.mz = \text{M}\bar{z};$$

$$\Sigma.m\frac{dx}{dt} = \text{M}\frac{d\bar{x}}{dt}, \qquad \Sigma.m\frac{dy}{dt} = \text{M}\frac{d\bar{y}}{dt}, \qquad \Sigma.m\frac{dz}{dt} = \text{M}\frac{d\bar{z}}{dt};$$

$$\Sigma.m\frac{d^2x}{dt^2} = \text{M}\frac{d^2\bar{x}}{dt^2}, \qquad \Sigma.m\frac{d^2y}{dt^2} = \text{M}\frac{d^2\bar{y}}{dt^2}, \qquad \Sigma.m\frac{d^2z}{dt^2} = \text{M}\frac{d^2\bar{z}}{dt^2}.$$

Firstly, let us take equations (34) and (35) which refer to instantaneous forces; then

$$\frac{dx}{dt}=\frac{d\bar{x}}{dt}+\frac{dx'}{dt}, \quad \frac{dy}{dt}=\frac{d\bar{y}}{dt}+\frac{dy'}{dt}, \quad \frac{dz}{dt}=\frac{d\bar{z}}{dt}+\frac{dz'}{dt}; \tag{59}$$

and these become in this case

$$v_x = \bar{v}_x + v_x', \quad v_y = \bar{v}_y + v_y', \quad v_z = \bar{v}_z + v_z',$$

and (34) become

$$\left.\begin{aligned} M\bar{v}_x &= \Sigma.mX, \\ M\bar{v}_y &= \Sigma.mY, \\ M\bar{v}_z &= \Sigma.mZ; \end{aligned}\right\} \tag{60}$$

which equations are of the same form as those of the motion of a material particle whose mass is M. Whence it appears that

The motion of translation of the mass-centre of a system of particles under the action of impulsive forces is the same as if the whole mass were collected into it, and all the impressed momenta were applied at it, each in a line parallel to its own line of action.

If $\bar{v}$ is the velocity of the mass-centre,

$$M\bar{v} = \{(\Sigma.mX)^2+(\Sigma.mY)^2+(\Sigma.mZ)^2\}^{\frac{1}{2}};$$

and the direction-cosines of the path which the mass-centre takes are given by (60).

Again, taking the first of (35), and making similar substitutions, we have

$$\Sigma.m\{(\bar{y}+y')(Z-\bar{v}_z-v_z')-(\bar{z}+z')(Y-\bar{v}_y-v_y')\} = 0;$$

which may be expressed as follows:

$$\bar{y}\Sigma.m(Z-\bar{v}_z)-\bar{y}\Sigma.mv_z'+\Sigma.my'(Z-v_z')-\bar{v}_z\Sigma.my'$$
$$-\bar{z}\Sigma.m(Y-\bar{v}_y)+\bar{z}\Sigma.mv_y'-\Sigma.mz'(Y-v_y')+\bar{v}_y\Sigma.mz' = 0;$$

and thus, omitting terms which vanish, we have

$$\Sigma.m\{y'(Z-v_z')-z'(Y-v_y')\} = 0;$$

similarly from the second and third of (35) we have

$$\left.\begin{aligned} \Sigma.m\{z'(X-v_x')-x'(Z-v_z')\} &= 0, \\ \Sigma.m\{x'(Y-v_y')-y'(X-v_x')\} &= 0. \end{aligned}\right\} \tag{61}$$

These are evidently the equations of the three couples of the lost momenta relatively to the axes of a system originating at the mass-centre; and the impressed momenta are the same as in the original equations (35): whence we infer that

If the motion of a system of material particles, under the action of impulsive forces, is resolved into a motion of translation of the mass-centre, and of rotation about an axis passing through that point, the motion of the mass-centre is the same as

if the masses of all the particles were collected at it, and all the impressed momenta were applied at it each in a line parallel to its own line of action; and the motion of rotation about the axis passing through the mass-centre is the same as if the mass-centre were a fixed point, and the system rotated about an axis passing through that point under the action of the impressed momenta which are actually applied to the several particles of the system.

And the mass-centre of the system is the only point which has this property; for there is no other point for which

$$\Sigma . m\frac{dx}{dt} = \Sigma . m\frac{dy}{dt} = \Sigma . m\frac{dz}{dt} = 0;$$

so that the terms omitted in (60) should disappear.

83.] Secondly, let us take equations (37) and (38), which D'Alembert's principle gives when the system of particles is subject to finite accelerating forces; then, differentiating (59), we have

$$\frac{d^2x}{dt^2} = \frac{d^2\bar{x}}{dt^2} + \frac{d^2x'}{dt^2}, \quad \frac{d^2y}{dt^2} = \frac{d^2\bar{y}}{dt^2} + \frac{d^2y'}{dt^2}, \quad \frac{d^2z}{dt^2} = \frac{d^2\bar{z}}{dt^2} + \frac{d^2z'}{dt^2};$$

so that (37) become

$$\left.\begin{aligned} \mathrm{M}\frac{d^2\bar{x}}{dt^2} &= \Sigma . m\mathrm{X}, \\ \mathrm{M}\frac{d^2\bar{y}}{dt^2} &= \Sigma . m\mathrm{Y}, \\ \mathrm{M}\frac{d^2\bar{z}}{dt^2} &= \Sigma . m\mathrm{Z}; \end{aligned}\right\} \tag{62}$$

which equations have the same form as those of the motion of a material particle whose mass is M. Whence it appears that

If the whole mass of a material system is collected into its mass-centre, and the several impressed momentum-increments are applied at it, each in a line parallel to its own line of action, the expressed momentum-increment, and therefore the motion of translation, of the mass thus condensed is that due to all the impressed momentum-increments thereat applied.

Again let us substitute in (38); then we have

$$\Sigma . m\left\{(\bar{y}+y')\left(\mathrm{Z} - \frac{d^2\bar{z}}{dt^2} - \frac{d^2z'}{dt^2}\right) - (\bar{z}+z')\left(\mathrm{Y} - \frac{d^2\bar{y}}{dt^2} - \frac{d^2y'}{dt^2}\right)\right\} = 0;$$

and this when expressed at length becomes

$$\bar{y}\Sigma . m\left(\mathrm{Z} - \frac{d^2\bar{z}}{dt^2}\right) - \bar{y}\Sigma . m\frac{d^2z'}{dt^2} + \Sigma . my'\left(\mathrm{Z} - \frac{d^2z'}{dt^2}\right) - \frac{d^2\bar{z}}{dt^2}\Sigma . my'$$

$$-\bar{z}\Sigma . m\left(\mathrm{Y} - \frac{d^2\bar{y}}{dt^2}\right) + \bar{z}\Sigma . m\frac{d^2y'}{dt^2} - \Sigma . mz'\left(\mathrm{Y} - \frac{d^2y'}{dt^2}\right) + \frac{d^2\bar{y}}{dt^2}\Sigma . mz' = 0;$$

and therefore, omitting terms which vanish by reason of preceding equations and conditions, we have

$$\left.\begin{aligned}&\Sigma.m\left\{y'\left(\text{Z}-\frac{d^2z'}{dt^2}\right)-z'\left(\text{Y}-\frac{d^2y'}{dt^2}\right)\right\}=0;\\ &\text{similarly from the other two equations of (38) we have}\\ &\Sigma.m\left\{z'\left(\text{X}-\frac{d^2x'}{dt^2}\right)-x'\left(\text{Z}-\frac{d^2z'}{dt^2}\right)\right\}=0,\\ &\Sigma.m\left\{x'\left(\text{Y}-\frac{d^2y'}{dt^2}\right)-y'\left(\text{X}-\frac{d^2x'}{dt^2}\right)\right\}=0.\end{aligned}\right\} \quad (63)$$

But these are evidently, in reference to the system of coordinate axes originating at the mass-centre, the three equations of the couples which arise from the excess of the impressed over the expressed momentum-increments; and the impressed momentum-increments are at each point the same as in the original equations (38); whence we infer that

If the motion of a system of material particles, under the action of finite accelerating forces, is resolved into a translation of the mass-centre, and a rotation about an axis passing through that point, the motion of the mass-centre is the same as if the mass of all the particles were collected at it, and all the impressed momentum-increments were applied at it, each in a line parallel to its own line of action; and the motion of rotation about an axis passing through the mass-centre is the same as if that centre were a fixed point, and the system rotated about an axis passing through it under the action of the impressed momentum-increments which are actually applied to the several particles of the system.

84.] Thus, when a body is projected in any direction, and moves under the action of gravity, which acts on all the particles of the body in parallel lines, the centre of gravity of the body describes a parabola in a vertical plane. Also, if a shell is projected and the shell bursts before it meets the earth, by the action of internal forces, these latter forces, being related in equal and opposite pairs, do not appear in the right-hand member of (62), and the centre of gravity of all the broken parts moves in the same parabolic path as before the explosion.

Hence the problems of Physical Astronomy are of two distinct classes: some are those of the motion of translation of the centres of gravity of the celestial bodies, others are of rotation of the bodies about axes passing through the centres of gravity. The Lunar and Planetary Theories are of the former class; the pro-

blems of Precession and Nutation of the Earth, and of the Libration of the Moon are of the latter.

This theorem is also true of the solar system as a whole; because it is a material system of the nature explained in Art. 65; so that if the solar system has a proper motion in space by the action of forces external to it, which have either acted once for all, or are finite and continuous, that proper motion will be shewn in the change of place of the mass-centre of the system; and if the path, velocity, &c., of the mass-centre can be determined by observation, the force to which it is due may also be determined. Now as the mass of the sun is so very much greater than that of the other constituent bodies of its system, and as they are arranged around it, we may, without great error, assume the centre of the sun to be the mass-centre of its system; and this being so, the result of the calculations of M. F. G. W. Struvè, founded on the studies of Argelander, O. Struvè, and Peters, is that the sun advances annually in space through 154,185,000 miles towards a point in the heavens situated in the constellation Hercules*. This result is arrived at from an estimation of the proper motion of the stars: but our knowledge of these motions is at present far too imperfect for us to decide how far the assigned velocity and direction of the solar motion deviates from exactness; and whether it continues uniform, or whether it shows any symptoms of deflection from rectilinearity. At present, says sir John Herschel, we require more precise and extensive knowledge, before we can hold out a prospect of being one day enabled to trace out an arc of the solar orbit, and to indicate the direction in which the preponderant gravitation of the sidereal firmament is urging the central body of our system.

Section 3.—*Principles of the conservation of the motion of the mass-centre, and of the conservation of moments of momenta and of areas. Laplace's invariable plane.*

85.] I propose now to consider certain theorems which arise out of equations (37) and (38), when the impressed momentum-increments are of certain particular forms; and, firstly, (37):

* See Études d'Astronomie Stellaire. St. Petersburg, 1847. Also a Paper "On the movement of the Solar System in Space," by Mr. Edwin Dunkin, Mem. Royal Astrom. Soc., Vol. XXXII, 1864.

Suppose a material system to have been put into motion by the action of instantaneous forces, so that the axial components of the velocity of its mass-centre are those given in equations (60); and let us suppose the forces which subsequently act on the system to be such that $\Sigma . m\text{X} = \Sigma . m\text{Y} = \Sigma . m\text{Z} = 0$;

the meaning of which condition is, that either the system is free from the action of any forces; or the forces are such that the momentum-increments impressed by them mutually destroy each other, when all are transferred to the mass-centre in lines parallel to their own lines of action; then, from (62), we have

$$\frac{d^2\bar{x}}{dt^2} = 0, \quad \frac{d^2\bar{y}}{dt^2} = 0, \quad \frac{d^2\bar{z}}{dt^2} = 0;$$

$$\therefore \quad \frac{d\bar{x}}{dt} = \bar{\text{v}}_x, \quad \frac{d\bar{y}}{dt} = \bar{\text{v}}_y, \quad \frac{d\bar{z}}{dt} = \bar{\text{v}}_z;$$

$$\bar{x}-a = \bar{\text{v}}_x t, \quad \bar{y}-b = \bar{\text{v}}_y t, \quad \bar{z}-c = \bar{\text{v}}_z t;$$

$$\therefore \quad \frac{\bar{x}-a}{\bar{\text{v}}_x} = \frac{\bar{y}-b}{\bar{\text{v}}_y} = \frac{\bar{z}-c}{\bar{\text{v}}_z}; \tag{64}$$

(a, b, c) being the place of the mass-centre when $t = 0$.

As (64) are the equations to a straight line, it follows that the mass-centre moves along a straight line, of which the direction-cosines are proportional to the axial components of its initial velocity; and its velocity is given in Art. 82.

If the mass-centre is initially at rest, so that

$$\bar{\text{v}}_x = \bar{\text{v}}_y = \bar{\text{v}}_z = 0,$$

it remains at rest during the whole motion of the system.

This theorem is called the principle of the conservation of the motion of the mass-centre; and by virtue of it, in all cases of motion of a free system of particles, and of a system which is subject to forces which mutually destroy each other, the mass-centre of the system either remains at rest or moves with a constant velocity along a determinate straight line.

86.] This principle supplies an answer to the question: If a man is on smooth ice, how can he move along it? or on a perfectly smooth horizontal table, how can he get off? He cannot move along the ice or get off the table, unless he can produce momentum in a direction other than that perpendicular to the ice or the table; whatever the motion of his body may be, the weight of it can produce motion only in a vertical straight line. If however he can throw any thing away from himself, and

thereby generate a horizontal momentum, he will give to his body a horizontal velocity in an opposite direction, such that the momentum of his body will be equal to that of the mass which he has thrown away, and this velocity will carry his body along the ice or off the table; similarly if a person of mass M walks upon a rough plank of mass m and length $2a$, which is placed on a smooth horizontal plane, the centre of gravity of the two bodies remains at rest, so that while the person walks from one end of the plank to the other, the plank will recede through the distance $\frac{2\,\mathrm{M}a}{\mathrm{M}+m}$, and he will have advanced through the distance $\frac{2ma}{\mathrm{M}+m}$.

Hence also the motion of the mass-centre of a system of particles is not altered by their mutual collision, whatever is their degree of elasticity, because a reaction always exists equal and opposite to the action. If an explosion takes place in a moving body, whereby it is broken into pieces, the line of motion and the velocity of the mass-centre of the whole are not changed by the explosion; thus the motion of the mass-centre of the earth is unaltered by earthquakes. The motion of the mass-centre of the solar system is not affected by the mutual and reciprocal action of its several members; it is only changed by the action of forces external to the system.

87.] If we take equations (35), and put them into the form

$$\left.\begin{aligned} \Sigma.m(y\mathrm{V}_z - z\mathrm{V}_y) &= \Sigma.m(y\mathrm{Z} - z\mathrm{Y}), \\ \Sigma.m(z\mathrm{V}_x - x\mathrm{V}_z) &= \Sigma.m(z\mathrm{X} - x\mathrm{Z}), \\ \Sigma.m(x\mathrm{V}_y - y\mathrm{V}_x) &= \Sigma.m(x\mathrm{Y} - y\mathrm{X}); \end{aligned}\right\} \quad (65)$$

the left-hand members of these are the axial components of the moments of the couples of the expressed momenta of all the particles of the system. And as the right-hand members are the similar quantities for the impressed momenta, the equality of the two is asserted in the equations. If therefore the system of particles moves at any time t with such momenta that the left-hand members of (65) express the axial components of the moments of the couples of the expressed momenta of all the particles, then $\Sigma.m\mathrm{X}$, $\Sigma.m\mathrm{Y}$, $\Sigma.m\mathrm{Z}$ are the momenta which impressed in a direction contrary to that of the motion and along their proper lines will destroy the rotation of the system; and moreover if $\Sigma.m\mathrm{X}$, $\Sigma.m\mathrm{Y}$, $\Sigma.m\mathrm{Z}$ are subject to the relations (34), the system will be brought to rest.

88.] Next let us take equations (40), and suppose the acting forces to be such that the impressed momentum-increments satisfy the following conditions, viz.

$$\Sigma . m(y\mathrm{Z} - z\mathrm{Y}) = \Sigma . m(z\mathrm{X} - x\mathrm{Z}) = \Sigma . m(x\mathrm{Y} - y\mathrm{X}) = 0. \qquad (66)$$

This is the case;

(1) When, for every particle of the system, $\mathrm{X} = \mathrm{Y} = \mathrm{Z} = 0$; that is, when the system is free of the action of all forces.

(2) When the members of the system are subject to forces, to each of which an equal and opposite one corresponds. Thus, for example, suppose m' and m'', situated at (x', y', z') and (x'', y'', z'') respectively, to be attracted towards each other by a certain force P, dependent on their invariable distance (r) from each other; then

$$\left.\begin{aligned} m'\mathrm{X}' &= \frac{x''-x'}{r}\mathrm{P}, & m'\mathrm{Y}' &= \frac{y''-y'}{r}\mathrm{P}, & m'\mathrm{Z}' &= \frac{z''-z'}{r}\mathrm{P}, \\ m''\mathrm{X}'' &= -\frac{x''-x'}{r}\mathrm{P}, & m''\mathrm{Y}'' &= -\frac{y''-y'}{r}\mathrm{P}, & m''\mathrm{Z}'' &= -\frac{z''-z'}{r}\mathrm{P}; \end{aligned}\right\} \qquad (67)$$

$$\therefore \quad m'(y'\mathrm{Z}' - z'\mathrm{Y}') + m''(y''\mathrm{Z}'' - z''\mathrm{Y}'') = 0; \qquad (68)$$

and similar results are true for every other pair of equal and opposite actions and reactions; and also for the other couples.

(3) When the lines of action of the forces acting on the several particles of the system pass through the origin; because in this case

$$\frac{\mathrm{X}}{x} = \frac{\mathrm{Y}}{y} = \frac{\mathrm{Z}}{z}. \qquad (69)$$

This includes the case, when the body moves about a fixed point, which is taken as the origin, and at which a pressure acts.

(4) When the forces would be in equilibrium, were the system on which they act brought suddenly to rest; because in that case (66) are the conditions of statical equilibrium.

In all these cases the right-hand members of (40) vanish, and integrating, we have

$$\left.\begin{aligned} \Sigma . m\left(y\frac{dz}{dt} - z\frac{dy}{dt}\right) &= h_1, \\ \Sigma . m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right) &= h_2, \\ \Sigma . m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) &= h_3, \end{aligned}\right\} \qquad (70)$$

where h_1, h_2, h_3 are certain constants of integration; and, as is evident from (65), are the axial components of the moments

of the couples of momenta of all the particles due to the instantaneous forces by which the system of particles is originally put into motion; or, as we may say, they are the axial components of the sum of the moments of the couples of all the momenta at any given epoch. Hence

If a system of material particles is put into motion by the action of instantaneous or other forces, and, when the action of these forces ceases, is acted on by forces which satisfy the conditions (66), then, notwithstanding the alteration in the expressed momenta of individual particles, the sum of the axial components of the moments of the couples of the expressed momenta of all the particles, at any time t, is constant.

Also the moment-axis of this resultant couple of all the expressed momenta is constant, and the direction of the rotation-axis is fixed; that is, these are independent of the time, and of any particular system of axes, and remain the same throughout the motion. Thus, if h is the moment-axis of the resultant couple, and α, β, γ are the direction-angles of the rotation-axis,

$$h^2 = h_1^2 + h_2^2 + h_3^2; \tag{71}$$

$$\frac{\cos\alpha}{h_1} = \frac{\cos\beta}{h_2} = \frac{\cos\gamma}{h_3} = \frac{1}{h}. \tag{72}$$

This axis is consequently called the invariable axis, and the plane through the origin which is perpendicular to it is called the invariable plane; and the theorem is called the principle of conservation of moments of momenta.

Also in reference to the origin at which the conditions (66) are satisfied, the line whose direction-cosines are proportional to h_1, h_2, h_3 is that with respect to which the sum of the moments of the momenta of the system is a maximum. For if we take any other line (l, m, n) passing through the origin and inclined to the former line at an angle θ: then if H is the sum of the moments of the momenta of the system about this latter line,

$$\begin{aligned} \text{H} &= l h_1 + m h_2 + n h_3, \\ &= h\cos\theta, \end{aligned}$$

so that H is always less than h; the line therefore whose direction-cosines are proportional to h_1, h_2, h_3 is that with respect to which the sum of the moments of the momenta of the system is greater than the sum with respect to any other axis passing through the same point.

89.] The cases in which the conditions (66) are satisfied are numerous enough to make the theorem of great importance. Thus, it is true when a collision takes place between two or more members of the system, because equal and opposite actions are generated thereby, whatever is the degree of elasticity. It is also true when two or more members become suddenly united; when parts of the system pass from the gaseous to the fluid state, or from the fluid to the solid state; provided that the causes by which a transmutation takes place produce equal and opposite actions. This is a remarkable case, because the forces may be functions of the time explicitly, but, as they disappear, the principle is true. Thus the moment of the couple of all the momenta of the earth, as well as the direction of its rotation-axis, would remain the same, supposing the earth to be cooled down, without loss of gravitating matter. And the principle is also true when the magnetic or electrical state of two particles or of two members of the system is altered, if the change is accompanied by an equal and opposite action. Thus, no alteration is caused either in the length of the day, or in the position of the earth's axis, that is, in the place of the polar star, by earthquakes, the rolling of the sea on the shore, the fall of avalanches, the continual friction of the wind against the surface of the earth, &c.; because all these actions are accompanied by equal and opposite reactions, and therefore are in accordance with the equations (66).

90.] Equations (70) have also received another interpretation, which is of a geometrical character. As a moment of momentum involves the product of a velocity and a perpendicular from a given point on its action-line, it is of two dimensions in space, and is so far homogeneous with, and may be adequately represented by, an area. Of this we have already had an instance in the case of a particle moving in a plane under the action of a central force. For if the centre of force is taken as the origin, the force has no moment about that point, consequently condition (66) is satisfied, and the moment of momentum about the axis perpendicular to the plane of motion is constant; the equation which expresses this fact is the last of (70), viz.

$$x\frac{dy}{dt} - y\frac{dx}{dt} = r^2\frac{d\theta}{dt} = h;$$

of which the geometrical interpretation is, as is well known, 'Equal sectorial areas are described in equal times.' Lagrange

and Laplace and other writers on Mechanics have interpreted on similar lines the three equations (70).

From the origin let radii vectores be drawn to each of the particles of the system; as the system moves, then, separating the motion of translation from that of rotation, if the origin is assumed not to move during the time dt, each radius describes an infinitesimal sectorial area, which is part of a conical surface; let these sectorial areas be projected on the coordinate planes; from (70) we shall infer that the aggregate of the products of each particle and the projection of the sectorial area described by its radius vector relatively to each of the coordinates planes, varies as the time.

Let r be the radius vector, drawn from the origin to the place of m, at the time t; let $d\text{A}$ be twice the infinitesimal sectorial area over which r passes in the time dt; and let $d.\text{A}_x$, $d.\text{A}_y$, $d.\text{A}_z$ be the projections of $d\text{A}$ on the planes of (y, z), (z, x), (x, y) respectively; then,

$$\left.\begin{aligned} d.\text{A}_x &= y\,dz - z\,dy, \\ d.\text{A}_y &= z\,dx - x\,dz, \\ d.\text{A}_z &= x\,dy - y\,dx; \end{aligned}\right\} \tag{73}$$

so that (70) become

$$\left.\begin{aligned} \Sigma.m\,d.\text{A}_x &= h_1\,dt, \\ \Sigma.m\,d.\text{A}_y &= h_2\,dt, \\ \Sigma.m\,d.\text{A}_z &= h_3\,dt. \end{aligned}\right\} \tag{74}$$

Now as these equations are true for an infinitesimal time dt, and for any point as centre of areas, provided that in case (3) of Art. 88 that point is the source of the central forces; so will they be true for a finite time, if the centre is fixed, or if the centre moves in a straight line; under either of these circumstances we may integrate (74), and we have

$$\left.\begin{aligned} \Sigma.m\,\text{A}_x &= h_1 t, \\ \Sigma.m\,\text{A}_y &= h_2 t, \\ \Sigma.m\,\text{A}_z &= h_3 t; \end{aligned}\right\} \tag{75}$$

the limits of integration being such that the areas and the time begin simultaneously. Thus, the sum of the products of the mass of every particle, and the projection of the sectorial area described by its radius vector on each coordinate plane, varies as the time; and for an unit of time is constant throughout the motion.

This theorem is called the principle of the conservation of areas, and is true whenever equations (66) are satisfied.

The signs of the areas are thus far determined by the signs of

the right-hand members of (73); they are therefore to be considered positive when the direction of rotation is positive according to the principle of Art. 44. Thus, for instance, for rotation about the axis of z, if the projection of the radius vector on the plane of (x, y) moves from the axis of x towards the axis of y, $d.A_z = x\,dy - y\,dx$; in which case $d.A_z$ is positive; and the signs of the other projections are to be taken on an analogous principle. If therefore the motion is retrograde, the areas will have negative signs.

91.] As the sum of the products of the mass of each particle and its projected sectorial area, varies as the time, and is constant for an unit of time, for each of the coordinate planes, so will it be also for every plane; the sum however of these products varies as the position of the plane on which they are projected varies: it is evident that there is an infinite number of planes, for which the sum vanishes; viz. all those planes, the direction-cosines of whose normals are l, m, n, and which satisfy the condition

$$l h_1 + m h_2 + n h_3 = 0: \qquad (76)$$

hence it is evident that all these planes may intersect along the line whose direction-cosines are proportional to h_1, h_2, h_3. And the plane which is perpendicular to this straight line has the peculiarity that the sum of the products of the masses, and of the projected areas, vanishes for all planes perpendicular to it; and, for any other plane, varies as the cosine of the angle at which the planes are inclined to each other. It is evident then that for this plane the sum is a maximum; and the position of it may thus be found:

Let l, m, n be the direction-cosines of the normal of the required plane; and let u be the sum of the products of each mass and its sectorial area projected on the required plane; then the theory of projections of areas gives us

$$u = l\Sigma.mA_x + m\Sigma.mA_y + n\Sigma.mA_z \qquad (77)$$

$$= (l h_1 + m h_2 + n h_3)t; \qquad (78)$$

and

$$1 = l^2 + m^2 + n^2; \qquad (79)$$

therefore

$$\mathrm{D}u = 0 = h_1\,dl + h_2\,dm + h_3\,dn,$$

$$0 = l\,dl + m\,dm + n\,dn;$$

$$\therefore \quad \frac{l}{h_1} = \frac{m}{h_2} = \frac{n}{h_3} = \frac{1}{h} = \frac{u}{h^2 t}; \qquad (80)$$

$$\therefore \quad u = ht = \{h_1^2 + h_2^2 + h_3^2\}^{\frac{1}{2}}\, t; \qquad (81)$$

$$l = \frac{h_1}{h}, \qquad m = \frac{h_2}{h}, \qquad n = \frac{h_3}{h};$$

whereby the maximum value of the products is determined, and also the direction-cosines of the plane for which the sum of the products of the masses and the projected sectorial areas is a maximum. And, if that plane passes through the origin, its equation is

$$h_1 x + h_2 y + h_3 z = 0.$$

Hence it appears that the position of it is independent of t; and is therefore the same throughout the motion. For this reason it is called the invariable plane. The preceding equation shews that if at any time the masses of the moving particles, their places with reference to a centre fulfilling the conditions (66), and their velocities are known, then h_1, h_2, h_3 may be calculated, and the position of the invariable plane will be completely determined.

92.] In the determination of the places and motion of the heavenly bodies astronomers are always subject to the difficulty that they have no fixed planes and no fixed lines to which they can refer them. It is true that they generally take the sun as a fixed centre and the plane of the ecliptic, that is, the plane in which the centre of the earth always is in its motion around the sun, to be a fixed plane. The proper motion of the stars however renders it almost more than probable, that their motions are in a great measure only apparent, and are due to a true proper motion of the sun: and the position of the plane of the ecliptic is subject to small variations by the disturbing effects of the moon, planets, and perhaps other members of the solar system.

Astronomers therefore are referring the places and motions of the planets to the sun, which is not a fixed centre, and to the ecliptic, which is a moving plane: herein lies what may be a fruitful source of uncertainty and inaccuracy; inferences from observations, and theory built upon them, are carried over long ages; and it would be of advantage to astronomy if a fixed point and a fixed plane could be determined, to which all observations and calculations could be conveniently referred; or if, the position of the latter being given in direction, the motion, rectilineal or other, of the former were known. Now it has been before observed that probably the sun has a proper motion in space; and that this is rectilineal, so far as our observations at present indicate, and with a known velocity. Thus far then, if the sun is taken as the centre of areas, the principle of areas

may be true for the solar system. The forces which act on the solar system are (1) chiefly internal forces of attraction which will disappear in the aggregate of the moving masses of the system; and (2) the external forces acting on the sun and other members of the system from stars and other bodies, some of which are perhaps not visible to us. As the mass of the sun however is so much larger than the masses of all the other bodies of the solar system, we may assume the sun's centre to be the centre of gravity of all the bodies of the system, and the external forces which act on the several members of the system to be applied at it, in accordance with the principle of Art. 82. We may reasonably suppose that these forces produce the sun's proper motion in space, and do not produce any sensible effect on the rotation of the bodies about it; that is, we shall assume these external forces, approximately and sensibly, to be such as satisfy the equations (66).

These forces therefore are such that the theory of the invariable plane is applicable to the solar system; and as its position is the same during the whole motion, being independent of the time, it is a plane to which the places and motions of the members of the system may be advantageously referred. The determination of its place however requires a knowledge of the masses of all the members of the system, and of the elements of their orbits. Approximate values of these are known for the planets and their satellites, but of the masses of the comets we are in total ignorance. As the mutual attractions and perturbations of the several planets however are sufficient for the explanation of all these inequalities, it is manifest that hitherto at least the action of the comets on the planetary system is insensible. The comet of 1770 approached so near to the earth, that the periodic time of the comet is calculated to have been increased by 2.046 days; and, if its mass had been equal to that of the earth, it would, according to Laplace, have increased the length of our year by nearly one hour and fifty-six minutes; but Laplace adds, that if an increase of only two seconds had taken place in the length of the year, it would have been detected; and as such an increase has not been detected, it follows that the mass of the comet must be less than $\frac{1}{5000}$dth part of the mass of the earth. The same comet passed through

the satellites of Jupiter in the years 1767 and 1779, without producing any effect. Thus, though comets are greatly disturbed by the action of the planets, it does not appear that they produce any sensible effect on the planets by their action. In the determination therefore of the position of the invariable plane of the solar system, their effect is insensible.

If therefore h_1, h_2, h_3 have been determined for the plane of the ecliptic, as that of (x, y), by observation, and θ is the inclination of the invariable plane to that of the ecliptic, and ψ is the longitude of its ascending node, from Art. 4 we have

$$h_1 = h \sin\psi \sin\theta, \qquad h_2 = -h\cos\psi \sin\theta, \qquad h_3 = h\cos\theta;$$

$$\therefore \quad \cos\theta = \frac{h_3}{h}, \qquad \tan\psi = -\frac{h_1}{h_2};$$

and thus the position of the invariable plane would be known.

It will be observed that h_1, h_2, h_3 are in (70) the sums of the axial components of the moments of the couples of the expressed momenta of all the particles due to an unit of time; and, in (75), are the sums of the products of every particle and the projected sectorial area of its radius vector about the origin in an unit of time. In calculating therefore these quantities for the determination of the position of the invariable plane of the solar system, since the planets and satellites rotate about their own axes, and the satellites revolve about their primaries, we cannot estimate their moments or their sectorial areas, as if they were single particles; but the required quantities must be calculated separately for each individual particle. Thus, as the sun rotates, the sectorial area corresponding to each of its elements has to be estimated. As satellites revolve about their primaries, and also rotate about their own axes, these have to be estimated. It would be out of place here to enter on these calculations, although they are of extreme interest, and of great importance in the calculations of accurate astronomy; I can do no more than refer the student to places where the mode of calculation is explained: (1) Laplace, Exposition du Système du Monde, 5me Ed. Paris, 1824, p. 199, lib. IV, ch. II. (2) Poinsot, Équateur du Système Solaire; appended to the Éléments de Statique; 8me Ed. 1842. (3) Poisson, Traité de Mécanique, 2nde Ed. 1833, Vol. II, p. 469. (4) A note, "Du plan invariable du Système du Monde," appended to the 3rd Vol. of Pontecoulant, "Système du Monde," Paris, 1834. The real dynamical things which are invariable,

and on which the position of the plane depends, are the momentum-moments; the products of the masses and the sectorial areas are geometrical representatives of them; and the theorem has been stated in the latter form probably because Kepler's Law of Areas becomes hereby generalised.

93.] If the equations of motion are expressed in the form and in terms of the symbols given in Art. 74, then the theorems proved in this section take the following forms, viz.

$$\frac{d\,\mathrm{v}_s}{dt} = 0, \quad \text{and} \quad \frac{d\,\mathrm{H}_s}{dt} = 0; \tag{82}$$

so that (1) v_s is constant; that is, the sum of the momenta of all the particles of the system resolved along any straight line s is constant, and is equal to $\mathrm{M}\,\bar{\mathrm{v}}_s$, where M is the mass of all the moving particles, and $\bar{\mathrm{v}}_s$ is the initial or other velocity of the mass-centre along that line. (2) H_s is constant; that is, the sum of the moments of the couples of the momenta of all the particles of the system about any line s is constant, and is equal to the sum of the moments of the momenta which were originally impressed on the particles about this line.

Hence it is evident that if the action-lines of all the forces acting on a system of particles are parallel, the sum of the moments of the momenta of all the particles about an axis parallel to the action-lines of the forces is constant, because none of the forces act to produce a change of moment of momentum about this axis.

SECTION 4. *On momentum, and moments of momenta.*

94.] The subjects of momentum, and especially of the moments of momenta of the particles of a system with respect of a given straight line, to which we have been led in the preceding sections, are so important that we must give to them separate and independent consideration.

As the momentum of a particle m, moving with a velocity v, is mv along the line of its motion, so is it $mv\cos\theta$ along a line making an angle θ with that line. Similarly, if there are many particles, each moving in its own line with its own velocity, $\Sigma.mv\cos\theta$ is the momentum of the system along a given line, $v\cos\theta$ being the velocity of m along that line. Thus $\Sigma.m\frac{dx}{dt}$, $\Sigma.m\frac{dy}{dt}$, $\Sigma.m\frac{dz}{dt}$ are the momenta of the system along the co-

ordinate axes. This result is also true of momentum-increments, as is evident from the investigations of the first section of this chapter.

Also whatever line is taken as an axis of rotation these momenta produce couples about it, the moments of which we may thus determine. If m be a particle at the point (x, y, z), and $\frac{ds}{dt}$ its velocity, the axial components of its momentum are $m\frac{dx}{dt}$, $m\frac{dy}{dt}$, $m\frac{dz}{dt}$; and the moments of the couples of this momentum about the three axes are respectively $m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)$, $m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)$, $m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$; so that for a system of particles the moments of the couples of these momenta about the three axes are respectively $\Sigma.m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)$, $\Sigma.m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)$, $\Sigma.m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)$.

For convenience we will denote these quantities severally by h_1, h_2, h_3; so that

$$\left.\begin{aligned}\Sigma.m\left(y\frac{dz}{dt}-z\frac{dy}{dt}\right)&=h_1,\\ \Sigma.m\left(z\frac{dx}{dt}-x\frac{dz}{dt}\right)&=h_2,\\ \Sigma.m\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)&=h_3;\end{aligned}\right\} \tag{83}$$

and consequently the equations (40) take the form

$$\frac{dh_1}{dt}=\text{L}, \qquad \frac{dh_2}{dt}=\text{M}, \qquad \frac{dh_3}{dt}=\text{N}; \tag{84}$$

also, if h is the moment of the resultant couple,

$$h_1^2+h_2^2+h_3^2=h^2;$$

so that h is the moment of the momenta about an axis whose direction-cosines are proportional to h_1, h_2, h_3, and the axial components of which are h_1, h_2, h_3.

In reference to (41), H_s is the general symbol for h_1, h_2, h_3, and K is the general symbol for L, M, N; so that h_1, h_2, h_3 are constants only when $\text{L}=\text{M}=\text{N}=0$. In the preceding sections, h_1, h_2 and h_3 were introduced as constants, and were treated as such, since $\text{K}=0$ for each of the coordinate axes to which these quantities severally refer.

If the motion of the system consists of rotation about an axis

passing through a fixed point with an angular velocity ω, of which the axial components are ω_x, ω_y, ω_z, then, since

$$\frac{dx}{dt} = z\,\omega_y - y\,\omega_z, \qquad \frac{dy}{dt} = x\,\omega_z - z\,\omega_x, \qquad \frac{dz}{dt} = y\,\omega_x - x\,\omega_y,$$

$$\left.\begin{aligned} h_1 &= \Sigma.m\,(y^2+z^2)\,\omega_x - \Sigma.m\,xy\,\omega_y - \Sigma.m\,zx\,\omega_z, \\ h_2 &= -\Sigma.m\,xy\,\omega_x + \Sigma.m\,(z^2+x^2)\,\omega_y - \Sigma.m\,yz\,\omega_z, \\ h_3 &= -\Sigma.m\,zx\,\omega_x - \Sigma.m\,yz\,\omega_y + \Sigma.m\,(x^2+y^2)\,\omega_z. \end{aligned}\right\} \quad (85)$$

These expressions will receive further explanation as to origin and meaning in the next chapter.

If the system consists of particles moving in one plane only, say in that of (x, y), or of particles each of which moves in a plane parallel to that of (x, y), then, as

$$\omega_x = \omega_y = 0, \quad \omega_z = \omega, \quad h_1 = h_2 = 0, \quad h_3 = h,$$

and
$$h = \Sigma.m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) = \omega\,\Sigma.m r^2,$$

where r is the distance from m from the axis of h.

95.] Let us now consider the changes which these undergo, when the system is referred to another origin, which may or may not have motion of translation, and through which as a fixed point an axis passes, about which the system rotates, as explained in Art. 61.

Let (x_0, y_0, z_0) be the new origin, of which let v be the velocity, and let u, v, w be the axial components of v. At the time t let (x, y, z) be the place of m in reference to the fixed axes, and (ξ, η, ζ) in reference to the new axes: so that

$$x = x_0 + \xi, \quad y = y_0 + \eta, \quad z = z_0 + \zeta;$$

$$\frac{dx}{dt} = u + \frac{d\xi}{dt}, \quad \frac{dy}{dt} = v + \frac{d\eta}{dt}, \quad \frac{dz}{dt} = w + \frac{d\zeta}{dt}:$$

and where, as the motion relative to the new origin is of rotation only,

$$\frac{d\xi}{dt} = \zeta\omega_\eta - \eta\omega_\zeta, \quad \frac{d\eta}{dt} = \xi\omega_\zeta - \zeta\omega_\xi, \quad \frac{d\zeta}{dt} = \eta\omega_\xi - \xi\omega_\eta.$$

Now in terms of these values, if M is the mass of the system,

$$\begin{aligned} h_1 &= \Sigma.m\left\{(y_0+\eta)\left(w + \frac{d\zeta}{dt}\right) - (z_0+\zeta)\left(v + \frac{d\eta}{dt}\right)\right\} \\ &= \text{M}\,(y_0 w - z_0 v) + \Sigma.m\left(\eta w - \zeta v + y_0\frac{d\zeta}{dt} - z_0\frac{d\eta}{dt}\right) \\ &\qquad + \Sigma.m\left(\eta\frac{d\zeta}{dt} - \zeta\frac{d\eta}{dt}\right); \quad (86) \end{aligned}$$

of which the first term is, in reference to the axis of h_1, the moment of momentum of the whole mass collected at the moving origin, and the last term is the moment of momentum of the system relative to the parallel axis through the moving origin. Hence h_1 is the sum of these quantities only when the intermediate term vanishes. This is the case when the moving origin is the mass-centre, because then $\Sigma.m\eta = \Sigma.m\zeta = 0$, and, using the notation of Art. 81,

$$h_1 = \text{M}(\bar{y}\bar{w}-\bar{z}\bar{v})+\Sigma.m\left(y'\frac{dz'}{dt}-z'\frac{dy'}{dt}\right),$$

$$h_2 = \text{M}(\bar{z}\bar{u}-\bar{x}\bar{w})+\Sigma.m\left(z'\frac{dx'}{dt}-x'\frac{dz'}{dt}\right),$$

$$h_3 = \text{M}(\bar{x}\bar{v}-\bar{y}\bar{u})+\Sigma.m\left(x'\frac{dy'}{dt}-y'\frac{dx'}{dt}\right).$$

Hence, as these quantities are subject to the laws of projection, we have the following general theorem:

The sum of the moments of the momenta of all the particles of a system about any axis is equal to the moments of the momentum of the whole mass collected at its mass-centre about the same axis, together with the sum of the moments of the momenta of all the particles about a parallel axis passing through the mass-centre.

This theorem is also thus evident. Through the origin O, and through the mass-centre G, let parallel lines be drawn, and through P, the place of any particle m, let a plane be drawn perpendicular to these lines; let r and r' be the distances from P on the lines through O and G, and let ρ be the perpendicular distance between these lines; then if θ is the angle between r' and ρ,

$$r^2 = \rho^2 - 2\rho r'\cos\theta + r'^2;$$

$$\therefore \quad \Sigma.mr^2 = \text{M}\rho^2 - 2\rho\,\Sigma.mr'\cos\theta + \Sigma.mr'^2;$$

but since r' is the perpendicular distance from m on a line passing through the mass-centre, $\Sigma.mr'\cos\theta = 0$:

$$\therefore \quad \Sigma.mr^2 = \text{M}\rho^2 + \Sigma.mr'^2;$$

which equation expresses the theorem enunciated above.

96.] A similar theorem is true, that is, the intermediate terms in (86), and in its two symmetrical expressions, vanish, when the new origin and its component velocities satisfy the following conditions, viz.

$$\left.\begin{aligned}\bar{\eta} w-\bar{\zeta} v+y_0 \frac{d\bar{\zeta}}{dt}-z_0 \frac{d\bar{\eta}}{dt} &= 0, \\ \bar{\zeta} u-\bar{\xi} w+z_0 \frac{d\bar{\xi}}{dt}-x_0 \frac{d\bar{\zeta}}{dt} &= 0, \\ \bar{\xi} v-\bar{\eta} u+x_0 \frac{d\bar{\eta}}{dt}-y_0 \frac{d\bar{\xi}}{dt} &= 0.\end{aligned}\right\} \qquad (87)$$

Let us investigate the meaning of these conditions.

Multiplying these equations severally by x_0, y_0, z_0, and adding, we have

$$u(\bar{\zeta} y_0-\bar{\eta} z_0)+v(\bar{\xi} z_0-\bar{\zeta} x_0)+w(\bar{\eta} x_0-\bar{\xi} y_0)=0,$$

which shews that the line of motion of the new origin lies in the plane containing the origin, the new origin, and the mass-centre.

Also, multiplying the equations severally by $\bar{\xi}, \bar{\eta}, \bar{\zeta}$ and adding, we have

$$\frac{d\bar{\xi}}{dt}(\bar{\zeta} y_0-\bar{\eta} z_0)+\frac{d\bar{\eta}}{dt}(\bar{\xi} z_0-\bar{\zeta} x_0)+\frac{d\bar{\zeta}}{dt}(\bar{\eta} x_0-\bar{\xi} y_0)=0, \qquad (88)$$

which shews that the line of motion of the mass-centre also lies in the aforesaid plane.

Hence the first condition for the co-existence of the equations (87) is that the absolute line of motion of the new origin, and the relative line of motion of the mass-centre lies in the plane which contains the origin, the new origin, and the mass-centre.

Also in reference to the origin, the equation to this plane is

$$x_0(w\bar{y}-v\bar{z})+y_0(u\bar{z}-w\bar{x})+z_0(v\bar{x}-u\bar{y})=0;$$

since

$$\frac{d\bar{\xi}}{dt}=\bar{\zeta}\omega_\eta-\bar{\eta}\omega_\zeta, \quad \frac{d\bar{\eta}}{dt}=\bar{\xi}\omega_\zeta-\bar{\zeta}\omega_\xi, \quad \frac{d\bar{\zeta}}{dt}=\bar{\eta}\omega_\xi-\bar{\xi}\omega_\eta,$$

$$\bar{\xi}\frac{d\bar{\xi}}{dt}+\bar{\eta}\frac{d\bar{\eta}}{dt}+\bar{\zeta}\frac{d\bar{\zeta}}{dt}=0;$$

and consequently the line of motion of the mass-centre is at right angles to the line joining the new origin and the mass-centre.

Putting (88) into the form

$$x_0\left(\bar{\zeta}\frac{d\bar{\eta}}{dt}-\bar{\eta}\frac{d\bar{\zeta}}{dt}\right)+y_0\left(\bar{\xi}\frac{d\bar{\zeta}}{dt}-\bar{\zeta}\frac{d\bar{\xi}}{dt}\right)+z_0\left(\bar{\eta}\frac{d\bar{\xi}}{dt}-\bar{\xi}\frac{d\bar{\eta}}{dt}\right)=0,$$

and replacing $\frac{d\bar{\xi}}{dt}, \frac{d\bar{\eta}}{dt}, \frac{d\bar{\zeta}}{dt}$ by their values given above, we have

$$(x_0\bar{\xi}+y_0\bar{\eta}+z_0\bar{\zeta})(\bar{\xi}\omega_\xi+\bar{\eta}\omega_\eta+\bar{\zeta}\omega_\zeta)-\bar{\rho}^2(x_0\omega_\xi+y_0\omega_\eta+z_0\omega_\zeta)=0;$$

where $\bar{\rho}$ is the distance of the mass-centre from the moving origin.

Now this condition is evidently satisfied if the rotation-axis is perpendicular to the lines which join the moving origin with the origin and with the mass-centre: because in these cases

$$x_0 \omega_\xi + y_0 \omega_\eta + z_0 \omega_\zeta = 0,$$

and

$$\bar{\xi} \omega_\xi + \bar{\eta} \omega_\eta + \bar{\zeta} \omega_\zeta = 0;$$

so that the rotation-axis is perpendicular to the plane above mentioned.

Also, since the lines of motion of the moving origin and of the mass-centre lie in this plane, there is some point in it which is an instantaneous centre; so that, if (ξ, η, ζ) is this point relative to the moving origin,

$$u = \zeta \omega_\eta - \eta \omega_\zeta, \quad v = \xi \omega_\zeta - \zeta \omega_\xi, \quad w = \eta \omega_\xi - \xi \omega_\eta;$$

$$\therefore \quad \frac{\xi \omega^2 - (v \omega_\zeta - w \omega_\eta)}{\omega_\xi} = \frac{\eta \omega^2 - (w \omega_\xi - u \omega_\zeta)}{\omega_\eta} = \frac{\zeta \omega^2 - (u \omega_\eta - v \omega_\xi)}{\omega_\zeta};$$

which are the equations to a straight line perpendicular to the plane and which intersects it at the instantaneous centre. This straight line is the central axis, see (61), Art. 49, and is that about which rotation takes place but along which in this case there is no motion of translation.

97.] If all the particles of the system move in one plane, or if each particle moves in a plane and all the planes are parallel, then if the plane of (x, y) is parallel to these planes, and h is the moment of momenta of all the particles about an axis perpendicular to it,

$$h = \Sigma . m \left(x \frac{dy}{dt} - y \frac{dx}{dt} \right).$$

Let (x_0, y_0) be the moving origin, u and v the axial components of its absolute velocity v, ω the angular velocity about the rotation-axis, (ξ, η) the place of m with reference to the moving origin; so that

$$\frac{dx}{dt} = u - \omega \eta, \qquad \frac{dy}{dt} = v + \omega \xi;$$

then

$$h = \Sigma . m \{ (x_0 + \xi)(v + \omega \xi) - (y_0 + \eta)(u - \omega \eta) \},$$

$$= \text{M} (x_0 v - y_0 u) + \omega \Sigma . m (\xi^2 + \eta^2)$$

$$+ \Sigma . m \{ \omega (x_0 \xi + y_0 \eta) + \xi v - \eta u \}.$$

Thus the moment of momenta about an axis perpendicular to the planes of motion of the particles is equal to the moment of momentum of the whole mass collected at a given point about the same axis, together with the moment of momenta of all the

particles about a parallel axis passing through the point, when the last term vanishes. Now this is the case when the new axis passes through the mass-centre, because then $\Sigma . m\xi = \Sigma . m\eta = 0$: so that if $\bar{v}$ is the velocity of the mass-centre, and p the perpendicular from the origin in its line of motion,

$$h = M\bar{v}p + \omega \Sigma . m(x'^2 + y'^2).$$

The last term also vanishes when

$$\omega(x_0 \bar{\xi} + y_0 \bar{\eta}) + \bar{\xi} v - \bar{\eta} u = 0.$$

But if (ξ_0, η_0) is the instantaneous centre,

$$\xi_0 = -\frac{v}{\omega}, \quad \eta_0 = \frac{u}{\omega}:$$

and substituting these values for x_0, y_0, the equation is satisfied.

Also substituting for u and v in terms of the coordinates of the instantaneous centre, the condition becomes

$$\bar{\xi}(x_0 - \xi_0) + \bar{\eta}(y_0 - \eta_0) = 0;$$

that is, the line joining the moving origin to the mass-centre is perpendicular to the line joining the moving origin to the instantaneous centre: or, in other words, the moving origin is on the circle described on the line joining the instantaneous centre and the mass-centre as its diameter. Consequently the moment of momentum of all the particles about any axis at right angles to the plane of motion is equal to the moment of momentum of the whole mass collected at a given point about the axis together with the moment of momentum of the system relative to a parallel axis through this point, provided that this point is situated on a circle in the plane of motion, of which the diameter is the line joining the mass-centre and the instantaneous centre.

98.] The following are instances in which the moment of momentum of a moving system is calculated.

Ex. 1. A straight rod of length a moves freely about an end, which is jointed to a fixed point. Find the moment of momentum of the rod with respect to a given plane.

Let O be the fixed end of the rod, and through O let a plane be drawn parallel to the given plane: also through O let a straight line be drawn perpendicular to the plane; and let θ be the angle between the straight line and the rod; let $d\phi$ be the angle described in the time dt, by the projection of the rod on the plane: let r be the distance of any element $\kappa\rho\, dr$ from O:

$$h = \int_0^a \kappa\rho r^2 (\sin\theta)^2 dr \frac{d\phi}{dt}$$
$$= \frac{\kappa\rho a^3}{3} (\sin\theta)^2 \frac{d\phi}{dt}.$$

Ex. 2. Four thin bars of equal length, jointed at their ends and forming a square, are placed on a smooth plane: one corner is fixed, and of the two bars abutting at it one moves with the angular velocity ω and the other with the angular velocity ω': find the moment of momentum of the system.

Let m be the mass, and $2a$ the length of each bar: let h be the moment of momentum about an axis passing through the fixed corner at right angles to the plane in which the bars move: then

$$h = \omega\int_0^{2a} \rho\kappa x^2 dx + \omega'\int_{-a}^{a} \rho\kappa x^2 dx + ma^2\{4\omega + \omega' + 2(\omega+\omega')\cos(\theta'-\theta)\}$$
$$+\omega\int_{-a}^{a} \rho\kappa x^2 dx + ma^2\{4\omega' + \omega + 2(\omega+\omega')\cos(\theta'-\theta)\}$$
$$+\omega'\int_0^{2a} \rho\kappa x^2 dx$$
$$= \frac{4ma^2}{3}\{5 + 3\cos(\theta'-\theta)\}(\omega+\omega').$$

Section 5. *Work: Energy: The Principle of Vis Viva: Lagrange's Principle of Least Action: Carnot's Theorems.*

99.] The effects of force in producing motion of matter may be estimated in two ways: either with respect to the time during which the force acts, or with respect to the distance over which it acts, or through which its point of application moves in the line of its action; these effects are respectively the time-integral and the space-integral of the force; the former is expressed in terms of momentum, which involves velocity in its first power; the latter in terms of vis viva, which involves velocity in its second power. Many instances of these effects have arisen in the former volume of the Treatise in cases of motion of a material particle. It is, however, to the space-integrals of forces acting on a system of material particles that our attention will mainly be given in this section, because these integrals have in late years assumed a much greater importance than heretofore, being the foundation of the principle of Work or Energy, whereby all forces in nature

are shewn to be not only correlated, but to have in them a principle of permanence or conservation, so that no energy can ever be lost or destroyed; energy can be transferred, and, as we shall see in the sequel, transmuted from one form into another, say from mechanical force into heat or electrical force, and vice versâ, but a principle of equivalence always rules it, and it can never be lost.

100.] Now this process of space-integration necessarily leads to the formation of quantities of the form $\text{F}\,ds$, where F denotes a force (impressed momentum-increment), and ds is the element of distance along the line of action of the force through which the force acts, or through which its point of application is moved; so that we have for consideration the product of a force and of the distance through which its point of application travels; this product is termed the Work of the force done in the distance. The appropriateness and the origin of the term are shewn by the following particular instance. If m is a mass lying on the earth's surface at a place where the acceleration due to the earth's action is g, so that the force acting on m is mg, which is also the weight of m, then exertion or effort or labour is required to raise the weight. We shall use the term effort. And the effort varies directly as the weight, and also directly as the height through which it is lifted; for p times as much effort is required to lift a weight of p pounds as to lift a weight of one pound; and also n times as much effort is required to lift a given mass through n feet vertical as to lift it through one foot vertical. Thus if m is the mass lifted, and h is the vertical height through which it is lifted, the effort varies as mgh; and effort admits of quantitative determination; and the name of *Work* is given to that particular measure of effort which is such that the work $= mgh$, that is, is equal to the product of the weight and the vertical distance through which the weight is lifted. In this case we assume h to be such that the force mg is constant throughout it. The conception, however, and the corresponding definition, may be extended to the case where the force is not constant, but varies from point to point; for we may divide the distance through which the force acts into infinitesimal elements, and consider the force to be constant through each of them, although varying from one to another.

Then if F denotes the force and ds the infinitesimal distance

through which it acts along its own line of action without variation, the work of the force in $ds = \text{F}\,ds$: and this is the element of work, and consequently $= d\text{W}$, if W is the general symbol for work, so that

$$d\text{W} = \text{F}\,ds;$$

and, taking these quantities between corresponding limits, we have

$$\text{W}_1 - \text{W}_0 = \int_0^1 \text{F}\,ds: \tag{89}$$

$\text{W}_1 - \text{W}_0$ being the quantity of work due to the force F in the change of position of the system from its first to its last place. If the body returns to its original position no work is done.

Thus as force acts in a certain direction, the point of application may be moved along the line of action either in the direction of or in the direction opposite to that of the action of the force; in the former case work is said to be done by the force, and in the latter case work is said to be done against the force; in the latter case of course some other force or agent acts, and works against the original force. Thus in the case cited of lifting a heavy body, work is done against mg (the earth's force) when the body is lifted by the lifting agent, whereas if the mass falls from a given height towards the earth work is done by the earth, and is measured by the product of the weight and the vertical distance through which the mass has fallen.

If the point of the application of the force is moved along a line which is inclined at an angle ϕ to the line of action of the force, and through a distance ds along that line, the work due to the force is equal to $\text{F} \cos\phi\, ds$, $ds \cos\phi$ being the projection of ds on the line of action of the force. So that if a force acts on a particle along a line at right angles to the line of motion of the particle it does no work on the particle; it may alter the direction of motion, but not the magnitude of the velocity. And if $\cos\phi$ is negative, work is done against the force. Thus when the point of application of a force is moved, the work due to the motion either may be done by the force, or may be zero, or may be done against the force.

If many forces act on a system the work due to each can be estimated in a similar manner, and the whole work is the sum of the several works due to the several forces. Thus if m were a particle of a system, and P were an accelerating force acting on m, and the system consisted of many particles of which m is the

type, the whole work would be equal to $\Sigma . m \int \text{P} \, dp$, dp being the distance in its line of action through which P acts, and the integral being definite.

As (89) is a definite integral, it follows that the work done in a system as it passes from one position to another, depends on the position of the points of application of the forces at the beginning and the end of the motion considered, and is independent of the time occupied in the passage of the system from its first to its last position. This circumstance arises from the fact of Work being a space-integral, and not a time-integral.

A change of configuration of the system will generally ensue on work being done by or against a force, because the points of application of the forces are moved and these are generally places of parts of the system.

101.] The following are examples of determining the amount of work due to the action of forces and the consequent change of position of a system.

Ex. 1. If a system of heavy bodies is lifted, each through a different height, the work done against the earth's attraction on the system $= g\bar{z}\Sigma . m$, where $\bar{z}$ is the vertical distance through which the centre of gravity of the system has been raised.

Hence we can determine the work done in raising a ladder from a horizontal into a vertical position; in drawing up a Venetian blind; in rolling up a cloth blind; in drawing up from a well a weight hanging from an axle by a heavy chain.

Ex. 2. A particle m moves towards a centre of force which varies inversely as the square of the distance: determine the amount of work done by the force in the motion from one place to another.

Let W be the work: then

$$\text{W} = -\int_{r_0}^{r} \frac{m}{r^2} dr$$

$$= m\left(\frac{1}{r} - \frac{1}{r_0}\right);$$

Hence if $r_0 = \infty$, $\text{W} = \frac{m}{r}$, and this is the work done in moving m from infinity to a place at a distance r from the centre of force. If m moves towards the centre of attraction work is done by the force on m, but if m is moved away from the centre by some external agent, work is done against the central force.

If a body or a system of particles is moving from infinity towards a centre of attraction which varies inversely as the square of the distance, the quantity $\Sigma . \frac{m}{r}$ is called the Potential of the system at the point from which the distances are measured. Hence the Potential is the amount of work done in the system in the motion of the particles, each from infinity to the distance r. This result is of great importance in subsequent enquiries.

Ex. 3. Prove that the work done by two particles m and m', which attract each other with forces varying directly as the mass and inversely as the square of the distance, in coming from rest at places at an infinite distance apart to places at the distance r apart, $= \frac{\mu m m'}{r}$.

Ex. 4. The work done in stretching an elastic string or bar of small transverse section from its natural length a to the length a' may thus be found.

We will assume the elasticity to be in accordance with Hooke's law, and E to be the modulus of elasticity: so that if x is the length of the string when F is the pulling force, a being its natural length,

$$x = a\left(1+\frac{\text{F}}{\text{E}}\right); \qquad \therefore \quad \text{F} = \text{E}\,\frac{x-a}{a};$$

$$\therefore \text{ the work } = \int_a^{a'} \text{F}\,dx = \frac{\text{E}}{2a}(a'-a)^2.$$

If T is the force which stretches the string to the length a',

$$\text{T} = \text{E}\,\frac{a'-a}{a};$$

$$\therefore \text{ the whole work } = \frac{\text{T}}{2}(a'-a),$$

and is equal to one half of the product of the stretching force and the extension.

Ex. 5. Shew that $\frac{w^2 a}{6\text{E}}$ is the work done, when a heavy elastic string of length a and weight w hanging vertically is stretched by its own weight.

102.] In the general case of doing work on a material system, let $m\text{P}$ be the force acting on a particle m at a point $\text{Q}\,(x, y, z)$, of which let the axial components be $m\text{X}$, $m\text{Y}$, $m\text{Z}$: let Q move to a point Q', and let $\text{QQ}' = ds$; let dp be the projection of ds on the

line of action of P: and let ϕ be the angle contained between ds and that line of action; so that $dp = ds\cos\phi$: then

$$\begin{aligned} m\text{P}\,dp &= m\text{P}\,ds\cos\phi \\ &= m\text{P}\,ds\frac{\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz}{\text{P}\,ds} \\ &= m(\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz), \end{aligned} \tag{90}$$

where dx, dy, dz are the projections of ds on the coordinate axes. And as this equation is true for the work done in every particle of the system on which force acts, so the work done by all the forces in the infinitesimal displacement will be

$$\Sigma . m(\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz).$$

Let us call this $d\text{W}$, so that

$$d\text{W} = \Sigma . m(\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz), \tag{91}$$

the summation extending to all the forces of the system, and therefore if the body receives a finite displacement, we have by integration

$$\text{W}_1-\text{W}_0 = \Sigma . m\int_0^1 (\text{X}\,dx+\text{Y}\,dy+\text{Z}\,dz); \tag{92}$$

the limits of the integral being given by the positions of the system at the beginning and the end of the motion; say at the times t and t_s.

Also from (91) we have

$$\frac{d\text{W}}{ds} = \Sigma . m\left(\text{X}\frac{dx}{ds}+\text{Y}\frac{dy}{ds}+\text{Z}\frac{dz}{ds}\right),$$

which is equal to the sum of the components of all the forces along the line s.

103.] The amount of work done is of course independent of any particular system of coordinates. The following are values of $d\text{W}$ in various systems.

If the forces act in one plane and motion takes place in that plane only, and position is expressed in terms of polar coordinates,

$$d\text{W} = \Sigma . m(\text{P}\,dr+\text{Q}\,r\,d\theta),$$

where $m\text{P}$ and $m\text{Q}$ are the radial and transversal forces respectively.

If the forces and positions are referred to spherical coordinates in space, then

$$d\text{W} = \Sigma . m(\text{P}\,dr+\text{Q}\,r\,d\theta+\text{R}\,r\sin\theta\,d\phi).$$

If the forces act in one plane, say that of (x, y), and the small

displacement is due to a rotation of the system through an angle $d\theta$ about an axis perpendicular to the plane, then $dx = -y\,d\theta$, $dy = x\,d\theta$,

$$\begin{aligned} \therefore\ d\mathrm{W} &= \Sigma.m(\mathrm{X}dx + \mathrm{Y}dy) \\ &= \Sigma.m(-\mathrm{X}y\,d\theta + \mathrm{Y}x\,d\theta), \\ \therefore\ \frac{d\mathrm{W}}{d\theta} &= \Sigma.m(\mathrm{Y}x - \mathrm{X}y), \qquad (93) \end{aligned}$$

= the moment of the forces about the axis.

Hence if G is the moment of the couple of a force, the work of G, which moves a body through an angle $d\theta$, is $\mathrm{G}\,d\theta$; that is, is the product of the moment of the couple into the angle through which the system is turned.

104.] Thus when a force is brought into action and does work or receives work, its capacity for future work is changed; it may either be diminished or increased; if a central attracting force draws a mass towards it, its power of work on the mass is diminished by a quantity which depends on the distance through which the force has acted; whereas if by the action of some other agent the mass is removed further from the centre of force, its power of work is increased by a quantity which depends on the distance through which the external agent has moved the mass. Now the capacity for work in a force is called its Energy; and if there be a system of forces, the whole capacity for work is called the energy of the system; thus we speak of the energy of the solar system. Sir W. Thomson has calculated this amount. So we speak of the energy of an electrical battery, and of the energy of accumulators. Similarly, if an elastic gas is compressed in a condenser, it is a store of energy. If the system receives work from an external agent its energy is increased; thus the energy of a clock is increased when it is wound up, and by exactly the amount of work which is spent on the winding: similarly when a watch is wound, its energy is increased exactly by the amount of work spent on the coiling of the spring, this being estimated by the work of the couple, as explained in the last example of the preceding Article; so the energy of a steam engine, including the boiler, is increased when heat is communicated to the water by the burning coal, heat being a special kind of work. If, however generally, the external agent, or in these particular cases the agent, is considered a part of the system, and is measured by the work which he does, or which he is capable of

doing, the whole energy of the system is unaltered; there has been a transference of energy from one member to another, but the sum of the whole remains the same. Energy is measured by work, and the unit of energy is the unit of work.

This principle is called the conservation or the permanence of energy or work; and states that energy or work is indestructible; its amount is permanent and can be neither increased nor diminished; what is taken from one member of a system is equivalently transferred to one or more of the other members; it may however be transmuted from one form into another. Formerly it was supposed that "force" could be *lost*, as by means of friction or the resistance of the air in a moving machine, and sometimes in a manner inexplicable; but in all these cases we now know that the "force" supposed to be "lost" is transmuted into heat or into electrical action, or into the work of the moving particles of air, and thence into heat, and the equivalence can be exactly determined; so that no work is lost, but the energy of the whole system remains unaltered, energy being the capacity for work in its many forms.

105.] When action takes place between one or more forces and a system of material particles, so that the forces either do work or receive work, a change in respect of velocity necessarily takes place in the material system and is equivalent to the change of work in the forces. In what manner however is this change of velocity exhibited? We shall shew that if m is the type particle of the system, and v and v_0 are the velocities of m respectively after and before the action of the force, the work done by or to the forces is equal to $\frac{1}{2}\{\Sigma.mv^2-\Sigma.mv_0^2\}$, that to one-half of the increase of the vis viva of the system; this one-half of the vis viva of the system, viz. $\Sigma.mv^2$, is called *kinetic energy*, and we shall shew that the work done by or to the forces is equal to the kinetic energy into which it is transmuted; and if the points of application of the forces are moved through only infinitesimal distances so that the increase of work, say $d\text{w}$, is infinitesimal, then we shall have

$$d\text{w} = \pm\tfrac{1}{2}d.\Sigma.mv^2,$$

the $\pm$ sign being introduced to shew that according as work is done by or to the forces, so will there be an increase or a diminution of kinetic energy.

Let us first take the case of a single force F acting on m, and let ds be the infinitesimal distance in its line of action through

which its point of application is moved; let $\frac{d^2s}{dt^2}$ be the consequent acceleration, then since $dW = F ds$, and $F = m\frac{d^2s}{dt^2}$,

$$dW = m\frac{d^2s}{dt^2}ds = \tfrac{1}{2}d.m\frac{ds^2}{dt^2};$$

and consequently if many forces act, and dW is the increase of work due to the action of all the forces, we may introduce the sign of summation, and

$$dW = \tfrac{1}{2}d.\Sigma.m\frac{ds^2}{dt^2} = \tfrac{1}{2}d.\Sigma.mv^2$$

$$= \text{the differential of kinetic energy.} \qquad (94)$$

This result may also thus be shewn.

106.] Let F be the force acting on the particle m, and let the point of application of F be moved over the distance (straight or curved) $s_n - s_0$; let this distance be divided into infinitesimal elements $s_1 - s_0$, $s_2 - s_1$, ... $s_n - s_{n-1}$, each being such that the variations of the force in it may be omitted; let F_1, F_2, ... F_n be the values of F, and $v_1, v_2, \ldots v_n$ be the values of the velocity in each one of these respectively; then, taking the first space-element $s_1 - s_0$, and assuming t_1 to be the time in which m moves through it,

$$F_1 t_1 = m(v_1 - v_0);$$

and as the variation of velocity in the course of t_1 will be very small we may take the arithmetic mean of the initial and terminal velocities to be its average value, so that

$$s_1 - s_0 = \tfrac{1}{2}(v_1 + v_0)t_1.$$

Hence, eliminating t_1, we have

$$F_1(s_1 - s_0) = \frac{m}{2}(v_1^2 - v_0^2).$$

Similarly for the next and succeeding intervals we have

$$F_2(s_2 - s_1) = \frac{m}{2}(v_2^2 - v_1^2),$$

. . . .

$$F_n(s_n - s_{n-1}) = \frac{m}{2}(v_n^2 - v_{n-1}^2);$$

and by addition

$$\int_0^n F\,ds = \frac{m}{2}(v_n^2 - v_0^2).$$

And as a similar result is true for each one of the acting forces, and as $dW = \Sigma.F ds$, if there are many acting forces, and W is the work of the whole system,

$$dW = \tfrac{1}{2}d.\Sigma.mv^2;$$

that is, the work done in the system by the forces is shewn by and is equal to the increase in the kinetic energy.

This investigation gives also the amount of work done by an impulse or a blow. Let Q be the momentum of the blow, and let m be the mass whose momentum is changed by it from mv_0 to mv_1, so that $Q = m(v_1 - v_0)$; now since a finite force (impressed momentum-increment) consists of a series of impulses succeeding one another at intervals of infinitesimal duration, we may take any one, say the first of the preceding equations, and apply it to the case of a finite impulse, that is, we may replace F_1 by Q, and consequently the work done by $Q = \frac{m}{2}(v_1^2 - v_0^2)$;

$$= \frac{v_1 + v_0}{2} Q.$$

107.] Let us now apply these principles to the general equation of motion as given in Art. 76. Here it will be observed that δx, δy, δz are arbitrary geometrical variations, parallel to the axes, of the points of application of the forces, whereby the system assumes a position arbitrarily chosen, though infinitesimally near to its former position. Now we are about to replace these variations by those which actually take place, and for which time is required; that is, the system is in its displaced position at an interval of dt after the time at which it was in its original position. Consequently δx, δy, δz cannot be replaced by dx, dy, dz, if the time t is explicitly contained in any term of the equation; and accordingly this process is inapplicable when the forces are functions of the time as well as of the positions.

M. Poisson has proved this restriction in the following way. Let $F_1 = 0$, $F_2 = 0$, ... $F_n = 0$ be the equations expressing conditions to which the system is subject; let $F = 0$ be a type of these, and we will suppose it to contain all the variables, and also t; then for the geometrical displacement the variation of this is

$$\left(\frac{dF}{dx_1}\right)\delta x_1 + \left(\frac{dF}{dy_1}\right)\delta y_1 + \left(\frac{dF}{dz_1}\right)\delta z_1 + \ldots + \left(\frac{dF}{dz_n}\right)\delta z_n = 0; \quad (95)$$

there being no variation of F in respect of t, because the shifting is a virtual geometrical displacement, for which no change of time is required. But if the changes in the coordinates are those which are due to the actual motion of the system, so that

we consider the system in two successive positions, then if F contains t, the total change in F is

$$\left(\frac{d\text{F}}{dx_1}\right) dx_1 + \left(\frac{d\text{F}}{dy_1}\right) dy_1 + \left(\frac{d\text{F}}{dz_1}\right) dz_1 + \ldots + \left(\frac{d\text{F}}{dz_n}\right) dz_n + \left(\frac{d\text{F}}{dt}\right) dt; \quad (96)$$

so that this cannot take the form of (95) when $\delta x_1, \delta y_1, \delta z_1, \ldots$ are replaced by $dx_1, dy_1, dz_1, \ldots$ unless $\left(\frac{d\text{F}}{dt}\right) = 0$, that is unless F does not contain t explicitly; that is, no equation of condition can contain t explicitly.

108.] Presuming, then, the logical legitimacy of substituting dx, dy, dz for δx, δy and δz respectively, and substituting accordingly in (43), Art. 76, we have

$$\Sigma . m \left\{ \frac{d^2 x}{dt^2} dx + \frac{d^2 y}{dt^2} dy + \frac{d^2 z}{dt^2} dz \right\} = \Sigma . m(\text{X} dx + \text{Y} dy + \text{Z} dz). \quad (97)$$

If v is the velocity of m at the time t, this may be expressed in the form

$$\tfrac{1}{2} d . \Sigma . m v^2 = \Sigma . m (\text{X} dx + \text{Y} dy + \text{Z} dz). \quad (98)$$

Now supposing $\Sigma . m (\text{X} dx + \text{Y} dy + \text{Z} dz)$ to be an exact differential, and thus to admit of integration, the conditions for which will be considered in a following Article, we have by integration

$$\frac{\Sigma . m v^2 - \Sigma . m v_0^2}{2} = \Sigma . m \int_0^1 (\text{X} dx + \text{Y} dy + \text{Z} dz); \quad (99)$$

the limits of integration of the two members corresponding to each other and to the times t and t_0 respectively. Hence, denoting the right-hand member by dW, as in (89), we have

$$\frac{\Sigma . m v^2 - \Sigma . m v_0^2}{2} = \text{W} - \text{W}_0, \quad (100)$$

which may be stated as the following theorem.

In the motion of a system of particles under the action of finite forces, the change of kinetic energy in passing from any one position to another is equal to the corresponding work done by the forces.

Or in other words,

Work done in a system is shewn by, and is equal to, the change of kinetic energy.

This theorem is called the principle of vis viva, or the principle of kinetic energy; and is one of the most important, as it is one of the most fruitful, in the whole range of Physical Science.

Hence, if forces act and impress work on a material system, the work is shewn and is measured by the increase in kinetic energy: whereas if kinetic energy is withdrawn from a material system, it is transferred to some force or other agent in the form of work. Kinetic energy is, it will be observed, a quantity independent of direction; that is, in the language of Quaternions, it is a scalar quantity.

109.] Now this investigation leads to very important results in the theory of energy and its conservation. Energy may exist in two forms: either energy developed into motion, and called kinetic energy; or energy capable of producing motion, and so called potential energy; the measure of potential energy being the quantity of kinetic energy which it is capable of producing. Thus, when force has acted on a material system, and work is done, it will generally be the case that that work is not the whole work which the force is capable of producing, and that the force is not exhausted in the kinetic energy which has been produced: there may be a further capacity and a further store for the production of which a further motion of the point of application is required: this is potential energy: and the whole energy of the force or of the system is the sum of the kinetic and the potential energies.

This theorem also follows from equation (99) of the preceding Article: both members of that equation are definite integrals having corresponding limits, those in the right-hand member being the values of the coordinates of the particles of the system in its initial and final states, say at the times t_0 and t; and as these are the only values of the coordinates which the expression contains, it follows that the increase or decrease of kinetic energy is due to the change of configuration or of position of the system, and not to the path, or to the time, which it has taken in passing from the one position to the other. Let us call the extreme positions of the system A and B, and let P be any intermediate position: then the whole kinetic energy is that which is developed in the motion of the system from A to B, and this is evidently made up of the kinetic energies developed in the motion from A to P, and in that from P to B. Consequently, if we consider the state of the system at P, the whole energy is partly developed into motion and partly is undeveloped; that is, is partly kinetic and partly potential: but the sum of these two is the whole energy; and

as the positions A, P and B are general, and subject to no special restrictions, we have the following general theorem;

The sum of the kinetic and potential energies in a system is constant.

The principle contained in this theorem is called the conservation of energy.

110.] The following is a simple illustration of it.

Let a weight mg be placed at any height, say h, above the surface of the earth, h being such that g is supposed not to vary throughout it; then the whole kinetic energy which it would acquire in falling to the earth is mgh: but suppose it to fall through a vertical distance z, which is less than h, then the kinetic energy in falling through z is mgz; and the energy which would be kinetically developed in falling through the remaining distance $h-z$ is $mg(h-z)$; but this is undeveloped, and is consequently potential energy: so that the kinetic energy + the potential energy

$$= mgz + mg(h-z)$$
$$= mgh,$$

which is the whole energy due to h, and is constant.

As the potential energy evidently depends on the position and configuration of a system, it is sometimes called the energy of position: and there is no change in this potential energy unless there is a change of position or of configuration. If a system has moved, and returned to its former position, its kinetic energy is the same as before, because the right-hand member of (99) vanishes, and the potential energy is also the same as before: whatever work has been done by the forces in the motion has also been recovered to the same amount. Hence if a system moves cyclically, the potential energy returns periodically to the same amount.

111.] The following are forms of vis viva and of kinetic energy in terms of various systems of reference.

For convenience, let 2T be the symbol for vis viva, so that T represents kinetic energy: then in all cases

$$2\text{T} = \Sigma . mv^2 = \Sigma . m\frac{ds^2}{dt^2}.$$

Hence if all particles of the system move in a plane, and are referred to rectangular axes, say of (x, y),

$$2\text{T} = \Sigma . m\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2}\right\}; \qquad (101)$$

and if they are referred to polar coordinates,

$$2\text{T} = \Sigma . m \left\{ \frac{dr^2}{dt^2} + r^2 \frac{d\theta^2}{dt^2} \right\}; \tag{102}$$

and these expressions are also true, when particles not in the plane of (x, y) move only in planes parallel to that of (x, y).

If the particles are referred to trilinear coordinates, and ∇ is the area of the triangle of reference,

$$2\text{T} = -\frac{abc}{4\nabla^2} \Sigma . m \left\{ a \frac{d\beta}{dt} \frac{d\gamma}{dt} + b \frac{d\gamma}{dt} \frac{d\alpha}{dt} + c \frac{d\alpha}{dt} \frac{d\beta}{dt} \right\}$$

$$= \frac{abc}{4\nabla^2} \Sigma . m \left\{ a \cos \text{A} \frac{d\alpha^2}{dt^2} + b \cos \text{B} \frac{d\beta^2}{dt^2} + c \cos \text{C} \frac{d\gamma^2}{dt^2} \right\}. \tag{103}$$

If the motion is in space, we have

for rectangular coordinates, $2\text{T} = \Sigma . m \left\{ \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2} \right\}$; (104)

for cylindrical coordinates, $2\text{T} = \Sigma . m \left\{ \frac{dr^2}{dt^2} + r^2 \frac{d\theta^2}{dt^2} + \frac{dz^2}{dt^2} \right\}$; (105)

for spherical coordinates,

$$2\text{T} = \Sigma . m \left\{ \frac{dr^2}{dt^2} + r^2 \frac{d\theta^2}{dt^2} + r^2 (\sin \theta)^2 \frac{d\phi^2}{dt^2} \right\}. \tag{106}$$

If a rigid body moves about a fixed axis with an angular velocity $\frac{d\theta}{dt}$, and r is the distance of the particle m from the axis,

$$2\text{T} = \Sigma . m r^2 \frac{d\theta^2}{dt^2} = \frac{d\theta^2}{dt^2} \Sigma . m r^2, \tag{107}$$

placing $\frac{d\theta}{dt}$ outside the sign of summation, since it is the same for every particle of the system; also taking the general values of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ due to rotation, which are given in Art. 53,

$$2\text{T} = \Sigma . m \{ (z\,\omega_y - y\,\omega_z)^2 + (x\,\omega_z - z\,\omega_x)^2 + (y\,\omega_x - x\,\omega_y)^2 \}$$

$$= \omega_x^2 \Sigma . m (y^2 + z^2) + \omega_y^2 \Sigma . m (z^2 + x^2) + \omega_z^2 \Sigma . m (x^2 + y^2)$$

$$- 2\,\omega_y \omega_z \Sigma . m y z - 2\,\omega_z \omega_x \Sigma . m z x - 2\,\omega_x \omega_y \Sigma . m x y. \tag{108}$$

112.] If the places of the particles are referred to a moving origin, the kinetic energy of the system at the time t may be expressed as follows.

Let the axes of the moving origin be parallel to the axes fixed in space; let (x_0, y_0, z_0) be the place of the moving origin, and (x, y, z) the place of m in reference to the fixed axes: let u, v, w be the component velocities of the moving origin, and (ξ, η, ζ) the place of m in reference to it: so that

$$x = x_0 + \xi, \qquad y = y_0 + \eta, \qquad z = z_0 + \zeta;$$
$$\frac{dx}{dt} = u + \frac{d\xi}{dt}, \quad \frac{dy}{dt} = v + \frac{d\eta}{dt}, \quad \frac{dz}{dt} = w + \frac{d\zeta}{dt};$$
$$\therefore\ 2\text{T} = \Sigma.m\left\{\left(u + \frac{d\xi}{dt}\right)^2 + \left(v + \frac{d\eta}{dt}\right)^2 + \left(w + \frac{d\zeta}{dt}\right)^2\right\}$$
$$= \text{M}\,\text{V}^2 + 2\Sigma.m\left(u\frac{d\xi}{dt} + v\frac{d\eta}{dt} + w\frac{d\zeta}{dt}\right) + \Sigma.mv^2; \quad (109)$$

where M is the mass of the whole system, V is the velocity of the moving origin, and $\Sigma.m\text{V}^2$ is the kinetic energy of the whole mass collected at that origin. This is the complete value of the kinetic energy.

It may also be expressed in the follow form: If $(\bar{x}, \bar{y}, \bar{z})$ is the place of the mass-centre relative to the fixed axes, and (x', y', z') is the place of m relative to the mass-centre,

$$x = \bar{x} + x', \qquad y = \bar{y} + y', \qquad z = \bar{z} + z';$$
$$\therefore\ \xi = \bar{x} + x' - x_0, \quad \eta = \bar{y} + y' - y_0, \quad \zeta = \bar{z} + z' - z_0:$$

and bearing in mind that $\Sigma.mx' = \Sigma.my' = \Sigma.mz' = 0$, the preceding value of 2T becomes

$$2\text{T} = \text{M}\text{V}^2 + 2\text{M}\left(u\frac{d\bar{x}}{dt} + v\frac{d\bar{y}}{dt} + w\frac{d\bar{z}}{dt} - \text{V}^2\right) + \Sigma.mv^2.$$

113.] These are the general values for kinetic energy; but take more simple forms under particular circumstances. Let us consider them.

When the mass-centre is the moving origin

$$\Sigma.m\xi = \Sigma.m\eta = \Sigma.m\zeta,$$

and
$$2\text{T} = \text{M}\bar{\text{V}}^2 + \Sigma.mv'^2:$$

that is, the kinetic energy of the whole system consists of that of a particle placed at the mass-centre whose mass is equal to that of the entire system, together with that of the system relative to the mass-centre.

If the motion relative to the mass-centre is wholly due to rotation about an axis passing through it, then if ω is the angular velocity, and r is the perpendicular distance from m on the rotation axis,
$$2\text{T} = \text{M}\bar{\text{V}}^2 + \omega^2\Sigma.mr^2.$$

Again, if the new origin lies on the instantaneous axis and is at rest, we have the simple form

$$2\text{T} = \omega^2\,\Sigma.m\,r^2,$$

where ω is the angular velocity about it. This axis is thus found: let (ξ, η, ζ) be a point in it in reference to the mass-centre, and

let u, v, w be the component velocities of the mass-centre, and ω_ξ, ω_η, ω_ζ the axial components of ω which is the angular velocity about the axis passing through the mass-centre, then if (ξ, η, ζ) is a point at rest

$$u - \zeta\omega_\eta + \eta\omega_\zeta = 0, \quad v - \xi\omega_\zeta + \zeta\omega_\xi = 0, \quad w - \eta\omega_\xi + \xi\omega_\eta = 0:$$

$$\therefore \frac{\xi\omega^2 - (v\omega_\zeta - w\omega_\eta)}{\omega_\xi} = \frac{\eta\omega^2 - (w\omega_\xi - u\omega_\zeta)}{\omega_\eta} = \frac{\zeta\omega^2 - (u\omega_\eta - v\omega_\xi)}{\omega_\zeta}, \quad (110)$$

which are the equations to the instantaneous axis. This axis is evidently at the instant the central axis of the system.

Also generally the intermediate term vanishes when

$$u\frac{d\bar{x}}{dt} + v\frac{d\bar{y}}{dt} + w\frac{d\bar{z}}{dt} - \text{v}^2 = 0.$$

As $\text{v}^2 = u^2 + v^2 + w^2$, this may be put into the form

$$u\left(\frac{d\bar{x}}{dt} - u\right) + v\left(\frac{d\bar{y}}{dt} - v\right) + w\left(\frac{d\bar{z}}{dt} - w\right) = 0;$$

and shews that the lines of the absolute velocity of the moving origin is at right angles to the line of the velocity of the mass-centre relative to the moving origin. It also admits of the follow interpretation.

If we take the mass-centre to be the moving origin, ω to be the angular velocity of the system about an axis through it, ω_x, ω_y, ω_z to be the components of ω, and (x, y, z) to be the place of the point for which the intermediate term disappears, then in this relation u, v, w are to be replaced by

$$z\omega_y - y\omega_z, \quad x\omega_z - z\omega_x, \quad y\omega_x - x\omega_y$$

respectively, and $\frac{d\bar{x}}{dt}, \frac{d\bar{y}}{dt}, \frac{d\bar{z}}{dt}$ by u, v, w, and then the preceding expression becomes

$$(z\omega_y - y\omega_z)^2 + (x\omega_z - z\omega_x)^2 + (y\omega_x - x\omega_y)^2$$
$$- u(z\omega_y - y\omega_z) - v(x\omega_z - z\omega_x) - w(y\omega_x - x\omega_y) = 0. \quad (111)$$

This is the equation to a circular right cylinder, which passes through the origin, and the axis of which is parallel to the rotation-axis through that point, which is the mass-centre of the system. Hence the rotation-axis is a generating line of the cylinder. Also the equation shows that the point whose coordinates are $\frac{v\omega_z - w\omega_y}{2\omega^2}$, $\frac{w\omega_x - u\omega_z}{2\omega^2}$, $\frac{u\omega_y - v\omega_x}{2\omega^2}$, is the centre of the circular section made by a plane passing through the origin: and consequently that the instantaneous axis (110) is also the generating line of the cylinder which is opposite to the rotation-

axis. For all generators of this circular cylinder the intermediate term of (109) vanishes, and the kinetic energy consists of only the first and last terms in (109).

114.] If the motion of all the particles takes place in one plane, say in that of (x, y), or every particle moves in one of a series of parallel planes, to which the plane of (x, y) is parallel, the kinetic energy will be due to motion of translation of a point in the plane and of motion of rotation about an axis perpendicular to the plane, so that using notation similar to that of the preceding Articles

$$x = x_0 + \xi, \qquad y = y_0 + \eta,$$

$$\frac{dx}{dt} = u - \omega\eta, \qquad \frac{dy}{dt} = v + \omega\xi;$$

$$\therefore \; 2\text{T} = \text{M}\text{V}^2 + 2\omega\Sigma.m(v\xi - u\eta) + \omega^2\Sigma.m(\xi^2+\eta^2),$$

which is the complete value of the kinetic energy.

This expression will take a more simple form in the following cases, viz. when

(1) The moving origin is the instantaneous centre; and then

$$2\text{T} = \omega^2\Sigma.m(\xi^2+\eta^2).$$

(2) The moving origin is the mass-centre; and then

$$2\text{T} = \text{M}\bar{\text{V}}^2 + \omega^2\Sigma.mr^2.$$

(3) The moving origin is on the circle described upon the line joining the mass-centre and the instantaneous centre as a diameter.

The last condition may thus be shewn shortly: the intermediate term will vanish when

$$v\bar{\xi} - u\bar{\eta} = 0;$$

but if (x_0, y_0) is the instantaneous centre,

$$v + \omega x_0 = 0, \qquad u - \omega y_0 = 0;$$

$$\therefore \; x_0\bar{\xi} + y_0\bar{\eta} = 0;$$

which shews that the straight lines drawn from the moving origin to the mass-centre and to the instantaneous centre respectively are at right angles to each other; and consequently the point of intersection lies on a circle, at the extremities of a diameter of which are the mass-centre, and the instantaneous centre. And for all points on this circle

$$2\text{T} = \text{M}\text{V}^2 + \omega^2\Sigma.mr^2.$$

115.] The quantity $\Sigma.m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz)$ which is found in the right-hand member of (99) requires further consideration.

If for every particle of the system $X = Y = Z = 0$, so that no force acts on the system and no work is communicated to it, then $\Sigma . m v^2 = \Sigma . m v_0^2$, and the vis viva of the system remains unaltered; the vis viva of particular particles may vary, but the sum of all remains the same: this theorem is called the principle of conservation of vis viva.

If one point of a system is fixed, and there are reactions or pressures at that point, these will not appear in the equation of vis viva, because the point of application does not admit of displacement, and consequently no work is done by the reaction.

If a system is subject to internal stresses, such as those which arise from the difference between the impressed and the expressed momentum-increments on any particle m, these will enter into the expression $\Sigma . m (X\,dx + Y\,dy + Z\,dz)$ in equal and opposite pairs, and as each pair acts at the same point and along the same line, the displacement of the point of application is the same for each, and consequently the work due to them will disappear in the summation.

There may, however, be pairs of equal and opposite stresses, whether tensions or pressures, which do not disappear in the quantity under consideration, because the displacements of the points of action of these stresses may not be equal. Thus in the solar system the vis viva is not always the same, even if we omit the action of forces external to the system, because a change arises from the internal forces producing an alteration of configuration, although they enter in equal and opposite pairs. If the configuration of the system is invariable, the internal stresses cancel each other; but if there are elastic connections or springs, and if expansions or contractions arise from such stresses, these will not disappear in the expression under consideration, and consequently a change of vis viva will be due to them. This observation will receive fuller development hereafter.

If the system or a particle of the system moves on a fixed smooth surface, the reaction of the surface does not appear in the expression for work, because the displacement is perpendicular to the line of action of the force, and consequently no work is done, and no change of vis viva is caused by it.

If, however, the surface is rough, and sliding takes place, then friction, which acts in the tangent plane to the surface at the point of contact, is brought into action, and a loss of work takes

place, and consequently a loss of vis viva. If, however, only rolling takes place, no loss of vis viva ensues.

116.] Whenever a material system passes through a position or configuration in which the forces would be in equilibrium, then at that instant, by the principle of Virtual Velocities,

$$\Sigma . m\,(\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz) = 0\,;$$

and therefore from (98), $d.\Sigma.mv^2 = 0$; so that there is no change of vis viva, and the vis viva consequently is either a maximum or a minimum, or is constant; the last being the case when no forces act, and no work is done on the system. Hence the vis viva of the system is a maximum or a minimum when the acting forces constitute an equilibrating system, and according as the equilibrium is stable or unstable.

Now in Articles 102, 103 and 107, Vol. III, it is proved that the equilibrium of a system is stable or unstable according as the radial moment, which is denoted by H, is a minimum or a maximum; but

$$\begin{aligned} d\mathrm{H} &= \Sigma . \mathrm{P}\,(\cos\alpha\,dx + \cos\beta\,dy + \cos\gamma\,dz), \\ &= \Sigma . m\,(\mathrm{X}\,dx + \mathrm{Y}\,dy + \mathrm{Z}\,dz) \\ &= \tfrac{1}{2}\,d.\Sigma.mv^2\,; \end{aligned}$$

so that the radial moment becomes in this notation kinetic energy; and consequently the vis viva of the system is a maximum or a minimum, according as the equilibrium state through which the system of forces passes is a state of stable or unstable equilibrium.

Let us illustrate this theorem by a system subject to the action of gravity only. Taking the horizontal plane to be that of (x, y), $\mathrm{X} = \mathrm{Y} = 0$; and supposing the mass to descend, $\mathrm{Z} = g$; therefore, if M is the mass of the system,

$$\begin{aligned} \frac{\Sigma.mv^2 - \Sigma.mv_0^{\,2}}{2} &= \Sigma.mg\,(z - z_0), \\ &= \mathrm{M}g\,(\bar{z} - \bar{z}_0), \end{aligned}$$

that is, the increase of vis viva depends only on the vertical distance through which the centre of gravity of the whole mass has descended; thus the vis viva is a minimum when the centre of gravity is in its highest position; also the vis viva is the same whenever the centre of gravity passes through a given horizontal plane.

Hence also it appears that in a system of heavy particles, such

as are many machines which are subject to gravitation, the vis viva of the system is the greatest when the centre of gravity has its lowest position; and the vis viva is the least when the centre of gravity has the highest position compatible with the constraints of the machine.

117.] Also that (99) may be interpretable and intelligible it is necessary that $\Sigma . m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz)$ should be an integrable expression; it must therefore be a function of x, y, z only, and must not explicitly contain any other variable. Thus the forces cannot be functions of t or of the velocity, such as is the case with friction or in motion through a resisting medium. Moreover $\Sigma . m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz)$ must be an exact differential. As this consists of a series of terms, each of which corresponds to a particular m, the term which belongs to each one, say

$$m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz),$$

must be an exact differential. Let us suppose this to be $d\text{U}$, so that

$$d\text{U} = m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz); \tag{112}$$

and consequently,

$$m\text{X} = \left(\frac{d\text{U}}{dx}\right), \qquad m\text{Y} = \left(\frac{d\text{U}}{dy}\right), \qquad m\text{Z} = \left(\frac{d\text{U}}{dz}\right); \tag{113}$$

hence also

$$\left(\frac{d\text{Y}}{dz}\right) = \left(\frac{d\text{Z}}{dy}\right), \quad \left(\frac{d\text{Z}}{dx}\right) = \left(\frac{d\text{X}}{dz}\right), \quad \left(\frac{d\text{X}}{dy}\right) = \left(\frac{d\text{Y}}{dx}\right). \tag{114}$$

The function U which satisfies the conditions (112) and (113) is called a force-function (Kraftefunction); and its differential is evidently the amount of work given to or received from the system at the point (x, y, z) through a distance ds, whose projections on the coordinate axes are dx, dy, dz. Thus the relation of U to W is as follows; viz.

$$d\text{W} = \Sigma . d\text{U}. \tag{115}$$

The name of force-function was given to U by Sir W. R. Hamilton and Jacobi independently of each other.

$-$W has been called by Rankine "Potential Energy," and by Clausius the "Ergal." The work done by a force has sometimes been called "the labouring force."

118.] In the special case of a particle m being attracted or repelled by other particles $m_1, m_2, \ldots$, when the force varies directly as the masses of the particle and inversely as the square of the distance between them, the work done by the forces on m in

dt is a complete differential, and the force-function is in this case called a potential or potential-function, the name being generally applied to this particular law and to none other, and denoted by V.

Thus if m the attracted particle is at (x, y, z), and m_1 the attracting particle is at (x_1, y_1, z_1), and r_1 is the distance between them, then, if P_1 is the force acting on m, $P_1 = \frac{m m_1}{r_1^2}$.

And if X_1, Y_1, Z_1 are the axial components of P_1,

$$X_1 = \frac{m m_1 (x_1 - x)}{r_1^3} = m \frac{d}{dx} \frac{m_1}{r_1};$$

$$Y_1 = \frac{m m_1 (y_1 - y)}{r_1^3} = m \frac{d}{dy} \frac{m_1}{r_1};$$

$$Z_1 = \frac{m m_1 (z_1 - z)}{r_1^3} = m \frac{d}{dz} \frac{m_1}{r_1};$$

and if other particles $m_2, m_3, \ldots$ act on m and give rise to similar values, then, if X, Y, Z are the whole axial components,

$$X = m \Sigma . \frac{d}{dx} \frac{m}{r}; \quad Y = m \Sigma . \frac{d}{dy} \frac{m}{r}; \quad Z = m \Sigma . \frac{d}{dz} \frac{m}{r}.$$

Let $$\frac{m_1}{r_1} + \frac{m_2}{r_2} + \ldots = \Sigma . \frac{m}{r} = V;$$

then $$X = m\left(\frac{dV}{dx}\right), \quad Y = m\left(\frac{dV}{dy}\right), \quad Z = m\left(\frac{dV}{dz}\right),$$

and $$X\,dx + Y\,dy + Z\,dz = m\,DV,$$

so that V is the force-function, and is called the potential of the system of particles which act on m.

If m_1 is the particle of a body lying outside of m, so that at no point does r vanish, and if ρ is the density of the attracting particle, $$V = \iiint \frac{\rho\,dx\,dy\,dz}{r}.$$

Several properties of the potential have already been demonstrated in Vol. III, Chapter VI, Section 2.

The potential is evidently the work done on an unit particle by the particles $m_1, m_2, \ldots$, moving from an infinite distance to the distances $r_1, r_2, r_3, \ldots$, respectively from the unit particle.

119.] Two particular cases wherein the conditions of a force-function are satisfied deserve special attention, (1) where the force is a central force and varies as some function of the distance of the particle m from the centre; (2) when two particles m and

m' attract or repel each other with a force which is a function of the distance between the particles.

(1) In the former case let (a, b, c) be the place of the centre of force; (x, y, z) the place of m, r the distance between them, and $f'(r)$ the force, then

$$\text{X} = \frac{x-a}{r} f'(r), \quad \text{Y} = \frac{y-b}{r} f'(r), \quad \text{Z} = \frac{z-c}{r} f'(r);$$

$$\therefore \quad m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz) = m f'(r)\,dr; \tag{116}$$

which is an exact differential; so that $mf(r)$ is the force-function, and the change of vis viva due to it is $2mf(r)$, the integrals being taken for assigned limits.

If $f'(r)$ is positive, so that the force is repulsive, and dr is also positive, then the vis viva is so far increased; but if dr is negative so that the particle approaches the centre of force, the vis viva is diminished. Similarly if the force is attractive the vis viva is increased or diminished according as the attracted particle approaches to or recedes from the centre of force.

(2) Let (x, y, z) and (x', y', z') be the places of m and m' respectively at the time t; let r be the distance between them, and let $f'(r)$ be the force of an unit particle acting upon them from one to the other; then

$$\left.\begin{aligned} \text{X} &= m'\frac{x-x'}{r} f'(r), & \text{Y} &= m'\frac{y-y'}{r} f'(r), & \text{Z} &= m'\frac{z-z'}{r} f'(r); \\ \text{X}' &= -m\frac{x-x'}{r} f'(r), & \text{Y}' &= -m\frac{y-y'}{r} f'(r), & \text{Z}' &= -m\frac{z-z'}{r} f'(r). \end{aligned}\right\}$$

Also $$r^2 = (x-x')^2 + (y-y')^2 + (z-z')^2;$$

$$\therefore \quad r\,dr = (x-x')(dx-dx') + (y-y')(dy-dy') + (z-z')(dz-dz');$$

therefore thus far

$$m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz) + m'(\text{X}'\,dx' + \text{Y}'\,dy' + \text{Z}'\,dz')$$

$$= \frac{mm'}{r}\{(x-x')(dx-dx') + (y-y')(dy-dy') + (z-z')(dz-dz')\} f'(r)$$

$$= mm'f'(r)\,dr, \tag{117}$$

which is an exact differential. Consequently the corresponding force-function is $mm'f(r)$.

Here, as in the former case, if the force is repulsive, the vis viva of the particles is increased or diminished according as the distance between them is increased or diminished; and if the force is attractive, the vis viva is increased or diminished according as the distance is diminished or increased.

Thus, if a system of particles, gaseous or solid, receives an increase of heat, whereby repulsive forces are brought into action, the particles are two and two repelled further from each other, and there is an increase of vis viva. If, on the other hand, heat is withdrawn, the particles are drawn nearer together, and a diminution takes place. Hence also if in a system of moving particles an explosion takes place, so that some of the particles are removed farther from each other, an increase of vis viva takes place.

A system of forces which is such that $\Sigma . m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz)$ is an exact differential and equation (99) is applicable, has been called by Sir William Thomson a conservative system, because no vis viva is lost in the passage of the system from any position to the same position again; or, in other words, whatever work is converted into kinetic energy as the system moves from the position or configuration of A to that of B, the kinetic energy may be recovered by the forces in the form of work as the system passes in the reverse direction from B to A.

120.] The following are simple illustrations of the principle of vis viva as exhibited in equation (100).

Ex. 1. A mass m in the form of a cylinder, with a circular base of radius b and of altitude $2a$, stands on a rough horizontal plane which moves with a velocity v; the plane is suddenly stopped by a fixed obstacle, and the cylinder is just overturned: the velocity of the plane may thus be determined. The vis viva of the cylinder, which is mv^2, becomes converted into work developed in the ascent of the centre of gravity of the cylinder through the vertical distance $(a^2+b^2)^{\frac{1}{2}} - a$; consequently we have

$$\frac{mv^2}{2} = mg\,\{(a^2+b^2)^{\frac{1}{2}} - a\};$$

$$\therefore\; v^2 = 2g\,\{(a^2+b^2)^{\frac{1}{2}} - a\}.$$

A seismometer has been constructed on this principle by Mr. Mallet. Let a series of heavy cylinders be made of a given height, but with bases the diameters of which gradually increase; let these be placed on a rough horizontal plane, the distance between any two cylinders being not less than the height of each. Let this plane be attached firmly to the earth. Now suppose a shock of earthquake to advance in a certain horizontal direction with a velocity v; it gives a jerk to the plane with that velocity, and all the cylinders are overturned up to

those in which $2g\{(a^2+b^2)^{\frac{1}{2}}-a\}$ is just equal to or greater than v^2: so that the velocity of the advancing disturbance is determined by means of the dimensions of the smallest cylinder that is not overturned; and the line in which the fallen cylinders lie determines the line of direction of the earthquake.

Ex. 2. If a heavy bar of length $2a$ revolves in a vertical plane about one end which is fixed, and is initially inclined at an angle α to the vertical through the fixed end, then if M is the whole mass $\Sigma . mr^2 = \text{M}\frac{4a^2}{3}$, and if $\frac{d\theta}{dt}$ is the angular velocity when the bar is inclined at an angle θ to the same vertical line, then $\text{M}\frac{4a^2}{3}\frac{d\theta^2}{dt^2} = 2\text{M}ga(\cos\theta - \cos\alpha)$. Consequently if the bar moves from the unstable vertical position into the stable position, and ω is the angular velocity in the last position $\omega^2 = \frac{3g}{a}$.

Ex. 3. A cylinder of altitude a and radius b rotating about its axis with an angular velocity ω, has enough energy to raise a weight equal to its own weight through a vertical distance $\frac{b^2\omega^2}{4g}$.

Ex. 4. A mass m of fluid describes a circular course of radius a with velocity u: another equal mass describes a concentric circular course of radius b with velocity v: the radius of one course increases and that of the other decreases, until each occupies the place of the other. Determine the work required for the change.

Let the velocity u become u', and v become v'; then the work required $= \frac{m}{2}(u'^2-u^2+v'^2-v^2)$: but by the principle of conservation of moments of momenta,

$$au = bu', \qquad bv = av';$$

therefore the work required $= \frac{m}{2}(b^2-a^2)\left(\frac{v^2}{a^2}-\frac{u^2}{b^2}\right)$.

121.] The question however may arise, what takes place in respect of kinetic energy or of work, if the acting forces, one or more, are not of a conservative character? Suppose, for instance, that there is sliding on a rough surface, or that one or more of the particles are rough and slide on each other, here a loss of kinetic energy ensues. What becomes of it? By the law of inertia, work or energy is indestructible; we can no more annihi-

late it than we can create it; and as it has not shown itself in the form of either kinetic energy which is energy of motion, or of potential energy which is energy of position, it must take another form, it has in fact become heat, which is another form of energy. Similarly mechanical work may be transmuted into light or sound, or into electrical energy as by a dynamo-machine. But in all these cases a common measure of work is required; and so far as it has been determined we have the means of comparing all these various forms of energy with mechanical work, and the principle of conservation of energy holds true. To carry this subject further leads us into the sciences of Thermodynamics, Electricity and Magnetism, and consequently beyond the range of our present work.

122.] If a system is in motion under the action of conservative forces, the space-integral of the whole momentum is a minimum as the system passes from one position to another. This space-integral of the momentum has been called by Lagrange the action of the system, and the theorem has been consequently called the principle of least action.

Let A be the action of the system, and let the definite integral be expressed as in the Calculus of Variations in Vol. II, so that

$$\text{A} = \Sigma . m \int_0^1 v\, ds. \tag{118}$$

Then taking the variation of this quantity

$$\delta \text{A} = \Sigma . m \int_0^1 (ds\, \delta v + v \delta . ds).$$

Now as the system of forces is conservative, by (98), Art. 108, for every particle we have,

$$v \delta v = \text{X}\, \delta x + \text{Y}\, \delta y + \text{Z}\, \delta z,$$

and also

$$\delta . ds = \frac{dx}{ds} \delta . dx + \frac{dy}{ds} \delta . dy + \frac{dz}{ds} \delta . dz;$$

so that

$$\delta \text{A} = \Sigma . m \int_0^1 \Big\{ \frac{ds}{v} (\text{X}\delta x + \text{Y}\delta y + \text{Z}\delta z) + \frac{v}{ds} (dx d. \delta x + dy d. \delta y + dz d. \delta z) \Big\}$$

$$= \Sigma . m \Big[v \Big(\frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y + \frac{dz}{ds} \delta z \Big) \Big]_0^1$$

$$+ \Sigma . m \int_0^1 \Big\{ \Big(\frac{\text{X} ds}{v} - d . \frac{v dx}{ds} \Big) \delta x + \Big(\frac{\text{Y} ds}{v} - d . \frac{v dy}{ds} \Big) \delta y + \Big(\frac{\text{Z} ds}{v} - d . \frac{v dz}{ds} \Big) \delta z \Big\};$$

since the limits of the integral are given and are fixed, the first

part vanishes of itself. And the second also vanishes identically; because

$$\Sigma . m\left(\frac{\mathrm{X}\,ds}{v} - d.\frac{v\,dx}{ds}\right) = \Sigma . m\left(\mathrm{X}\,dt - d.\frac{dx}{dt}\right)$$
$$= \Sigma . m\left(\mathrm{X} - \frac{d^2x}{dt^2}\right)dt = 0.$$

Similarly, each of the other terms vanishes; therefore $\delta\mathrm{A} = 0$; and the action is either a minimum, or is constant, for evidently it cannot be a maximum.

Since $\Sigma . m\int_0^1 v\,ds = \Sigma . \int_0^1 m\,v^2\,dt$, and this quantity is a minimum, the principle may be called, "the principle of least vis viva": and we may then say, that the vis viva acquired by the system during the time of its passage from one position to another is less than it would be under any other law of connexion between momenta impressed and momenta expressed. It is necessary that the first and last positions of the system should be given, because we have assumed the variations of the coordinates which correspond to them to vanish.

The definition of action leads to the following value for it

$$\mathrm{A} = \Sigma . m\int\left(\frac{dx}{dt}dx + \frac{dy}{dt}dy + \frac{dz}{dt}dz\right).$$

This principle of least action is useless as a method of solution of dynamical problems; because, assuming it to be true from a priori or other reasoning, it gives only the equations of motion (37), Art. 73, which are derived more satisfactorily from D'Alembert's principle; and if the variations of x, y, and z had been taken in the most general forms, which are given in Art. 76, being due to not only a motion of translation, but also to that of rotation, we should from the principle infer the equations (38) as well as (37) of Art. 73. It is merely then a formula which includes them. The other principles, however, which we have proved in the preceding Articles, are more useful; under certain circumstances, they give us actual integrals of the equations of motion: thus, for a conservative system of forces, the equation of vis viva is a first integral, and that from which the time may be found by a single integration. So, if no forces act on the system, or only internal forces which have equal and opposite ones, the principles of conservation of the mass-centre and of the conservation of moments of momenta give integrals of the equations of motion.

123.] Another problem in connection with the present subject which requires consideration is that of the change in the vis viva of a system of material particles which is caused by the action of internal instantaneous forces; so that this question includes those of collisions amongst the particles of a mass and of explosions, which are of importance in the kinetic theory of gases, for therein the particles are assumed to be in a state of continual motion, and from time to time to collide with each other.

The equation of motion on which this enquiry is founded is (44), Art. 76, viz.

$$\Sigma . m\{(u'-u)\delta x+(v'-v)\delta y+(w'-w)\delta z\} = \Sigma .(\text{X}\,\delta x+\text{Y}\delta y+\text{Z}\delta z), \quad (119)$$

where u, v, w are the axial components of the velocity of m before the action of the impulsive forces, and u', v', w' are the like quantities after the action, so that $m(u'-u)$, $m(v'-v)$, $m(w'-w)$ are the axial components of the increase in the momentum of m due to the forces.

The general theorem is that stated by M. Sturm to the French Academy of Sciences, December 6, 1841, and printed in the Comptes Rendus, Tome XIII, p. 1046.

If a system of material particles in motion is subject to certain connexions or restraints (*liaisons*), and these restraints are suddenly and abruptly changed, so that some of the particles are compelled to take new courses, vis viva is lost by the system, and the loss is equal to that due to the velocities of the particles lost in the passage from one state to the other. That is, if $\Sigma . m\text{v}^2$ and $\Sigma . m\text{v}'^2$ are respectively the vis viva of the system before and after the change of restraint,

$$\Sigma . m\text{v}^2 - \Sigma . m\text{v}'^2 = \Sigma . m\{(u-u')^2+(v-v')^2+(w-w')^2\}.$$

This is sometimes expressed in the form, the vis viva lost is the relative vis viva.

The restraints are supposed to depend on the position of the particles, and to be independent of t; and the changes in them are supposed to constitute a system of forces in equilibrium, so that the right-hand member of (119) is equal to zero; hence the fundamental equation is

$$\Sigma . m\{(u'-u)\delta x+(v'-v)\delta y+(w'-w)\delta z\} = 0.$$

Now let us take for the virtual displacements the actual displacements in the time dt, which follows the instant at which the forces act, viz. $u'dt$, $v'dt$, $w'dt$; then

$$\Sigma . m \{(u'-u)u' + (v'-v)v' + (w'-w)w'\} = 0;$$

$$\therefore \quad \Sigma . m \mathrm{v}^2 - \Sigma . m \mathrm{v}'^2 = \Sigma . m \{(u-u')^2 + (v-v')^2 + (w-w')^2\};$$

which proves the proposition.

As the right-hand member is a quantity essentially positive, $\Sigma . m \mathrm{v}^2$ is greater than $\Sigma . m \mathrm{v}'^2$, and consequently vis viva is lost by the change in the system of restraints.

Also suppose that there is a succession of similar changes in the restraints, as shewn in the following series,

$$\Sigma . m \mathrm{v}^2 - \Sigma . m \mathrm{v}_1^2 = \Sigma . m \{(u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2\},$$

$$\Sigma . m \mathrm{v}_1^2 - \Sigma . m \mathrm{v}_2^2 = \Sigma . m \{(u_1-u_2)^2 + (v_1-v_2)^2 + (w_1-w_2)^2\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Sigma . m \mathrm{v}_{n-1}^2 - \Sigma . m \mathrm{v}_n^2 = \Sigma . m \{(u_{n-1}-u_n)^2 + (v_{n-1}-v_n)^2 + (w_{n-1}-w_n)^2\};$$

therefore, by addition,

$$\Sigma . m \mathrm{v}^2 - \Sigma . m \mathrm{v}_n^2 = \Sigma . m \{(u-u_1)^2 + (v-v_1)^2 + (w-w_1)^2\},$$
$$+ \cdots\cdots\cdots\cdots$$
$$+ \Sigma . m \{(u_{n-1}-u_n)^2 + (v_{n-1}-v_n)^2 + (w_{n-1}-w_n)^2\};$$

so that the whole lost vis viva is the sum of those which are lost at each successive change of restraint.

If the changes in the restraints are due to the collision of the particles of the system with each other, and the particles are perfectly hard and inelastic, so that no compression takes place, then the conditions are satisfied under which the preceding theorems are found, and the loss of vis viva is that which is expressed in the preceding equations. This theorem was first stated by Carnot, in his "Essai sur les machines en général," Basle, 1797.

If the material system consists of particles imperfectly elastic, and these collide, the reactions between any two will be equal and act in opposite directions at the instant when the compression is a maximum; we may therefore take this as the instant when the right-hand member of (119) vanishes, and replace δx, δy, δz by the actual displacements* which occur in dt just after; that it is by $\frac{u'+eu}{1+e}dt$, $\frac{v'+ev}{1+e}dt$, $\frac{w'+ew}{1+e}dt$, where e is the constant of elasticity; whence we have

$$\Sigma . m \{(u'-u)(u'+eu) + (v'-v)(v'+ev) + (w'-w)(w'+ew)\} = 0;$$

$$\therefore \quad \Sigma . m \mathrm{v}^2 - \Sigma . m \mathrm{v}'^2 = \frac{1-e}{1+e} \Sigma . m \{(u'-u)^2 + (v'-v)^2 + (w'-w)^2\},$$

* See Art. 263, Vol. III.

which expresses the loss of vis viva. This theorem is due to Duhamel; see Journal de l'École Polytechnique, Cahier XXIV. 1835.

If $e = 0$, this equation gives the theorem of Carnot established in the preceding paragraph.

If $e = 1$, the right-hand member vanishes. In the case, then, of the collision of perfectly elastic particles, there is no loss of vis viva: that is, whatever loss has arisen in the compression an equal amount has been recovered in the return of the figure of the particles.

The right-hand member of (119) will also vanish when an explosion takes place in the moving system, because the reactions due to the explosion are equal and opposite: hence, taking the time dt just before the explosion, we may replace δx, δy and δz by $u\,dt$, $v\,dt$ and $w\,dt$, whence we have

$$\Sigma . m \{(u'-u)u+(v'-v)v+(w'-w)w\} = 0,$$

$$\Sigma . m\mathrm{v}'^2 - \Sigma . m\mathrm{v}^2 = \Sigma . m\{(u'-u)^2+(v'-v)^2+(w'-w)^2\},$$

and consequently vis viva is always gained by the explosion.

In all these cases the colliding particles or bodies have been supposed to be smooth, so that there is no loss of vis viva by friction; if, however, they are rough, and the impact in the case of collision is oblique, terms corresponding to friction will enter equation (119), and the results will require modification accordingly.

124.] Under what circumstances the vis viva of a system of particles may be expressed in terms of the impressed momentum-increments, is a question of great importance in the kinetic theory of gases, and generally in molecular physics. The following problem is due to Clausius, having been given by him in the Philosophical Magazine, August, 1870, and is now known as Clausius' Theorem.

Let there be a system of particles of which m is the type; and let (x, y, z) be its place at the time t; let it be subject to a system of forces. Then

$$\frac{d}{dt}\Sigma . m x^2 = 2\Sigma . m x\frac{dx}{dt},$$

$$\frac{d^2}{dt^2}\Sigma . m x^2 = 2\Sigma . m x\frac{d^2x}{dt^2} + 2\Sigma . m\frac{dx^2}{dt^2},$$

and as similar results are true for $\frac{d^2}{dt^2}\Sigma . m y^2$ and $\frac{d^2}{dt^2}\Sigma . m z^2$, we

have by addition,

$$\tfrac{1}{2}\frac{d^2}{dt^2}\Sigma.m(x^2+y^2+z^2) = \Sigma.m\left(x\frac{d^2x}{dt^2}+y\frac{d^2y}{dt^2}+z\frac{d^2z}{dt^2}\right) + \Sigma.m\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right);$$

but since X and $\frac{d^2x}{dt^2}$ act at the same point (x, y, z),

$$\Sigma.mx\frac{d^2x}{dt^2} = \Sigma.mx\text{X};$$

and, as a similar result is true for the other components,

$$\tfrac{1}{2}\frac{d^2}{dt^2}\Sigma.m(x^2+y^2+z^2) = \Sigma.m(x\text{X}+y\text{Y}+z\text{Z})+\Sigma.mv^2.$$

Now suppose the system of particles to be such that $\Sigma.mr^2$, where r is the distance of m from the origin, does not vary with the time, then the left-hand member vanishes, and we have

$$-\Sigma.m(x\text{X}+y\text{Y}+z\text{Z}) = \Sigma.mv^2.$$

The quantity $\Sigma.m(x\text{X}+y\text{Y}+z\text{Z})$ has been called by Clausius the virial function, or the virial of the forces, and consequently the last equation asserts that the vis viva of the system is equal to the virial of the system taken negatively. This is Clausius' theorem.

125.] But what is the nature of the system of particles when $\frac{d^2}{dt^2}\Sigma.mr^2 = 0$? This is the case:

(1) When the distance of each particle from the origin remains unchanged during the motion; as, for instance, when a rigid system pirouettes about the origin.

(2) If the system is a homogeneous fluid whose particles are in relative motion, the whole of it occupying the same space during the motion: because all the particles are equal, and if one moves from its place, that place is immediately occupied by another equal particle; or

(3) When the particles do not move far from their original places, but each in its motion describes periodically a small orbit about its original place, in a time which is infinitesimal in comparison of the time during which the vis viva and the virial function are estimated, this motion being such as is supposed to take place in the molecules of a body in a state of heat. This is an average result.

All these kinds of motion are called stationary; and they are

evidently such that the particles do not continually move farther and farther in the same direction from their original places, but oscillate either regularly or irregularly about these places.

One particular case of the virial deserves consideration, that, viz., in which an internal action of either attraction or repulsion exists between two particles, m and m', which are at the points (x, y, z) and (x', y', z'), at a distance r apart. Let $f(r)$ be the attraction. Then

$$\text{X} = -m'\frac{x-x'}{r}f(r), \quad \text{Y} = -m'\frac{y-y'}{r}f(r), \quad \text{Z} = -m'\frac{z-z'}{r}f(r);$$

$$\text{X}' = \ \ m\frac{x-x'}{r}f(r), \quad \text{Y}' = \ \ m\frac{y-y'}{r}f(r), \quad \text{Z}' = \ \ m\frac{z-z'}{r}f(r).$$

Therefore the virial, so far as the pair m and m' affects it,

$$= m\text{X} + m\text{Y} + m\text{Z} + m'\text{X}' + m'\text{Y}' + m'\text{Z}',$$
$$= -m m' r f(r).$$

Hence, so far as internal stresses are concerned, the whole virial $= -\Sigma . m m' r f(r)$, this comprising every pair of particles, r being the distance apart, and $f(r)$ the stress of each pair.

If $f(r) = \frac{\mu}{r^2}$, the virial $= \Sigma . \frac{\mu m m'}{r}$, and is the potential of the system.

Section 6.—*Lagrange's Equations of Motion; and the Hamiltonian Equations.*

126.] Although the theory of Lagrange's equations of motion and of the Hamiltonian equations which are closely allied with them will be found fully discussed by Professor Donkin in the last chapter of this volume, yet the subject is so intimately connected with the theory of vis viva and of work and with the principle of least action, that our consideration of these subjects would not be complete without shewing how they lead to those equations. Accordingly I propose to deduce them in two ways; firstly from the equation of vis viva, and secondly from the principle of least action.

The equations of Lagrange express in the most general form, and in terms of the most general variables, without any reference to their geometrical meaning, relations which exist between certain functions of these variables, and the momenta or the momen-

tum-increments which are impressed in the system. They are indeed the most general equations of motion imaginable.

The equation of vis viva, as given in the preceding section, in terms of x, y, z and their t-differentials is the complete value of the work done in a system in moving from one position to another; and we may take it in this respect as our point of departure. According to the notation of that section, as in Art. 111, we shall denote the vis viva by the symbols 2T, so that, as in (104),

$$2\text{T} = \Sigma.m\left\{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{dz^2}{dt^2}\right\}.$$

For convenience of notation we shall use accents to denote complete differentiation in respect of t, the time; and in this section we shall not use accents in any other meaning. Thus $\frac{dx}{dt}$ will be denoted by x', $\frac{dy}{dt}$ by y', $\frac{d^2x}{dt^2}$ by x'', and so on.

Thus
$$2\text{T} = \Sigma.m\,(x'^2+y'^2+z'^2). \tag{120}$$

Let us now introduce new variables $\xi, \eta, \zeta, \ldots$, and suppose them to be connected with x, y, z by equations such as

$$\left.\begin{array}{l} x = f(\xi, \eta, \zeta, \ldots t), \\ y = \phi(\xi, \eta, \zeta, \ldots t), \\ \cdots\cdots \end{array}\right\} \tag{121}$$

the right-hand members of which may be functions of t, as well as of $\xi, \eta, \zeta, \ldots$, but are not necessarily so; but no time-differentials, such as $\xi', \eta', \ldots$, must be contained in them. The number of these new variables will be as many as the conditions of the problem require; they are supposed to be independent of each other, and the problem is to determine each of them in terms of t, so that their values may be substituted in (121), whereby we shall know x, y, z and all the circumstances of the problem. These new variables are called coordinates, as by their combined values the position of the system at any time t is determined, and they are called generalised coordinates, as they are most general in form, and become if need be any system that we wish to apply; we shall presently deduce from them some of the systems which are in ordinary use.

127.] Since

$$2\text{T} = \Sigma.m\,(x'^2+y'^2+z'^2),$$

$$\frac{d\text{T}}{d\xi'} = \Sigma.m\left\{x'\frac{dx'}{d\xi'}+y'\frac{dy'}{d\xi'}+z'\frac{dz'}{d\xi'}\right\}; \tag{122}$$

as x, y, z are functions of the variables $\xi, \eta, \zeta, \ldots t$, but do not contain any t-differentials of them such as $\xi', \eta', \ldots$, we have from (121)

$$\left.\begin{aligned} x' &= \left(\frac{dx}{dt}\right) + \left(\frac{dx}{d\xi}\right)\xi' + \left(\frac{dx}{d\eta}\right)\eta' + \ldots, \\ y' &= \left(\frac{dy}{dt}\right) + \left(\frac{dy}{d\xi}\right)\xi' + \left(\frac{dy}{d\eta}\right)\eta' + \ldots, \\ &\ldots\ldots\ldots\ldots \end{aligned}\right\} \tag{123}$$

the brackets indicating that the quantities which they enclose are partial differential coefficients. Taking the first of (123) as the type-expression and differentiating it with regard to $\xi', \eta', \ldots$,

$$\left.\begin{aligned} \left(\frac{dx'}{d\xi'}\right) &= \left(\frac{dx}{d\xi}\right); \quad \left(\frac{dx'}{d\eta'}\right) = \left(\frac{dx}{d\eta}\right), \ldots, \\ \left(\frac{dy'}{d\xi'}\right) &= \left(\frac{dy}{d\xi}\right); \quad \left(\frac{dy'}{d\eta'}\right) = \left(\frac{dy}{d\eta}\right), \ldots; \end{aligned}\right\} \tag{124}$$

$\therefore$ substituting in (122),

$$\frac{d\text{T}}{d\xi'} = \Sigma . m\left\{x'\frac{dx}{d\xi} + y'\frac{dy}{d\xi} + z'\frac{dz}{d\xi}\right\}. \tag{125}$$

Now taking the t-differential of this, we have

$$\begin{aligned} \frac{d}{dt}\frac{d\text{T}}{d\xi'} = {} & \Sigma . m\left\{x''\frac{dx}{d\xi} + y''\frac{dy}{d\xi} + z''\frac{dz}{d\xi}\right\} \\ & + \Sigma . m\left\{x'\frac{d}{dt}\frac{dx}{d\xi} + y'\frac{d}{dt}\frac{dy}{d\xi} + z'\frac{d}{dt}\frac{dz}{d\xi}\right\}; \end{aligned} \tag{126}$$

but as the operations of time-differentiation and space-differentiation are commutative, that is, as their order is interchangeable,

$$\left.\begin{aligned} \frac{d}{dt}\frac{dx}{d\xi} &= \frac{d}{d\xi}\frac{dx}{dt} = \frac{dx'}{d\xi}; \\ \text{similarly} \quad \frac{d}{dt}\frac{dy}{d\xi} &= \frac{dy'}{d\xi}, \quad \frac{d}{dt}\frac{dz}{d\xi} = \frac{dz'}{d\xi}; \end{aligned}\right\} \tag{127}$$

$$\begin{aligned} \therefore \quad \frac{d}{dt}\left(\frac{d\text{T}}{d\xi'}\right) = {} & \Sigma . m\left\{x''\frac{dx}{d\xi} + y''\frac{dy}{d\xi} + z''\frac{dz}{d\xi}\right\} \\ & + \Sigma . m\left\{x'\frac{dx'}{d\xi} + y'\frac{dy'}{d\xi} + z'\frac{dz'}{d\xi}\right\}; \end{aligned} \tag{128}$$

but the last term in this expression $= \dfrac{d\text{T}}{d\xi}$, by (120);

$$\therefore \quad \frac{d}{dt}\left(\frac{d\text{T}}{d\xi'}\right) - \left(\frac{d\text{T}}{d\xi}\right) = \Sigma . m\left\{x''\frac{dx}{d\xi} + y''\frac{dy}{d\xi} + z''\frac{dz}{d\xi}\right\} \tag{129}$$

$$= \frac{\Sigma . m(\text{X}\,dx + \text{Y}\,dy + \text{Z}\,dz)}{d\xi} \tag{130}$$

by D'Alembert's principle.

As the system of forces is conservative, we will suppose them to be derived from a force-function $-\text{V}$, so that the right-hand member of (130) is $-\left(\frac{d\text{V}}{d\xi}\right)$; accordingly we have

$$\frac{d}{dt}\left(\frac{d\text{T}}{d\xi'}\right)-\left(\frac{d\text{T}}{d\xi}\right)=-\left(\frac{d\text{V}}{d\xi}\right); \tag{131}$$

similarly

$$\frac{d}{dt}\left(\frac{d\text{T}}{d\eta'}\right)-\left(\frac{d\text{T}}{d\eta}\right)=-\left(\frac{d\text{V}}{d\eta}\right), \tag{132}$$

and so on, for the other variables. These are called Lagrange's equations of motion in generalised coordinates. The right-hand and the left-hand members respectively represent the impressed and the expressed momentum-increments, as they arise from an infinitesimal displacement of a system of forces and of matter on which the forces act, due to the variation of a general variable; and as no restriction has been put upon the character of the displacement, so they include all kinds of displacement; and as the variations of the coordinates represent one kind of displacement as well as another, rotation as well as translation, so are all equations of motion contained in these equations. The system of forces is in all cases a conservative system; D'Alembert's principle supplies the transition from (129) to (130); and the principle of virtual velocities gives the equation of vis viva (120) from which we started.

The type-equation (131) may be written in the form

$$\left(\frac{d\text{T}}{d\xi'}\right)'-\frac{d}{d\xi}(\text{T}-\text{V})=0; \tag{133}$$

and if $\text{T}-\text{V}=\text{U}$, since V does not contain t or t-differentials, (133) takes the form

$$\left(\frac{d\text{U}}{d\xi'}\right)'-\left(\frac{d\text{U}}{d\xi}\right)=0. \tag{134}$$

128.] These equations also follow immediately from the principle of least action, as explained and proved in Art. 122.

Let A, as in Article 122, be the action of the system acquired in its passage from its place at the time t_0 to its place at the time t_1: then

$$\text{A}=\int_0^1 \Sigma\,.\,mv^2\,dt$$

$$=2\int_0^1 \text{T}\,dt. \tag{135}$$

Now let the vis viva be expressed in terms of generalised coordinates, then T is a function of ξ, η, ζ, and of the first

t-differential coefficients of these variables; but none of a higher order are found in it; it may also be a function of t. So that

$$2\mathrm{T} = f(\xi, \eta, \zeta, \xi', \eta', \zeta', t), \tag{136}$$

where f denotes a function depending on the configuration of the system.

Let us operate on (135) and (136) according to the principles of the Calculus of Variations, since A is a minimum: then as $\delta\mathrm{A} = 0$, we have

$$\begin{aligned} 0 &= \int_0^1 (\delta\mathrm{T}\, dt + \mathrm{T}\,\delta . dt) \\ &= \int_0^1 (\delta\mathrm{T}\, dt + \mathrm{T}\, d . \delta t) \\ &= \Big[\mathrm{T}\,\delta t\Big]_0^1 + \int_0^1 (\delta\mathrm{T}\, dt - d\mathrm{T}\,\delta t); \end{aligned}$$

but from (136),

$$\begin{aligned} d\mathrm{T} &= \Big(\frac{d\mathrm{T}}{dt}\Big) dt + \Big(\frac{d\mathrm{T}}{d\xi}\Big) d\xi + \Big(\frac{d\mathrm{T}}{d\eta}\Big) d\eta + \Big(\frac{d\mathrm{T}}{d\zeta}\Big) d\zeta + \dots \\ &\quad + \Big(\frac{d\mathrm{T}}{d\xi'}\Big) d\xi' + \Big(\frac{d\mathrm{T}}{d\eta'}\Big) d\eta' + \Big(\frac{d\mathrm{T}}{d\zeta'}\Big) \delta\zeta' + \dots, \\ \delta\mathrm{T} &= \Big(\frac{d\mathrm{T}}{dt}\Big) \delta t + \Big(\frac{d\mathrm{T}}{d\xi}\Big) \delta\xi + \Big(\frac{d\mathrm{T}}{d\eta}\Big) \delta\eta + \Big(\frac{d\mathrm{T}}{d\zeta}\Big) \delta\zeta + \dots \\ &\quad + \Big(\frac{d\mathrm{T}}{d\xi'}\Big) \delta\xi' + \Big(\frac{d\mathrm{T}}{d\eta'}\Big) \delta\eta' + \Big(\frac{d\mathrm{T}}{d\zeta'}\Big) \delta\zeta' + \dots; \end{aligned}$$

therefore

$$0 = \Big[\mathrm{T}\,\delta t\Big]_0^1 + \int_0^1 \Big\{\Big(\frac{d\mathrm{T}}{d\xi}\Big)(\delta\xi dt - d\xi\,\delta t) + \Big(\frac{d\mathrm{T}}{d\xi'}\Big)(d\xi' dt - d\xi' dt) + \dots\Big\}, \tag{137}$$

these being the terms corresponding to ξ, and there being similar terms corresponding to η, ζ, ...; let us take as types of all those corresponding to ξ.

Now $$\delta\xi' dt - d\xi'\,\delta t = \frac{d}{dt}(\delta\xi\, dt - d\xi\,\delta t);$$

therefore

$$0 = \Big[\mathrm{T}\,\delta t + \frac{d\mathrm{T}}{d\xi'}(\delta\xi dt - d\xi\,\delta t)\Big]_0^1 + \int_0^1 \Big\{\Big(\frac{d\mathrm{T}}{d\xi}\Big) - \frac{d}{dt}\Big(\frac{d\mathrm{T}}{d\xi'}\Big)\Big\}(\delta\xi - \xi'\delta t)\, dt. \tag{138}$$

As the values of the variables at the initial and final positions of the system are fixed, the integrated portion of this expression vanishes identically at each limit. As to the part under the sign of integration, it consists of a series of terms corresponding to η, ζ, ..., similar to the set corresponding to ξ. If all these variables were independent, each set would separately vanish: but they are not so, being subject in their variations to the expression of virtual moments, in respect to the impressed momen-

tum-increments as derived from the force-function. Now to form the most general equations of condition we have supposed a time-variation, but for our present purpose we may without loss of generality assume that there is no such variation, so that $\delta t = 0$; then the part under the sign of integration which is to vanish gives a series of terms which may be expressed in the form

$$\Sigma.\Big\{\Big(\frac{d\,\mathrm{T}}{d\,\xi}\Big)-\frac{d}{d\,t}\Big(\frac{d\,\mathrm{T}}{d\,\xi'}\Big)\Big\}\delta\,\xi = 0,$$

and we have also from the force-function

$$\Sigma.\Big(\frac{d\,\mathrm{V}}{d\,\xi}\Big)\delta\,\xi = 0.$$

But these quantities and their variations are subject to the equation of vis viva, viz., $\mathrm{T}-\mathrm{V} =$ a constant;
and consequently the expressions for the variations of T and V are equal: hence we have a series of terms of the form

$$\Sigma\Big\{\Big(\frac{d\,\mathrm{T}}{d\,\xi}\Big)-\frac{d}{d\,t}\Big(\frac{d\,\mathrm{T}}{d\,\xi'}\Big)-\Big(\frac{d\,\mathrm{V}}{d\,\xi}\Big)\Big\}\delta\,\xi = 0,$$

but as we assume that $\xi, \eta, \zeta, \ldots$ are all independent of each other, the coefficients of the variations will separately vanish; and we have

$$\left.\begin{aligned}\frac{d}{d\,t}\Big(\frac{d\,\mathrm{T}}{d\,\xi'}\Big)-\Big(\frac{d\,\mathrm{T}}{d\,\xi}\Big) &= -\Big(\frac{d\,\mathrm{V}}{d\,\xi}\Big),\\ \frac{d}{d\,t}\Big(\frac{d\,\mathrm{T}}{d\,\eta'}\Big)-\Big(\frac{d\,\mathrm{T}}{d\,\eta}\Big) &= -\Big(\frac{d\,\mathrm{V}}{d\,\eta}\Big);\end{aligned}\right\} \quad (139)$$

which are Lagrange's equations of motion.

129.] The advantage of these expressions, as pointed out by Lagrange, is that they include all equations of motion, whatever is the system of reference; and consequently leave us at liberty to choose that which is most simple, and especially that which most easily admits of integration; and to shew how this is the case let us take some particular instances of the ordinary equations of motion.

Ex. 1. For motion in one plane and in terms of polar coordinates $d\,s^2 = d\,r^2 + r^2 d\,\theta^2$; and for motion of one particle m,

$$2\,\mathrm{T} = m\,(r'^2 + r^2\theta'^2);$$

$$\therefore\quad \frac{d\,\mathrm{T}}{d\,r'} = m\,r', \qquad \frac{d\,\mathrm{T}}{d\,\theta'} = m\,r^2\theta',$$

$$\frac{d}{d\,t}\frac{d\,\mathrm{T}}{d\,r'} = m\,r'', \qquad \frac{d}{d\,t}\frac{d\,\mathrm{T}}{d\,\theta'} = m\,(r^2\theta')',$$

$$\frac{d\,\mathrm{T}}{d\,r} = m\,r\,\theta'^2, \qquad \frac{d\,\mathrm{T}}{d\,\theta} = 0.$$

so that Lagrange's expression becomes

$$m(r''-r\theta'^2)\delta r+m(r^2\theta')'\delta\theta.$$

Now if P and Q are the central force and the transversal moment respectively, and are so connected with the force-function V, that they are of the form $-\left(\frac{dV}{dr}\right)$ and $-\left(\frac{dV}{d\theta}\right)$ respectively: then

$$P\delta r+Q\delta\theta=-DV,$$

and we have

$$m(r''-r\theta'^2)=P, \qquad m(r^2\theta')'=Q.$$

The corresponding problem in spherical coordinates will be found in a subsequent Article.

Ex. 2. The motion of a single particle in a plane referred to tangential and normal resolution.

$$2T=ms'^2, \qquad \frac{dT}{ds'}=ms', \qquad \frac{dT}{ds}=0;$$

$$\therefore \quad \frac{d}{dt}\frac{dT}{ds'}-\frac{dT}{ds}=ms'';$$

which is the tangential component of the expressed momentum-increment.

Again, $$2T=m(x'^2+y'^2).$$

Let dn be the length-element of the normal; then

$$\frac{dT}{dn'}=0, \qquad \frac{dT}{dn}=m\frac{d^2x\,dx+d^2y\,dy}{dn\,dt^2},$$

$$\frac{d}{dt}\frac{dT}{dn'}=0; \quad \text{but } \frac{dx}{dn}=\frac{dy}{ds}, \quad \text{and } \frac{dy}{dn}=-\frac{dx}{ds};$$

$$\therefore \quad \frac{dT}{dn}=m\frac{d^2x\,dy-d^2y\,dx}{ds\,dt^2}$$

$$=m\frac{v^2}{\rho},$$

if ρ is the radius of curvature of the curvilinear path of the particle at the point (x, y). Hence $m\frac{v^2}{\rho}$ is the normal component of the expressed momentum-increment.

130.] The equation of vis viva is deducible from the general equations of motion in the following way, as shewn by Lagrange:

Let the equations (139) be multiplied respectively by $d\xi$, $d\eta$, ..., and added, then since

$$\left(\frac{dV}{d\xi}\right)d\xi+\left(\frac{dV}{d\eta}\right)d\eta+\ldots=dV.$$

we have

$$\left(\frac{d\text{T}}{d\xi'}\right)' d\xi + \left(\frac{d\text{T}}{d\eta'}\right)' d\eta + \ldots - \left\{\left(\frac{d\text{T}}{d\xi}\right) d\xi + \left(\frac{d\text{T}}{d\eta}\right) d\eta + \ldots\right\} = -d\text{V}. \quad (140)$$

Now, since T, if it does not involve t, is a homogeneous function of the second degree in terms of ξ', η', ..., we have by Euler's Theorem

$$2\text{T} = \left(\frac{d\text{T}}{d\xi'}\right)\xi' + \left(\frac{d\text{T}}{d\eta'}\right)\eta' + \ldots;$$

$$\therefore\ 2\text{T}' = \xi'\left(\frac{d\text{T}}{d\xi'}\right)' + \eta'\left(\frac{d\text{T}}{d\eta'}\right)' + \ldots$$

$$+ \left(\frac{d\text{T}}{d\xi'}\right)\xi'' + \left(\frac{d\text{T}}{d\eta'}\right)\eta'' + \ldots;$$

$\therefore$ substituting in (140) we have

$$2\text{T}'dt - \left\{\left(\frac{d\text{T}}{d\xi'}\right)\xi''dt + \left(\frac{d\text{T}}{d\eta'}\right)\eta''dt + \ldots\right.$$

$$\left. + \left(\frac{d\text{T}}{d\xi}\right)d\xi + \left(\frac{d\text{T}}{d\eta}\right)d\eta + \ldots\right\} = -d\text{V};$$

but since T is a function of $\xi, \eta, \ldots, \xi', \eta', \ldots,$

$$\text{T}'dt = \left(\frac{d\text{T}}{d\xi}\right)d\xi + \left(\frac{d\text{T}}{d\eta}\right)d\eta + \ldots + \left(\frac{d\text{T}}{d\xi'}\right)\xi''dt + \left(\frac{d\text{T}}{d\eta'}\right)\eta''dt + \ldots;$$

$$\therefore\ 2\text{T}'dt - \text{T}'dt = -d\text{V};$$

$$\therefore\ \text{T} + \text{V} = \text{a constant}$$

$$= \text{T}_0 + \text{V}_0,$$

which is the equation of vis viva, and states that the sum of the kinetic and the potential energies is constant.

131.] Sir W. R. Hamilton has by means of certain substitutions transformed these equations, and deduced from them some theorems which are of great importance in the general theory of dynamical equations, though of not much use in the solution of particular problems.

Let us take the type-equation (134), and make the following substitutions; viz. let

$$\left(\frac{d\text{U}}{d\xi'}\right) = u, \quad \left(\frac{d\text{U}}{d\eta'}\right) = v, \ldots; \quad (141)$$

then from (134) we have

$$u' - \left(\frac{d\text{U}}{d\xi}\right) = 0, \quad v' - \left(\frac{d\text{U}}{d\eta}\right) = 0, \ldots; \quad (142)$$

also let

$$\text{H} = -\text{U} + u\xi' + v\eta' + \ldots; \quad (143)$$

then if DH is the total differential of H,

$$\begin{aligned}\text{DH} = &-\left(\frac{d\text{U}}{d\xi}\right)d\xi - \left(\frac{d\text{U}}{d\eta}\right)d\eta - \dots \\ &-\left(\frac{d\text{U}}{d\xi'}\right)d\xi' - \left(\frac{d\text{U}}{d\eta'}\right)d\eta' - \dots \\ &+u\,d\xi' \qquad +v\,d\eta' + \dots \\ &+\xi' du \qquad +\eta' dv + \dots ;\end{aligned} \tag{144}$$

but by reason of (141) the second and third rows cancel each other, so that

$$\begin{aligned}\text{DH} = &-\left(\frac{d\text{U}}{d\xi}\right)d\xi - \left(\frac{d\text{U}}{d\eta}\right)d\eta - \dots \\ &+\xi' du + \eta' dv + \dots ;\end{aligned} \tag{145}$$

also by means of (141) ξ', η', ... may be replaced in H in terms of ξ, η, ζ, ... u, v, w, ..., so that H is a function of these quantities only, with or without t, and consequently

$$\begin{aligned}\text{DH} = &\left(\frac{d\text{H}}{d\xi}\right)d\xi + \left(\frac{d\text{H}}{d\eta}\right)d\eta + \dots \\ &+\left(\frac{d\text{H}}{du}\right)du + \left(\frac{d\text{H}}{dv}\right)dv + \dots :\end{aligned} \tag{146}$$

comparing this with (145) and equating coefficients, since the two expressions are identical, we have

$$\left(\frac{d\text{H}}{d\xi}\right) = -\left(\frac{d\text{U}}{d\xi}\right), \quad \left(\frac{d\text{H}}{du}\right) = \xi', \tag{147}$$

with similar equations for each of the other coordinates.

So that from (142) and (147),

$$u' + \left(\frac{d\text{H}}{d\xi}\right) = 0, \qquad \xi' - \left(\frac{d\text{H}}{du}\right) = 0; \tag{148}$$

with similar equations for each of the other coordinates; where

$$u = \left(\frac{d\text{U}}{d\xi'}\right), \qquad v = \left(\frac{d\text{U}}{d\eta'}\right), \dots ; \tag{149}$$

and

$$\text{H} = -\text{U} + u\xi' + v\eta' + \dots . \tag{150}$$

Thus, as in (148), there are two differential equations corresponding to each coordinate: so that the whole number is twice that of the Lagrangian equations; but as each equation in (148) is of the first order, and each Lagrangian equation is of the second order, if n is the number of independent coordinates, there are $2n$ equations of the first order instead of n equations of the second order. Now in the integration of these equations $2n$ constants will be introduced whether they arise from $2n$ equations of the first order or from n equations of the second order.

Equations (148) are called Hamiltonian Equations; and the function H, given in (143), has been called the Hamiltonian Function.

132.] With respect to H. Since H and U are so related by (148) that

$$\left(\frac{d\text{U}}{d\xi'}\right) = u, \quad \left(\frac{d\text{H}}{du}\right) = \xi',$$

with similar relations for the other coordinates, H and U have been called Reciprocal Functions, each being the reciprocal of the other. In the case where T does not contain t, but is a homogeneous function of the second order in terms of $\xi', \eta', \zeta', \ldots$, so that by Euler's Theorem

$$2\text{T} = \left(\frac{d\text{T}}{d\xi'}\right)\xi' + \left(\frac{d\text{T}}{d\eta'}\right)\eta' + \ldots; \tag{151}$$

then since $\text{U} = \text{T} - \text{V}$,

$$\begin{aligned} \text{H} &= -\text{U} + u\xi' + v\eta' + \ldots \\ &= -\text{T} + \text{V} + \left(\frac{d\text{T}}{d\xi'}\right)\xi' + \left(\frac{d\text{T}}{d\eta'}\right)\eta' + \ldots \qquad (152) \\ &= -\text{T} + \text{V} + 2\text{T} \\ &= \text{T} + \text{V}; \end{aligned}$$

so that H is the sum of the kinetic and potential energies, and is thus the whole energy of the system.

For further information on this subject I must refer the reader to Sir W. R. Hamilton's original Memoirs in the Philosophical Transactions 1834 and 1835.

SECTION 7.—*Gauss' theorem of least constraint.*

133.] In Art. 110, Vol. III, a statical theorem is given which is there deduced from the principle of virtual velocities. It is however only a particular form of a very general proposition which includes all dynamics. It is useless for the direct solution of problems, but in the same way as the principle of least action is useless: yet, as it comprises the equations of motion given in (37) and (38) of Art. 73, and gives a new meaning to them, it deserves attention. If we could either assume its truth, or prove it to be true by general reasoning, we might deduce from it all the equations of motion: but it is better for us to take an opposite course; to state the principle, and then to shew that it is deducible from the equations of motion.

The theorem is due to Gauss, and is now called "Gauss' principle of least constraint:" it was first given in Crelle's Journal, Vol. IV, 1829; and a French translation of the memoir is inserted as a note to the 2nd volume of the Mécanique Analytique of Lagrange, edited by M. Bertrand, Paris. 1855. A full explanation of it with examples is given in "Zeitschrift für Mathematik und Physik von Schlömilch und Witzschel," III. Band, Leipzig, 1858, by Dr. Hermann Scheffler. The following is the enunciation of the theorem:

If a system of material particles is in motion, under the action of given finite accelerating forces, the sum of the products of each particle and of the square of the distance between its place at the end of an infinitesimal time, say dt, and the place which it would have under the action of the given forces, and in the same initial circumstances, if it were free, is a minimum.

If therefore we measure constraint by the square of the distance between the actual place of m, and the place which it would have if it were under the action of the same forces and were a single unconstrained particle, then the theorem is, that the sum of the products of each particle and its constraint is a minimum.

Let m be the type of the particles of the system; and let (x, y, z) be its place at the time t; let u, v, w be the axial components of the velocity of m, and let X, Y, Z be the axial components of the impressed velocity-increments. Then, at the time $t+dt$, the coordinates of m are respectively

$$x+u\,dt+\frac{1}{2}\frac{du}{dt}dt^2, \quad y+v\,dt+\frac{1}{2}\frac{dv}{dt}dt^2, \quad z+w\,dt+\frac{1}{2}\frac{dw}{dt}dt^2; \qquad (153)$$

but if m were unconstrained, and moved with the same initial circumstances, the coordinates of its place would be

$$x+u\,dt+\frac{\text{X}}{2}dt^2, \qquad y+v\,dt+\frac{\text{Y}}{2}dt^2, \qquad z+w\,dt+\frac{\text{Z}}{2}dt^2. \qquad (154)$$

Let us suppose $(x+\xi, y+\eta, z+\zeta)$ to be any other place which it were possible for m to take; and let U be the sum of the products of every particle, and the square of the distance between the possible place and the place which the particle would have if it were unconstrained; so that

$$\text{U} = \Sigma.m\left\{(\xi - u\,dt - \frac{\text{X}}{2}dt^2)^2 + (\eta - v\,dt - \frac{\text{Y}}{2}dt^2)^2 + (\zeta - w\,dt - \frac{\text{Z}}{2}dt^2)^2\right\}.$$

Then Gauss' theorem consists in the assertion that U is a minimum when the possible place is that given by the coordinates (153).

134.] Let us differentiate U; then, if the total differential vanishes,

$$\text{DU} = 0 = 2\Sigma.m\left(\xi - u\,dt - \frac{\text{X}}{2}dt^2\right)d\xi + 2\Sigma.m\left(\eta - v\,dt - \frac{\text{Y}}{2}dt^2\right)d\eta$$
$$+ 2\Sigma.m\left(\zeta - w\,dt - \frac{\text{Z}}{2}dt^2\right)d\zeta; \quad (155)$$

and as $d\xi$, $d\eta$, $d\zeta$ are independent of each other, the coefficient of each must separately vanish; therefore

$$\left.\begin{aligned} \Sigma.m\left(\xi - u\,dt - \frac{\text{X}}{2}dt^2\right) &= 0, \\ \Sigma.m\left(\eta - v\,dt - \frac{\text{Y}}{2}dt^2\right) &= 0, \\ \Sigma.m\left(\zeta - w\,dt - \frac{\text{Z}}{2}dt^2\right) &= 0. \end{aligned}\right\} \quad (156)$$

But by the equations (37), Art. 73,

$$\left.\begin{aligned} \Sigma.m\text{X} &= \Sigma.m\frac{d^2x}{dt^2} = \Sigma.m\frac{du}{dt}, \\ \Sigma.m\text{Y} &= \Sigma.m\frac{d^2y}{dt^2} = \Sigma.m\frac{dv}{dt}, \\ \Sigma.m\text{Z} &= \Sigma.m\frac{d^2z}{dt^2} = \Sigma.m\frac{dw}{dt}; \end{aligned}\right\} \quad (157)$$

so that (156) become

$$\left.\begin{aligned} \Sigma.m\left(\xi - u\,dt - \frac{1}{2}\frac{du}{dt}dt^2\right) &= 0, \\ \Sigma.m\left(\eta - v\,dt - \frac{1}{2}\frac{dv}{dt}dt^2\right) &= 0, \\ \Sigma.m\left(\zeta - w\,dt - \frac{1}{2}\frac{dw}{dt}dt^2\right) &= 0; \end{aligned}\right\} \quad (158)$$

and therefore it is necessary that the possible place should coincide with the actual place of which the coordinates are given in (153).

Since U is the sum of a series of positive quantities, it is evident that it does not admit of a maximum; neither generally is it a constant: it must therefore be a minimum; so that the sum of the products of each particle and its constraint is a minimum.

And that it is a minimum, we may also thus demonstrate. Differentiating again (155) we have,

$$\text{D}^2\text{U} = 2\Sigma.m\,d\xi^2 + 2\Sigma.m(\xi - u\,dt - \frac{\text{X}}{2}dt^2)\,d^2\xi + \ldots + \ldots$$
$$= 2\Sigma.m(d\xi^2 + d\eta^2 + d\zeta^2),$$

by reason of (156); and this is necessarily a positive quantity; so that U is a minimum for the particular values of ξ, η, ζ given in (158).

The minimum value of U is evidently

$$\text{U} = \frac{1}{4}\Sigma.m\left\{\left(\text{X} - \frac{du}{dt}\right)^2 + \left(\text{Y} - \frac{dv}{dt}\right)^2 + \left(\text{Z} - \frac{dw}{dt}\right)^2\right\}dt^4. \quad (159)$$

In statics m has no velocity, so that $u = v = w = 0$; and the theorem takes the following form:

If a system of particles of invariable form is in equilibrium under the action of given pressures, and is disturbed; the sum of the products of each particle and the square of the distance between the original place, and the place which it would have in the displacement, if it had been free, is a minimum.

Let us apply this principle to the following example. Two inelastic particles, m and m', impinge on each other; it is required to find their common velocity after impact.

Let v and v' be the velocities of m and m' respectively before impact; and let V be the common velocity after impact: then, if U is the sum of the products of each mass and the square of the distance between the place which it has actually, and the place which it would have if it were unconstrained at the end of dt,

$$\text{U} = m(v - \text{V})^2 dt^2 + m'(\text{V} - v')^2 dt^2;$$
$$-2m(v - \text{V})\,dt^2 + 2m'(\text{V} - v')\,dt^2 = 0,$$

if

$$m(v - \text{V}) = m'(\text{V} - v');$$
$$\therefore \quad \text{V} = \frac{mv + m'v'}{m + m'}.$$

It is hardly necessary to observe the close analogy which exists between the principle of least constraint and the method of least squares. As that method gives the most likely value to quantities determined by observation, which are subject to small accidental errors, so, if we apply to dynamics similar considerations, this theorem of least constraint would assign the most plausible position to a system of constrained particles. We have however deduced this theorem from the ordinary laws of motion, and

from D'Alembert's principle; from laws that are incontestably and otherwise true; so that the places which would be assigned to the particles as the most plausible by the method of least squares are proved to be their actual places.

Notwithstanding the disparaging remarks which are not unfrequently made on the principle of least action, and which Lagrange thought it worth while to reply to; and although similar remarks may be made on this theorem, inasmuch as it is, like that of least action, the expression of a metaphysical idea, yet it commends itself to every mathematician on account of its elegance and its comprehensiveness. The end of all science is a knowledge of laws which govern phænomena: and therefore a theorem which includes all mechanical phænomena, and gives a new meaning to them, and indicates that there is no waste of power, is not to be discarded as useless. In the solution of particular problems we may indeed apply less general laws: but the more general law cannot fail of exciting curiosity and of creating a desire to know the details which it contains.

SECTION 8.—*Newton's principle of similitude.*

135.] As in the present Chapter we are investigating those theorems and principles which follow immediately from the equations of motion of Art. 73, and which will be, each in its own degree, applied in the following Chapters; so shall we here introduce a principle of similitude which is given by Newton in Book II, Section VII, Prop. XXXII, of the Principia; and of which a proof and some applications are given by M. Bertrand in Cahier XXXII of the Journal de l'Ecole Polytechnique, Paris, 1848.

The problem which the principle of similitude solves is this: A system of material particles of a certain form is in motion under the action of certain forces; we have a new system exactly similar, and either larger or smaller: in what proportions are the masses, their expressed velocities, the time of motion, and the impressed momenta or the acting forces to be changed, that is, either increased or diminished, so that the motion of the new system may be similar to that of the old system? In other words, a machine "works" on a given scale; in what proportion are its parts to be changed, that it should "work" on another

given scale? A model succeeds; a machine made after the model breaks into pieces: what is the cause of this? Here is a distinguishing point between geometry and mechanics; whatever in geometry is true of a triangle on a small scale, is equally true of a triangle on a large scale: in mechanics it is not so; if a large machine is made with all its parts geometrically proportional to all the parts of a small one, the "working" of the large one cannot be inferred from that of the small one. Now the proportion in which the parts are to be changed is rightly given by Newton in the proposition above mentioned, and of which the enunciation is;

Si corporum systemata duo similia ex æquali particularum numero constent et particulæ correspondentes similes sint et proportionales, singulæ in uno systemate singulis in altero, et similiter sitæ inter se, ac datam habeant rationem densitatis ad invicem, et inter se temporibus proportionalibus similiter moveri incipiant (ea inter se quæ in uno sunt systemate, et ea inter se quæ sunt in altero), et si non tangant se mutuo, quæ in eodem sunt systemate, nisi in momentis reflexionum, neque attrahant, vel fugent se mutuo nisi viribus acceleratricibus quæ sint ut particularum correspondentium diametri inverse et quadrata velocitatum directe, dico quod systematum particulæ illæ pergent inter se temporibus proportionalibus similiter moveri.

According to this theorem, to any system of particles of a certain form, the number of similar systems is infinite. A proportionality however must exist between five elements of the two systems, each to each; the lengths, the times, the velocities, the acting forces, and the masses; instead of between the lengths only, as the principle of similitude in geometry requires.

Let us take the general equation of motion (43), Art. 76, as given by the principle of virtual velocities; and modify it so far as to make X, Y, Z the impressed momentum-increments; then we have

$$\Sigma . \left\{ \left(\mathrm{X} - m\frac{d^2x}{dt^2} \right) \delta x + \left(\mathrm{Y} - m\frac{d^2y}{dt^2} \right) \delta y + \left(\mathrm{Z} - m\frac{d^2z}{dt^2} \right) \delta z \right\} = 0; \quad (160)$$

and this equation, as we have shewn in that Article, includes all possible cases of motion. Suppose now that all the circumstances of motion of the system are deduced from this equation and that the place of every particle at the time t is given in terms of t. Also, let us suppose that we have a second system of par-

ticles of a form similar to the first; so that when the moti begins, the position of the particles in one system is similar a similarly situated to that of those in the other; and let bo systems be subject to similar constraints, so that both ha similar equations of condition; and let the equation of moti which applies to this latter system be

$$\Sigma.\left\{\left(X'-m'\frac{d^2x'}{dt'^2}\right)\delta x'+\left(Y'-m'\frac{d^2y'}{dt'^2}\right)\delta y'+\left(Z'-m'\frac{d^2z'}{dt'^2}\right)\delta z'\right\}=0. (\ $$

Let
r = the ratio of the similar lines in the two systems,
k = the ratio of the masses of corresponding particles,
ρ = the ratio of the corresponding impressed momenta,
n = the ratio of the times in each,
σ = the ratio of the velocities of corresponding particles in each;
so that

$$\left.\begin{aligned}\frac{x'}{x}=\frac{y'}{y}=\frac{z'}{z}=r=\frac{\delta x'}{\delta x}=\frac{\delta y'}{\delta y}=\frac{\delta z'}{\delta z};\\ \frac{X'}{X}=\frac{Y'}{Y}=\frac{Z'}{Z}=\rho;\\ m'=km,\quad t'=nt,\quad \frac{ds'}{dt'}=\sigma\frac{ds}{dt};\end{aligned}\right\} \quad (16$$

then it is evident that, if these quantities are substituted i (161), the result is identical with (160) if

$$\rho=\frac{kr}{n^2}; \quad (16$$

by which equation, any three quantities being given, we can de termine the fourth. Thus, for instance, if all the elements of system are changed in the ratios given in (162), the time vari directly as the square root of the linear distances, directly a the square root of the masses, and inversely as the square ro of the acting force. Newton's principle of similitude consists i equation (164).

Also $$\sigma=\frac{ds'}{dt'}\frac{dt}{ds}=\frac{r}{n}=\left(\frac{\rho r}{k}\right)^{\frac{1}{2}}. \quad (16$$

Thus, in all questions of Dynamics, if the motion or "working of any system is to be inferred from that of a similar system: the linear distances, the times, and the masses are increase in the ratios $r:1$, $n:1$, $k:1$ respectively; then the velocitie and the impressed momenta or acting forces, must be increase in the ratios respectively of $r:n$ and of $kr:n^2$. Let us tak

the following case of this: A large locomotive works; what conditions are necessary that a small similar locomotive should also work?

Let the ratio of the linear dimensions of the parts in the small machine be to that in the larger machine as $r:1$, where r is a proper fraction; then the ratio of the masses is as $r^3:1$; so that $k=r^3$; also, since gravity acts in both cases, and since the ratio of the weights of the several parts are as the masses, therefore the ratio of the forces is as $r^3:1$; and this is to be constant with all the forces, so that $\rho=r^3$: thus, from (164), $n=r^{\frac{1}{2}}$; and, from (165), $\sigma=r^{\frac{1}{2}}$: thus the times and the velocities will both be in the ratio $r^{\frac{1}{2}}:1$. Let us consider other forces: the pressure of steam on the piston will be diminished in the ratio of $r^3:1$. As the resistance of the air varies as the square of the velocity and as the area of the surface, it is diminished in the ratio of $r^3:1$. Sliding frictions, being proportional to the pressures, are diminished in the same ratio. Rolling frictions, which are found to vary directly as the pressures, and inversely as the radii of the rolling wheels, must be diminished in the ratio of $r^2:1$. Thus, in a small machine, this ratio may be very great, and consequently rolling friction may be very great.

I subjoin two simple examples of this principle of similitude; others will be found in the Memoir of M. Bertrand, which has already been alluded to.

Ex. 1. Two equal particles at rest are attracted towards a centre of force, which varies directly as the distance; to prove that, whatever are their initial distances, they arrive at the centre simultaneously, and that their distances at any instant from the centre are proportional to their initial distances.

Let x_0 and x'_0 be the initial distances of the particles from the centre of force; so that $x'_0=rx_0$: also, if μx is the attractive force, then $\mu x'_0=\rho\mu x_0$; therefore $\rho=r$; also $k=1$; so that (164) gives $n=1$; therefore the times are equal in the two cases and both arrive simultaneously at the centre of force. And as the systems are similar at first, they are similar throughout the motion; and thus, if x and x' are the distances of the particles from the centre of force at the time t,

$$\frac{x}{x_0}=\frac{x'}{x_0{}'};$$

which is the second part of the theorem.

These results are manifestly also true, when the motion takes place in a medium of which the resistance varies directly as the velocity.

Ex. 2. Two simple pendulums, with equal weights, whose lengths are l and l' respectively, are moved from their vertical position through equal angles, and are under the action of the gravities g and g' respectively; it is required to compare the times of oscillation.

These are evidently two similar systems: $l'=rl$, $g'=\rho g$, $k=1$.

$$\therefore \quad n^2 = \frac{r}{\rho} = \frac{l'g}{lg'};$$

therefore, if T and T′ are the times of oscillation,

$$\text{T} : \text{T}' :: \left(\frac{l}{g}\right)^{\frac{1}{2}} : \left(\frac{l'}{g'}\right)^{\frac{1}{2}};$$

which is the well known result.

Section 9.—*On Units.*

136.] The conclusions of the preceding sections of this Chapter are drawn from certain equations which have their origin in D'Alembert's principle and in the principle of virtual velocities, and which are expressed in terms of velocity, acceleration, momentum, momentum-increments, &c. These lead to quantities called vis viva, work, force-function, &c., and to equations connecting these quantities. Now as the science of Mechanics is rigorous and precise, these equations must be intelligible, interpretable and measurable; these conditions affect both their qualitative and quantitative characters.

In respect of the former, viz. the qualitative character, as three and only three capital and fundamental elements, viz., space (linear space or length), time, and matter enter into these equations, and as these are independent of each other, all the terms must be homogeneous in respect of each of these elements; for were they not so, the form of the equation might vary as the unit of each element, which is arbitrary, varied, and the law expressed by the equation would change, and consequently would not have the attributes of permanence and invariability which the science requires. All the symbols, however, are not immediately referable to these simple elements; there are certain

secondary and derived ones, and the dimensions of these in reference to the three primary ones must be determined, ere we can conveniently test for homogeneity the several terms of the equations. The determination of these is the subject of the following Articles. It is to be observed that whenever the word "space" is used, it means linear space or length.

137.] In the first place it is evident that geometrical surface or area is of (2) dimensions in space; and that geometrical volume or content is of (3) dimensions in space.

As velocity is connected with space and time by the equation $v = \frac{ds}{dt}$, velocity is of (1) dimension in space, and of (-1) dimension in time.

Hence the unit of velocity is the unit of space passed through uniformly in an unit of time t.

As to acceleration f, since $f = \frac{d^2 s}{dt^2}$, acceleration is of (1) dimension in space, and of (-2) dimensions in time.

This result is also thus evident: let f act uniformly during the time, and let v be the velocity thereby generated; then $v = ft$; and thus f is of (1) dimension in velocity and of (-1) dimension in time; but since velocity is of (1) dimension in space and of (-1) dimension in time, it follows that acceleration is of (1) dimension in space and of (-2) dimensions in time.

Hence the unit of acceleration is that acceleration which acting uniformly during an unit of time generates an unit of velocity.

As density $= \frac{\text{mass}}{\text{volume}}$, density is of (1) dimension in mass, and of (-3) dimensions in space.

Hence the unit of density is the unit of mass contained in an unit of volume.

As expressed momentum-increment $= m\frac{d^2 s}{dt^2}$, it is of (1) dimension in mass, of (1) dimension in space, and of (-2) dimensions in time.

Consequently as moving force P is, or is measured by, expressed momentum-increment, it is of (1) dimension in mass, of (1) dimension in space, and of (-2) dimensions in time.

Consequently also weight, which is equal to mg, is of one

dimension in mass, of (1) dimension in space, and of (−2) dimensions in time. This is the dynamical definition of weight.

Hence the unit of moving force is that which impresses on an unit-mass an unit of acceleration. This is Gauss' absolute unit of force.

As work $= \frac{m}{2}(v^2 - v_0^2)$, it is of one dimension in mass, of (2) dimensions in space, and of (−2) dimensions in time.

This is also evident, since $d\text{W} = \text{P}\,ds = m\frac{d^2s}{dt^2}ds$.

Hence the unit of work is the work done by an unit of force when its point of application is moved through an unit of space on the line along which the force acts.

The secondary units thus determined are absolute units, independent of any arbitrarily chosen units of space, time, and mass; their truth rests immediately on the principles of the science of Mechanics, and hence they are called absolute kinetic units. Let us apply them to some examples, before we proceed to the consideration of the systems of primary units which are principally in use.

138.] Take the general equation (160) as given in Art. 135; here X, Y, Z are impressed momentum-increments and are consequently of (1) dimension in mass, of (1) dimension in space, and of (−2) dimensions in time; as these are also the dimensions of the expressed momentum-increments, the whole equation is homogeneous, and, including the arbitrary displacements, is of (1) dimension in mass, of (2) dimensions in space, and of (−2) dimensions in time; consequently the several terms express work, and the whole equation is an equation of work.

Again, take the first of Euler's equations of rotation; see equations (95), Art. 174, viz.

$$\text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{B})\,\omega_2\omega_3 = \text{L}:$$

as A, B, C are moments of inertia, each is of one dimension in mass and of (2) dimensions in space: $\frac{d\omega_1}{dt}$ is of (−2) dimensions in time, ω_2 and ω_3 are each of (−1) dimension in time, so that the left-hand member is of (1) dimension in mass, of (2) dimensions in space, and of (−2) dimensions in time. Also as the right-hand member is the moment of a force (impressed momentum-increment), it is of (1) dimension in mass, of (2) dimensions in space,

and of (-2) dimensions in time. Hence the equation satisfies the test of homogeneity.

The preceding rules of homogeneity afford a presumptive test, which is easily applied, of the correctness of operations performed on mathematical expressions. If these are homogeneous in their original state, this quality of homogeneity continues, whatever are the processes to which they are subjected; and consequently if they are homogeneous in their final state, it is a presumption that the processes have been correctly performed.

Further applications of these principles will occur in Hydromechanics, in the theory of Elasticity, in Heat, in Electricity and in Magnetism; and indeed in all sciences where work is done in consequence of acting forces.

139.] As to the three primary units of time, space, and mass, they are entirely arbitrary; but as the equations in which they occur have to be applied quantitatively, it is convenient to choose those particular values which are best suited to the circumstances of the science. Two such systems are principally in use; one by English engineers, which is called the British system; the other by French engineers, which is called the French metrical system; or, derived from it, the C. G. S. system, which is now generally used by all Physicists.

140.] In the British system a second is taken as the unit of time, being such that $86400''$ are equal to a mean solar day; were this unit lost, it would be recoverable by means of astronomical observations and a clock.

The British unit of length is one foot, being one-third part of the British standard yard. The British standard yard is the distance at 62° F. between two parallel lines in gold plugs in the bronze bar deposited in the Standards Office at the Board of Trade in London; copies being also deposited at the Royal Mint, at the apartments of the Royal Society at Burlington House, at the Royal Observatory at Greenwich, and at the New Palace at Westminster. This standard is arbitrary, and is not founded on any natural length or distance; it was chosen so that its length might be as nearly as possible the same as that of the best yards as formerly used in England. Its authority is derived from an enactment in Parliament (18 and 19 Vict. c. 72, July 30, 1855). An inch is one thirty-sixth part of this standard yard; and 1760 standard yards are a mile.

Were this standard yard and all the copies of it lost, it might possibly be recovered from the accurately measured and now known length between two points fixed in the earth, which are the extremities of a base used in a Trigonometrical Survey.

The British unit of mass is the imperial standard pound avoirdupois: being the quantity of matter contained in a certain disc of platinum which is deposited in the Standards Office at the Board of Trade, London; copies being deposited also at the places where copies of the standard yard are kept. One seven-thousandth part of this pound is a grain, and 5760 grains are a pound Troy.

Thus, in the British system, the unit of force (impressed momentum-increment) is that force which, acting on a pound, produces an acceleration of one foot-per second-per second. This unit is called a poundal.

Hence, as the earth's attraction, acting on a body near its surface, produces an acceleration of 32·19 feet-per second-per second, the weight of a pound is 32·19 poundals: and so on for any number of pounds under the action of the earth's attraction.

Thus also, in the British system, if the force producing work, or the force against which work is done, is gravitation, or the earth's attraction, the unit of work is the work required to lift one pound through one vertical foot. This unit of work is called a foot-pound. As g varies at different places on the earth's surface, the foot-pound also varies; and consequently, in measurements of great accuracy, the place must be given where they are made. Thus the foot-pound is a gravitation measure of work, and, not being absolute, is not useful as a scientific measure.

The rate of doing work is generally termed power; and horse-power is the unit of power adopted by British engineers; it is taken to be 33000 foot-pounds per minute, or 550 foot-pounds per second.

141.] In the French system, as in the British, a second is the unit of time. The unit of space is one centimètre, being the one-hundredth part of a standard called a mètre, which is the length of a certain platinum bar at the temperature 4° C., this temperature being chosen as that at which the density of water is the greatest. This bar was made by Borda*; its length is

* Borda was a member of a Commission consisting of Borda, Condorcet, Monge, Lagrange, and Laplace, appointed soon after the French Revolution to prepare a new system of weights and measures.

39·37043 inches, that is 3·280869 feet, being supposed to be the ten millionth part of a quadrant of the earth's meridian according to measurements of the same made by Delambre and Mèchain; the object was to obtain a natural standard of length, in contra-distinction to an artificial one—one, that is, which would be recognised by all nations, and not be the national standard of only one country. Subsequent surveys, however, have shewn that the measurements on which the length was founded are not sufficiently exact for the purpose; and the mètre has now become an artificial length, quite as much so as the English foot, and is the length of the aforesaid platinum bar at 4° C. The bar is kept at Paris. It was established as the standard of length by a law of the French Republic in the year 1795. Multiples of mètres are called by names derived from the Greek: as, e. g., deca-mètre, hecto-mètre, kilo-mètre; sub-multiples by names derived from the Latin: as e. g., decimètre, centimètre, millimètre. The relations between the French and English measures of length and volume are as follows:

1 cm. = 0·3937043 inches,
1 mètre = 39·37043 inches,
1 kilo-mètre = 1093·62311 yards;
1 inch = 2·53998 cm.,
1 foot = 30·4797 cm.,
1 litre = 1·760725376 pints.

The French unit of mass is a gramme, being the one-thousandth part of a certain mass of platinum, in the form of a cylinder, whose height is equal to the diameter of its base, made by Borda and kept at Paris, which is called the Kilogramme des Archives. It was originally intended that this mass should be equal to that of a litre, viz. a cubic decimètre, of distilled water at its greatest density, that is, at 4° C., and thus that it should be a natural standard: it is found, however, that this is not exactly the case, the mass of the cubic centimètre of water being 1·000013 gm., and not 1 gm. exactly, according to a comparison of standards made by Professor Kupffer. Thus this standard of mass is just as arbitrary as the English pound, since the authority for it depends on a law of the French State. The relation between the English and French units of mass are as follows:

1 ounce avoirdupois = 28·3495 gm.,
1 pound do. = 453·5927 gm.,

1 gramme = 0·03527 oz.,
" = 15·43234874 grains,
1 kilogramme = 2·20462 pounds.

142.] This system of units is known as the metrical system, having been founded on the mètre as its base unit. In the system now generally used by Physicists, the centimètre, the gramme, and the second are the fundamental units, and the system is called the C. G. S. system. It has the great advantage over the British system of having an uniform decimal method of multiplication and sub-multiplication in all its elements, their multiples and sub-multiples being formed in all cases by 10 and powers of 10.

The following are derived units in the C. G. S. system:

The unit of velocity is one centimètre per second.

The unit of acceleration is an unit of velocity per second.

The unit of force (impressed momentum-increment) is the force which acting on a gramme produces an accleration of one centimètre-per second-per second. This unit of force is called a dyne; so that a dyne = 1 gm. × 1 cm.-per second-per second. One million dynes is called a megadyne, and is equal to 10^6 dynes.

The relation between the English and the C. G. S. units of force, that is between the poundal and the dyne, is thus determined:

A poundal
= 1 pound × 1 foot-per second-per second,
= 453·5926 gm. × 30·4797 cm.-per second-per second,
= 13825·38 dynes.

Hence also the weight of a gramme may be expressed in dynes as follows, if g = 32·19:

Weight of 1 gm.
= 1 gm. × 32·19 feet-per second-per second,
= 1 gm. × 32·19 × 30·4797 cm.-per second-per second,
= 32·19 × 30·4797 dynes,
= 981·1142 dynes.

The unit of work is the work done when a dyne acts through a centimètre along its line of action. This unit of work is called an erg, so that an erg = 1 gm. × 1 cm.-per second-per second × 1 cm. A million ergs is called a megalerg and is equal to 10^6 ergs. Hence also a foot-poundal, corresponding to a given value of g, = 421393·8 ergs. The relation between the foot-pound and the erg may thus be found:

A foot-pound
= 1 lb. × 32·19 feet-per second-per second × 1 foot,
= 453·592 gm. × 32·19 × 30·4797 cm.-per sec.-per sec. × 30·4797 cm.,
= 453·592 × 32·19 × $(30·4797)^2$ ergs,
= 13·565 megalergs.

Another unit of work in the metrical system is the kilogramme-mètre, which is the amount of work done in lifting a kilogramme through a vertical mètre near to the earth's surface. This is also a gravitation unit. It is equal to 7·233 foot-pounds.

The kilogramme-mètre, corresponding to a given value of gravitation, say g = 32·19 feet-per second-per second, may thus be expressed in terms of ergs.

The kilogramme-mètre
= 1000 gm. × 32.19 feet-per second-per second × 100 cm.,
= 100000 × 32·19 × 30·4797 gm. cm.-per sec.-per sec. . cm.,
= 98114000 ergs.

Dr. Joule of Manchester found by a series of very exact experiments, made with great care, that the heat required to raise one pound of water from 58° F. to 59° F. is equivalent to 772 foot-pounds of work at Manchester. This is called the mechanical equivalent of heat in the British system. Now the corresponding equivalent in the metrical system is the quantity of work to be transmuted into heat so as to raise the temperature of one kilogramme of water by 1° C. This can be deduced from the British equivalent by the following process:

The work which raises 1 lb. of water by 1° F.
= 772 foot-pounds,
or, expressing these quantities in the metrical system,

the work which raises 453·5926 gm. by $\frac{5°}{9}$ C.
= 772 × 30·4797 cm. × 453·5927 gm. weight.

∴ work which raises 1000 gm. by 1° C.
= 423·545 kilogramme-mètres.

143.] In the preceding Articles the three fundamental units of time, space, and mass have been assumed to be entirely independent of each other, and the dimensions of the several terms of the equations have been estimated on this assumption. Assuming however matter to have the property that two particles attract each other directly as their masses and inversely as the

square of the distance between them, the following relation exists between mass, space, and time. Let unit-mass be placed at P, and a particle of mass m at Q, and let the distance PQ $= r$; then by the law of attraction,

$$\frac{d^2 r}{dt^2} = -\frac{m}{r^2};$$

$$\therefore \quad m = -r^2 \frac{d^2 r}{dt^2};$$

and consequently mass is of (3) dimensions in space, and of (−2) dimensions in time; wherever therefore mass occurs in the preceding determinations of the dimensions of an expression, (1) dimension in mass may be replaced by (3) dimensions in space, and (−2) dimensions in time. Hence we have the following dimensions:

Density is of (0) dimensions in space and of (−2) dimensions in time.

Force (impressed momentum-increment) is of (4) dimensions in space, and of (−4) dimensions in time. These are also the dimensions of weight.

Work is of (5) dimensions in space and of (−4) dimensions in time.

From the preceding value of mass it follows that an unit of mass is that mass which by its attraction on an unit of mass produces an unit of acceleration at an unit of distance. This is an absolute unit.

CHAPTER IV.

THE EQUATIONS OF MOTION OF A RIGID BODY EXPRESSED IN TERMS OF ANGULAR VELOCITIES AND THEIR INCREMENTS. PRINCIPAL AXES AND MOMENTS OF INERTIA.

SECTION 1.—*The transformation of the equations of motion.*

144.] THE most general motion of a system of material particles of invariable form may be, as we have already proved, resolved into a motion of translation of any point, and a motion of rotation about an axis passing through that point. Generally the position and direction of that axis undergoes a continual change, and the axis may be considered to be constant during only an infinitesimal time-element dt; for it is only in a few cases that the axis is fixed during the whole motion.

From the nature of angular velocities, which have been explained in Chapter II, it is evident that they admit of increase and decrease, either continuously or discontinuously; and, in the general motion of a body, there will generally be a continuous variation of angular velocity, whether the rotation-axis is permanently fixed, or has the same position for only an infinitesimal time-element; and the angular velocity may either increase or decrease. In motion however about an axis, even though no momentum is impressed by any external force, yet certain centrifugal forces are developed which may produce a couple, and thereby cause not only a change of angular velocity but also a change of position of the rotation-axis both in the body and in space; but these changes will be contained within certain limits which are fixed by the principles of conservation of kinetic energy and of moments of momenta; they will generally be periodical; but they can no more exceed these limits than kinetic energy can be acquired without the communication of work. If ever a change takes place outside these limits some force acts to produce that change; and the relation between the change of angular velocity and the producing forces will be the subject of our inquiry in the present Chapter. We shall demonstrate the relation indirectly at first, by a transformation of the preceding equations of motion: but we shall introduce direct

proofs as occasion arises in the course of our inquiry. Thus, while the process of transformation will enable us to conduct our treatise in a systematic form, the direct proofs will remove all intermediate operations, shew the close dependence of our results on first principles, and thus enable us to view the relations as they are in themselves. Thus, as I conceive, we have the advantage of both the analytical and the synthetical processes, of which such admirable examples are given respectively in the Mécanique Analytique of Lagrange, and the Nouvelle Théorie de Rotation of Poinsot.

145.] As we shall have for the most part to consider the changes of angular velocity which take place in an infinitesimal time dt, the position of the rotation-axis will be assumed to be unchanged during that time, because any change which it undergoes will be infinitesimal, and will be expressed in infinitesimals of a higher order than those which express the changes in the angular velocities and are contained in the same equations. Thus if ω is the angular velocity at the time t about an axis whose direction-angles are a, β, γ in reference to axes fixed in space, and $\omega_x, \omega_y, \omega_z$ are the axial components of ω, then

$$\omega = \omega_x \cos a + \omega_y \cos \beta + \omega_z \cos \gamma;$$

$$\therefore \quad \frac{d\omega}{dt} = \frac{d\omega_x}{dt}\cos a + \frac{d\omega_y}{dt}\cos\beta + \frac{d\omega_z}{dt}\cos\gamma$$

$$+ \omega_x \frac{d.\cos a}{dt} + \omega_y \frac{d.\cos\beta}{dt} + \omega_z \frac{d.\cos\gamma}{dt};$$

but since $\omega_x, \omega_y, \omega_z$ are proportional to $\cos a$, $\cos \beta$, $\cos \gamma$, and $(\cos a)^2 + (\cos \beta)^2 + (\cos \gamma)^2 = 1$, the sum of the last three terms is equal to zero; and thus $\frac{d\omega}{dt}$ does not involve any variation of the position of the axis of ω, so far as first differentials are concerned.

This theorem is also evident from the following considerations: since

$$\omega^2 = \omega_x^2 + \omega_y^2 + \omega_z^2;$$

$$\therefore \quad \omega \frac{d\omega}{dt} = \omega_x \frac{d\omega_x}{dt} + \omega_y \frac{d\omega_y}{dt} + \omega_z \frac{d\omega_z}{dt},$$

$$\therefore \quad \frac{d\omega}{dt} = \frac{\omega_x}{\omega}\frac{d\omega_x}{dt} + \frac{\omega_y}{\omega}\frac{d\omega_y}{dt} + \frac{\omega_z}{\omega}\frac{d\omega_z}{dt}$$

$$= \cos a \frac{d\omega_x}{dt} + \cos\beta \frac{d\omega_y}{dt} + \cos\gamma \frac{d\omega_z}{dt},$$

which is the same result as above.

Hence if the axis of ω becomes coincident with one of the coordinate axes, say, the axis of x, not only does ω become ω_x, but $\frac{d\omega}{dt}$ is also equal to $\frac{d\omega_x}{dt}$, for in this case $\alpha = 0$, $\beta = \gamma = \frac{\pi}{2}$, and

$$\omega = \omega_x, \quad \frac{d\omega}{dt} = \frac{d\omega_x}{dt};$$

and similar results are true for the other axes.

These theorems may also be proved from general kinematical principles.

In all cases therefore of an infinitesimal variation of an angular velocity about a given axis, the position of the axis will be taken to be unchanged. This is a theorem of considerable importance in subsequent parts of our work.

146.] If, by the action of an impulsive force, the angular velocity of a body is abruptly changed, or if a body at rest receives a finite angular velocity, we consider only the whole velocity which is communicated to be the effect of the force: we do not inquire into the law of communication, which would assign the rate at which, in successive time-elements, the communication took place, but as the whole process is completed in an infinitesimal time, we take the whole at once.

If however a finite force acts, whereby the angular velocity of the body about the given axis continuously varies, then there are two cases to be considered, according as equal or unequal angular velocities are communicated (or abstracted) in equal time-elements; these two cases corresponding to those of a constant and of a variable force in the linear motion of material particles respectively. Let us first take the case in which equal angular velocities are impressed in equal times. Let Φ be the angular velocity impressed in an unit of time; and let ω be the angular velocity impressed and also expressed in t units of time: then

$$\omega = \Phi t. \tag{1}$$

As equal angular velocities are impressed in equal times, the force which impresses them is called a constant angular force.

If the body moves with an angular velocity Ω, before the force which impresses Φ acts, and if ω is its angular velocity, when the force has acted for t units of time,

$$\omega = \Omega + \Phi t; \tag{2}$$

and if Φ is impressed in a direction contrary to that of Ω,

$$\omega = \Omega - \Phi t. \tag{3}$$

Now from Art. 33, equation (4), $\frac{d\theta}{dt} = \omega$; so that, if θ is the angle through which the body rotates in the time t, and if $\theta = 0$, when $t = 0$, then generally from (2)

$$\frac{d\theta}{dt} = \Omega + \Phi t;$$

$$\therefore \quad \theta = \Omega t + \frac{1}{2}\Phi t^2; \tag{4}$$

which gives the angle through which the body rotates in the time t under the action of a constant angular force.

Next let us suppose unequal angular velocities to be impressed in equal time-elements; then the force is called a variable angular force. Let us however suppose it to be such at the time t, that an angular velocity Φ would be impressed by it in an unit of time, if the force were constant during that unit of time; and to be such at the time $t + dt$, that an angular velocity $\Phi + d\Phi$ would be impressed by it under the same supposition as to constancy; let ω be the angular velocity at the time t, and $\omega + d\omega$ at the time $t + dt$; then, if e is a symbol for a proper fraction, $\Phi + e\,d\Phi$ will express the *mean* or average value of the impressed angular velocity due to an unit of time during dt; and as $d\omega$ is the angular velocity expressed in dt, we have from (1)

$$d\omega = (\Phi + e\,d\Phi)\,dt;$$

and neglecting $d\Phi \times dt$, which is an infinitesimal of the second order,

$$d\omega = \Phi\,dt; \tag{5}$$

$$\therefore \quad \Phi = \frac{d\omega}{dt} = \frac{d}{dt}\frac{d\theta}{dt} = \frac{d^2\theta}{dt^2}, \tag{6}$$

if t is equicrescent; and this supposition we shall make throughout the treatise, unless it is expressly stated that t is not equicrescent.

What we have called an angular force, is a force which produces rotation, and is consequently a couple, of which the moment is given.

Hence, if Φ is given in terms of either θ or t, we can deduce from (6) by means of two integrations the relation between θ and t, and thus determine the angle through which the body rotates in the time t.

As $d\omega$ is an angular velocity, although it is infinitesimal, it is capable of resolution and composition according to the laws

which have been investigated in Chapter II. This observation is important.

Let thus much suffice for angular velocity-increments; we return to the equations for rotatory motion which have been found in Art. 73, with the purpose of expressing them in terms of angular velocities, and angular velocity-increments.

147.] Let us first take equations (35), Art. 73; and replace the linear velocities in them in terms of the angular velocities about the three coordinate axes, these angular velocities being due to the acting forces.

Let us take any point of the body for the origin; and let three rectangular axes fixed in space originate at it: the origin we may consider to be fixed, while we calculate the rotation about an axis through it. We will assume the body to be initially at rest. Let a, β, γ be the direction-angles of the rotation-axis; let Ω be the angular velocity due to the acting forces; let Ω_x, Ω_y, Ω_z be the axial components of Ω; then

$$\left.\begin{array}{c} \Omega_x = \Omega \cos a, \qquad \Omega_y = \Omega \cos \beta, \qquad \Omega_z = \Omega \cos \gamma; \\ \Omega^2 = \Omega_x{}^2 + \Omega_y{}^2 + \Omega_z{}^2. \end{array}\right\} \tag{7}$$

Let L, M, N be the moments of the axial components of the couple of the impressed momenta; so that

$$\left.\begin{array}{l} \text{L} = \Sigma . m(y\text{Z} - z\text{Y}), \\ \text{M} = \Sigma . m(z\text{X} - x\text{Z}), \\ \text{N} = \Sigma . m(x\text{Y} - y\text{X}); \end{array}\right\} \tag{8}$$

then from (35), Art. 73, we have

$$\left.\begin{array}{l} \Sigma . m(y\text{v}_z - z\text{v}_y) = \text{L}, \\ \Sigma . m(z\text{v}_x - x\text{v}_z) = \text{M}, \\ \Sigma . m(x\text{v}_y - y\text{v}_x) = \text{N}. \end{array}\right\} \tag{9}$$

Now, by (72), Art. 53, we have the following values for the axial components of the linear velocity of m, which is due to the angular velocity Ω,

$$\left.\begin{array}{l} \text{v}_x = z\,\Omega_y - y\,\Omega_z = \Omega\,(z \cos \beta - y \cos \gamma), \\ \text{v}_y = x\,\Omega_z - z\,\Omega_x = \Omega\,(x \cos \gamma - z \cos a), \\ \text{v}_z = y\,\Omega_x - x\,\Omega_y = \Omega\,(y \cos a - x \cos \beta); \end{array}\right\} \tag{10}$$

then substituting in (9), we have

$$\left.\begin{array}{l} \Omega\{\quad \cos a\,\Sigma . m(y^2+z^2) - \cos\beta\,\Sigma . m\,xy - \cos\gamma\,\Sigma . m\,zx\} = \text{L}, \\ \Omega\{-\cos a\,\Sigma . m\,xy + \cos\beta\,\Sigma . m(z^2+x^2) - \cos\gamma\,\Sigma . m\,yz\} = \text{M}, \\ \Omega\{-\cos a\,\Sigma . m\,zx - \cos\beta\,\Sigma . m\,yz + \cos\gamma\,\Sigma . m\,(x^2+y^2)\} = \text{N}; \end{array}\right\} \tag{11}$$

Ω, $\cos a$, $\cos \beta$, $\cos \gamma$ having been placed outside the summatory

symbols, because they are the same for all particles of the body. And these three equations are, in terms of the resultant angular velocity and the direction-angles of the rotation-axis, the equivalents of (35), Art. 73; and by these the angular velocity Ω and the position of the instantaneous rotation-axis are to be determined.

Let us multiply them severally by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and add; then

$$\Omega\Sigma.m\{(y^2+z^2)(\cos\alpha)^2+(z^2+x^2)(\cos\beta)^2+(x^2+y^2)(\cos\gamma)^2 \\ -2yz\cos\beta\cos\gamma-2zx\cos\gamma\cos\alpha-2xy\cos\alpha\cos\beta\} \\ = \text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma; \quad (12)$$

which may be expressed as follows;

$$\Omega\Sigma.m\{(z\cos\beta-y\cos\gamma)^2+(x\cos\gamma-z\cos\alpha)^2+(y\cos\alpha-x\cos\beta)^2\} \\ = \text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma; \quad (13)$$

but if r is the perpendicular distance from (x, y, z), the place of m, to the rotation-axis,

$$r^2=(z\cos\beta-y\cos\gamma)^2+(x\cos\gamma-z\cos\alpha)^2+(y\cos\alpha-x\cos\beta)^2; \quad (14)$$

so that (13) becomes

$$\Omega.\Sigma.mr^2 = \text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma; \quad (15)$$

$$\therefore \quad \Omega = \frac{\text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma}{\Sigma.mr^2}; \quad (16)$$

which gives the angular velocity about the instantaneous axis.

148.] The right-hand member of this equation requires accurate and close examination. The numerator of the fraction is the moment of the couple of the impressed momenta of all the particles about the rotation-axis; for L, M, N are the moments of the axial components of the couples of the impressed momenta, and the numerator is the sum of the parts of those axial components which are effective about the rotation-axis. The denominator is the sum of the products of every moving particle and the square of its distance from the rotation-axis: and in the case of a continuous body it becomes the integral of r^2dm, the integration extending over and including all the mass-elements of the body. This quantity is called the *moment of inertia* of the body or of the moving system, relatively to the particular rotation-axis, and the geometrical definition of it is that just given. It appears also from (15) that it is the factor by which the angular velocity Ω is multiplied, and thus equated to the moment of the couple of the impressed momenta about the rotation-axis. This last is the dynamical definition of it.

The name "moment of inertia" has been given for the following reason. Let us compare (15) with the fundamental theory of Art. 257, Vol. III, of the motion of translation of a material particle m, which is acted on by an impulsive force. It appears that if V is the expressed velocity of m, and if P is the momentum impressed by the instantaneous force, then

$$m\text{V} = \text{P}; \tag{17}$$

so that m, which symbolises the mass, is the factor by which V is multiplied, and so equated to the impressed momentum; and as in (15) $\Sigma . mr^2$ is the factor by which Ω is multiplied, and thus equated to the moment of the impressed momentum, so the old mechanicians compared the m in (17) with the $\Sigma . mr^2$ in (15); and as they were wont to say that a body's inertia was proportional to or identical with its mass, so, by an analogy somewhat rough, they called $\Sigma . mr^2$ the moment of inertia. It seems difficult to demonstrate the correctness of the term; but as it is undesirable to introduce a new name, except by urgent necessity, I shall retain the old one, and call $\Sigma . mr^2$ the moment of inertia of the body or system of particles relatively to the rotation-axis. It is a quantity which is evidently of one dimension in mass and of two dimensions in space. The determination of it is the first step in the solution of a problem which depends on the equation (16); and it is otherwise of great importance. Hereafter many properties of moments of inertia will be investigated, and I shall calculate the moments of inertia of bodies and moving systems in many particular cases.

Sometimes the moment of inertia is expressed in the following manner: Let M be the mass of the moving system, and let us suppose the whole system to be condensed into a particle of mass M, at a distance k from the rotation-axis, so that the moment of inertia of the system thus condensed may be the same relatively to the axis as that of the moving system: then, as the moment of inertia of M in this imaginary and condensed state is $\text{M}k^2$, so by our assumption,

$$\text{M}k^2 = \Sigma . r^2 dm; \tag{18}$$

k is called the radius of gyration of the body relatively to the particular rotation-axis.

Hence, if a continuous body is referred to three rectangular axes in space, and if ρ is the density of the particle at (x, y, z),

$$dm = \rho\, dx\, dy\, dz, \tag{19}$$

and the moments of inertia of the body relatively to the three coordinate axes of x, y, and z are severally

$$\iiint \rho(y^2+z^2)\,dx\,dy\,dz, \iiint \rho(z^2+x^2)\,dx\,dy\,dz, \iiint \rho(x^2+y^2)\,dx\,dy\,dz;$$

the integrals being definite, and including all the elements of the body.

149.] As (16) is the fundamental equation of rotation of a body under the action of instantaneous forces, it is worth while to deduce it immediately from the first principles of motion.

For the sake of simplicity, let us take the rotation-axis to be the coordinate axis of z, and suppose the line of action of the impressed momentum to be perpendicular to this axis. Let N be the moment of the couple impressed about the axis of z; let r be the distance of m from the axis of z; and let Ω be the expressed angular velocity, and V the expressed velocity due to the instantaneous force; so that

$$\mathrm{V} = r\Omega; \tag{20}$$

hence the expressed momentum is $m\mathrm{V} = mr\Omega$, of which the moment, relatively to the rotation-axis, is $mr^2\Omega$; so that the excess of the moment of the couple of the impressed momentum over that of the expressed momentum in the case of the particle m is

$$\mathrm{N} - mr^2\Omega;$$

and as by D'Alembert's principle these taken throughout the moving mass are in equilibrium, we have

$$\Sigma.\mathrm{N} - \Sigma.mr^2\Omega = 0. \tag{21}$$

As Ω is the same for all mass-elements, it may be placed outside the sign of summation; also let G be the moment of the couple of the impressed momenta of all the particles, then we have

$$\Omega.\Sigma.mr^2 = \mathrm{G}; \tag{22}$$

$$\therefore \quad \Omega = \frac{\mathrm{G}}{\Sigma.mr^2}; \tag{23}$$

which is the same equation as (16); for if the axis of z is the rotation axis, then in (16) $\cos\alpha = \cos\beta = 0$, $\cos\gamma = 1$; and we have

$$\Omega = \frac{\mathrm{N}}{\Sigma.mr^2}; \tag{24}$$

where N is the moment of the couple of the impressed momenta about the rotation-axis, and is the same as G in (23).

Equations (11) are so close on the first principles of motion, as explained in Art. 53 and of the measure of couples, that further explanation is unnecessary.

150.] The direction-cosines of the instantaneous-axis are proportional to Ω_x, Ω_y, Ω_z; and these latter quantities may be thus found:

Let us once for all make the following abbreviations; let

$$\Sigma.m(y^2+z^2) = A, \quad \Sigma.m(z^2+x^2) = B, \quad \Sigma.m(x^2+y^2) = C; \quad (25)$$

$$\Sigma.mx^2 = A', \quad \Sigma.my^2 = B', \quad \Sigma.mz^2 = C'; \quad (26)$$

$$\Sigma.myz = D, \quad \Sigma.mzx = E, \quad \Sigma.mxy = F. \quad (27)$$

These nine quantities are of great importance in the following investigations, and the substitutions which are here made will be continued throughout the treatise.

A, B, C are the sums of the products of each particle of the moving mass and the square of its distance from the axes of x, y, z respectively: in other words, A, B, C are the moments of inertia of the moving system relatively to the axes of x, y, z respectively.

A′, B′, C′, are severally the sum of the products of each particle and the square of the x-, y-, z-coordinate of its place.

D, E, F will have full explanation in the following section, although (27) evidently exhibit their meaning.

I may observe, that

$$A = B'+C', \quad B = C'+A', \quad C = A'+B'; \quad (28)$$

$$A' = \frac{B+C-A}{2}, \quad B' = \frac{C+A-B}{2}, \quad C' = \frac{A+B-C}{2}; \quad (29)$$

whereby A, B, C are severally determined in terms of A′, B′, C′; and A′, B′, C′ in terms of A, B, C. Also

$$A+A' = B+B' = C+C' = A'+B'+C' = I \text{ (say)}. \quad (30)$$

Thus $B+C-A$, $C+A-B$, $A+B-C$ are all positive quantities; as are also $BC-D^2$, $CA-E^2$, $AB-F^2$.

151.] Now, using in (11) these abbreviating symbols, we have, by means of (7),

$$\left.\begin{aligned} A\Omega_x - F\Omega_y - E\Omega_z &= L, \\ -F\Omega_x + B\Omega_y - D\Omega_z &= M, \\ -E\Omega_x - D\Omega_y + C\Omega_z &= N; \end{aligned}\right\} \quad (31)$$

whence

$$\left.\begin{aligned} \Omega_x &= \frac{L(BC-D^2)+M(DE+CF)+N(DF+BE)}{ABC-AD^2-BE^2-CF^2-2DEF}, \\ \Omega_y &= \frac{L(ED+CF)+M(CA-E^2)+N(EF+AD)}{ABC-AD^2-BE^2-CF^2-2DEF}, \\ \Omega_z &= \frac{L(FD+BE)+M(FE+AD)+N(AB-F^2)}{ABC-AD^2-BE^2-CF^2-2DEF}; \end{aligned}\right\} \quad (32)$$

from which, and from (7), a, β, γ may be determined. The equations to the instantaneous-axis are

$$\frac{x}{\Omega_x} = \frac{y}{\Omega_y} = \frac{z}{\Omega_z}. \tag{33}$$

152.] The form of the preceding expressions at once suggests a geometrical interpretation, and it is desirable to work it out because further and important mechanical conceptions will be thereby exhibited which it is desirable to have present to our minds.

Let us take equation (12) and replace the coefficients by their equivalent symbols which are given in (25) and (27). Let K be the moment of the impressed couple about the instantaneous rotation-axis, so that

$$\text{L} \cos a + \text{M} \cos \beta + \text{N} \cos \gamma = \text{K}; \tag{34}$$

then from (12) we have

$$\text{A} (\cos a)^2 + \text{B} (\cos \beta)^2 + \text{C} (\cos \gamma)^2 - 2\,\text{D} \cos \beta \cos \gamma - 2\,\text{E} \cos \gamma \cos a - 2\,\text{F} \cos a \cos \beta = \frac{\text{K}}{\Omega}. \tag{35}$$

Along the instantaneous-axis from the origin take a distance r, such that

$$\frac{\mu}{r^2} = \frac{\text{K}}{\Omega}, \tag{36}$$

where μ is an undetermined constant, which may be unity, if such a value is convenient. Let (x, y, z) be the extremity of r, then

$$\frac{x}{\cos a} = \frac{y}{\cos \beta} = \frac{z}{\cos \gamma} = r = \left(\frac{\mu \Omega}{\text{K}}\right)^{\frac{1}{2}}; \tag{37}$$

and from (35),

$$\text{A} x^2 + \text{B} y^2 + \text{C} z^2 - 2\,\text{D} yz - 2\,\text{E} zx - 2\,\text{F} xy = \mu. \tag{38}$$

As A, B, C are all positive, and are related to D, E, F by the inequalities given in Art. 150, this equation represents an ellipsoid; and also a series of concentric and coaxial ellipsoids, since μ is undetermined.

153.] Since Ω_x, Ω_y, Ω_z are proportional to $\cos a$, $\cos \beta$, $\cos \gamma$, that is to x, y, z, it follows from (31) that L, M, N are proportional to $\text{A}x - \text{F}y - \text{E}z$, $-\text{F}x + \text{B}y - \text{D}z$, $-\text{E}x - \text{D}y + \text{C}z$; that is, to the x-, y-, z-partial differential coefficients of (38); and as these last are proportional to the direction-cosines of the normal to the plane which touches (38) at the point (x, y, z), it follows that the axis of the impressed couple is perpendicular to the plane which touches the ellipsoid at (x, y, z); in other words, the central

radius vector to the point (x, y, z) is conjugate to the plane of the couple; but this radius vector is the instantaneous-axis; hence we have the following important theorem: If an impulsive couple, such as that due to a blow, is impressed on a body or a system of material particles whereby rotation is produced, the instantaneous-axis is the axis conjugate to the plane of the impressed couple with respect to the ellipsoid (38).

Also since

$$\Omega = \frac{\kappa}{\mu} r^2, \tag{39}$$

the instantaneous angular velocity is proportional to the square of the radius vector of the ellipsoid which coincides with the instantaneous-axis.

Hence also inversely if a body or a system of particles has rotation about an axis at a given time, the system may be brought to rest as to rotation by means of an impulsive couple whose plane is conjugate to the instantaneous axis with respect to the ellipsoid (38).

As μ does not enter into the equations to an axis which is conjugate to a given plane, or into the equation to a plane which is conjugate to a given axis, any value that is convenient may be given to it in (38). The ellipsoid is evidently of great importance in these enquiries; its form and shape depend on the coefficients, and these are again dependent on the configuration of the material system.

As the ellipsoid has three principal axes, the conjugate planes to which are perpendicular to them, it follows that there are three lines at every point mutually perpendicular to each other, which being axes of impulsive couples are also corresponding instantaneous axes; and no other axis has this property; hence, if an impulsive couple has any other line as its axis, it produces rotation about a line which does not coincide with its own axis. If the axis of the couple is a principal axis of the ellipsoid, and if κ is the moment of the couple, then from (39)

$$\Omega = \frac{\kappa}{\mu} r^2, \tag{40}$$

where r is the length of the coincident principal axis; so that the instantaneous angular velocity is a maximum, a mean, or a minimum, according as r has like values. In all these respects the principal axes form an unique system.

If the equation to the ellipsoid given in (38) is transformed, according to the process of Art. 6, to principal axes, it takes the form

$$\text{A}x^2+\text{B}y^2+\text{C}z^2 = \mu, \tag{41}$$

where A, B, C are not the same as in (38), but are functions of them and of D, E, F, as determined in Art. 6. They are also the moments of inertia of the system about the three new axes of x, y, z; and if the moment of inertia is greatest about the z-axis and least about the x-axis, so that $\text{C} > \text{B} > \text{A}$, then for an impulsive couple of given intensity K, which is coaxial with a principal axis of the ellipsoid, the resulting angular velocity Ω is greatest about the x-axis, and least about the z-axis; being $\frac{\text{K}}{\text{A}}$ about the former and $\frac{\text{K}}{\text{C}}$ about the latter; it is $\frac{\text{K}}{\text{B}}$ about the mean axis of y; and all other angular velocities are contained within these limits.

154.] Next let us consider equations (38), Art. 73. Let any point of the body be the origin; and at it let three rectangular coordinate axes fixed in space originate; and let us consider the body at the time t, and during dt, so that the rotation-axis may be considered fixed during that time. Let α, β, γ be the direction-angles of the rotation-axis, and let ω be the angular velocity about it at the time t; of which let $\omega_x, \omega_y, \omega_z$ be the axial-components; so that

$$\omega^2 = \omega_x^{\ 2} + \omega_y^{\ 2} + \omega_z^{\ 2};$$

$$\omega_x = \omega\cos\alpha, \qquad \omega_y = \omega\cos\beta, \qquad \omega_z = \omega\cos\gamma. \tag{42}$$

Let the moments of the axial-components of the couples of the impressed momentum-increments at the time t be L, M, N; so that

$$\left.\begin{aligned} \Sigma.m(y\text{Z}-z\text{Y}) &= \text{L}, \\ \Sigma.m(z\text{X}-x\text{Z}) &= \text{M}, \\ \Sigma.m(x\text{Y}-y\text{X}) &= \text{N}. \end{aligned}\right\} \tag{43}$$

Then, taking the form of the equations as given in (40), Art. 74, we have

$$\left.\begin{aligned} \frac{d}{dt}\Sigma.m\Big(y\frac{dz}{dt}-z\frac{dy}{dt}\Big) &= \text{L}, \\ \frac{d}{dt}\Sigma.m\Big(z\frac{dx}{dt}-x\frac{dz}{dt}\Big) &= \text{M}, \\ \frac{d}{dt}\Sigma.m\Big(x\frac{dy}{dt}-y\frac{dx}{dt}\Big) &= \text{N}; \end{aligned}\right\} \tag{44}$$

and from (72), Art. 53, we have

$$\left.\begin{aligned}\frac{dx}{dt} &= \omega\,(z\cos\beta - y\cos\gamma),\\ \frac{dy}{dt} &= \omega\,(x\cos\gamma - z\cos\alpha),\\ \frac{dz}{dt} &= \omega\,(y\cos\alpha - x\cos\beta);\end{aligned}\right\} \tag{45}$$

whence we have also

$$x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt} = 0; \tag{46}$$

$$\cos\alpha\frac{dx}{dt} + \cos\beta\frac{dy}{dt} + \cos\gamma\frac{dz}{dt} = 0; \tag{47}$$

(46) shewing that m at (x, y, z) remains at the same distance from the origin during dt, and (47) shewing that the line of motion of m during dt is perpendicular to the instantaneous axis.

Now introducing these substitutions and conditions,

$$\begin{aligned}\frac{d}{dt}\Sigma.m\Big(y\frac{dz}{dt} - z\frac{dy}{dt}\Big) &= \frac{d}{dt}\Sigma.m\omega\,\{(y^2+z^2)\cos\alpha - xy\cos\beta - xz\cos\gamma\},\\ &= \frac{d}{dt}\Sigma.m\omega\,\{(x^2+y^2+z^2)\cos\alpha - x\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\},\\ &= \Sigma.m\frac{d\omega}{dt}\{(x^2+y^2+z^2)\cos\alpha - x\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\}\\ &\quad - \Sigma.m\omega^2\{(x\cos\alpha + y\cos\beta + z\cos\gamma)(z\cos\beta - y\cos\gamma)\}.\end{aligned}$$

Therefore, substituting in (44), and placing ω and $\frac{d\omega}{dt}$ outside the sign of summation, as they, as well as the direction-cosines of the rotation-axis, are the same for all the particles, we have

$$\begin{aligned}&\frac{d\omega}{dt}\cos\alpha\,\Sigma.m\,(x^2+y^2+z^2) - \frac{d\omega}{dt}\Sigma.mx\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\\ &\qquad - \omega^2\Sigma.m\{(x\cos\alpha + y\cos\beta + z\cos\gamma)(z\cos\beta - y\cos\gamma)\} = \text{L}; \quad (48)\\ &\frac{d\omega}{dt}\cos\beta\,\Sigma.m\,(x^2+y^2+z^2) - \frac{d\omega}{dt}\Sigma.my\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\\ &\qquad - \omega^2\Sigma.m\{(x\cos\alpha + y\cos\beta + z\cos\gamma)(x\cos\gamma - z\cos\alpha)\} = \text{M}; \quad (49)\\ &\frac{d\omega}{dt}\cos\gamma\,\Sigma.m\,(x^2+y^2+z^2) - \frac{d\omega}{dt}\Sigma.mz\,(x\cos\alpha + y\cos\beta + z\cos\gamma)\\ &\qquad - \omega^2\Sigma.m\{(x\cos\alpha + y\cos\beta + z\cos\gamma)(y\cos\alpha - x\cos\beta)\} = \text{N}. \quad (50)\end{aligned}$$

The complete solution of the problem requires that ω, $\cos\alpha$, $\cos\beta$, $\cos\gamma$ should be expressed in terms of t; L, M, and N being functions of these five quantities: now, as a relation exists between α, β, γ, the preceding equations contain only three inde-

pendent quantities which are to be expressed in terms of t; for this purpose the number of the equations is sufficient; but as they generally do not admit of integration, we can apply them only to particular cases, and have recourse to such artifices as a particular problem suggests.

155.] Let us multiply (48), (49), and (50), severally, by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and let us add them; then

$$\frac{d\omega}{dt}\Sigma . m\{x^2+y^2+z^2-(x\cos\alpha+y\cos\beta+z\cos\gamma)^2\}$$
$$= \text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma; \quad (51)$$

but if r is the perpendicular distance from (x, y, z), the place of m at the time t, on the rotation-axis,

$$r^2 = x^2+y^2+z^2-(x\cos\alpha+y\cos\beta+z\cos\gamma)^2;$$

so that (51) becomes

$$\frac{d\omega}{dt}\Sigma . mr^2 = \text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma; \quad (52)$$

$$\therefore \quad \frac{d\omega}{dt} = \frac{\text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma}{\Sigma . mr^2}; \quad (53)$$

which gives the angular velocity-increment about the rotation-axis which is due to the impressed momentum-increments.

Now this equation, like (16), requires careful attention; it is that from which, by integration, the increase or diminution of the angular velocity in a finite time is to be found. The numerator of the right-hand member is the moment of the couple of the impressed momentum-increments of all the particles relatively to the rotation-axis; for L, M, N are the axial components of the moments of these couples; and $\text{L}\cos\alpha+\text{M}\cos\beta+\text{N}\cos\gamma$ is the sum of the parts of these axial components which are effective about the rotation-axis. The denominator is the moment of inertia of the body or moving mass relatively to the rotation-axis; and the remarks made in Art. 148 are applicable equally to that and this case.

156.] If the rotation-axis of the body has the same position during the whole motion, either because two or more points in it are fixed, or because it bears a certain relation to the particles of the moving mass, then α, β, γ are constant, and are known, and the numerator of (53) is given at the time t; and, if the integration can be performed, the angular velocity will be determined. If, however, the position of the rotation-axis changes continuously from time to time, so that it can be considered

fixed only for an infinitesimal time-element dt, then α, β, γ are functions of t, and equation (53) cannot generally be integrated as it stands. In this case we must return to equations (48), (49), (50); in them let us replace $\omega\cos\alpha$, $\omega\cos\beta$, $\omega\cos\gamma$, severally, by ω_x, ω_y, ω_z, and use the abbreviating symbols of Art. 150; then they become

$$\left.\begin{aligned} A\frac{d\omega_x}{dt}+(C-B)\,\omega_y\omega_z-D(\omega_y^2-\omega_z^2)-E\left(\frac{d\omega_z}{dt}+\omega_x\omega_y\right)-F\left(\frac{d\omega_y}{dt}-\omega_x\omega_z\right)&=L;\\ B\frac{d\omega_y}{dt}+(A-C)\,\omega_z\omega_x-E(\omega_z^2-\omega_x^2)-F\left(\frac{d\omega_x}{dt}+\omega_y\omega_z\right)-D\left(\frac{d\omega_z}{dt}-\omega_y\omega_x\right)&=M;\\ C\frac{d\omega_z}{dt}+(B-A)\,\omega_x\omega_y-F(\omega_x^2-\omega_y^2)-D\left(\frac{d\omega_y}{dt}+\omega_z\omega_x\right)-E\left(\frac{d\omega_x}{dt}-\omega_z\omega_y\right)&=N;\end{aligned}\right\}\quad(54)$$

from which three equations ω_x, ω_y, ω_z are to be determined in terms of t: the integration however is beyond our power, except in a very few special cases, which we shall consider hereafter.

157.] As these last are the fundamental equations of rotation of a solid body, or of a material system of invariable form, and will be employed in all our subsequent investigations, they require close examination. We have arrived at them by transformations from expressions involving velocities of translation into those involving angular velocities. I will now shew that they may be found more directly by D'Alembert's principle: and, in the course of the inquiry, I shall dissect the equations, and shew the independent origin of their several terms; and I shall also exhibit other properties of these equations of rotating rigid systems besides those of the preceding pages.

As the particles of the moving system are in a state of relative rest, the moments of the forces acting on them, relatively to every and any axial line, satisfy the conditions of statical equilibrium; and thus, by D'Alembert's principle, the moments of the stresses, which arise from the excess of the impressed over the expressed momentum-increments, must satisfy the laws of equilibrium when they are taken throughout the whole system. In reference to any axis and for any one particle m, we have the following moments: (1) the moment of the impressed momentum-increments; (2) the moment of the expressed angular momentum-increment; (3) the moment of the centrifugal force which is due to the motion of the body about the instantaneous axis at the time t; and the moment of the stress, which is effective at m, is the excess of the moments of the impressed

momentum-increment and of the centrifugal force over the moment of the expressed momentum-increment; and the moment of all these stresses vanishes for every axis.

Let us employ the same notation as heretofore; m is the type-particle of the system; (x, y, z) is its place at the time t; ω is the angular velocity at the time t about the instantaneous axis, of which ω_x, ω_y, ω_z are the axial components; α, β, γ are the direction-angles of the instantaneous axis; r is the perpendicular distance from (x, y, z) on the instantaneous axis; L, M, N are the axial components of the moments of the couples of the impressed momentum-increments on all the particles of the system.

158.] Let X′, Y′, Z′ be the axial components of the expressed momentum-increments of all the particles of the system due to the increments of the angular velocities at the time t; and let P′ be the resultant of these; then, from Art. 53, we have

$$\left.\begin{aligned} \Sigma.m\left(z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}\right) &= \text{X}', \\ \Sigma.m\left(x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}\right) &= \text{Y}', \\ \Sigma.m\left(y\frac{d\omega_x}{dt}-x\frac{d\omega_y}{dt}\right) &= \text{Z}'; \end{aligned}\right\} \tag{55}$$

and
$$\text{P}'^2 = \text{X}'^2+\text{Y}'^2+\text{Z}'^2. \tag{56}$$

If the origin moves, P′ is proportional to the expressed velocity-increment of it, which is due to the increments of the angular velocities at the time t; and if the origin is absolutely fixed, P′ produces a pressure on it.

Let L′, M′, N′ be the axial components of the moments of the couples which arise from these expressed momentum-increments; and let G′ be the resultant moment of these; then, as the axial components of the expressed momentum-increment of m are

$$m\left(z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}\right), \quad m\left(x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}\right), \quad m\left(y\frac{d\omega_x}{dt}-x\frac{d\omega_y}{dt}\right),$$

so the axial components of the moment of the couple of this expressed-momentum of m are respectively

$$\left.\begin{aligned} &\frac{d\omega_x}{dt}m(y^2+z^2)-\frac{d\omega_y}{dt}mxy-\frac{d\omega_z}{dt}mxz, \\ &\frac{d\omega_y}{dt}m(z^2+x^2)-\frac{d\omega_z}{dt}myz-\frac{d\omega_x}{dt}myx, \\ &\frac{d\omega_z}{dt}m(x^2+y^2)-\frac{d\omega_x}{dt}mzx-\frac{d\omega_y}{dt}mzy; \end{aligned}\right\} \tag{57}$$

and taking the aggregates of these for all the particles of the system, and using the abridging symbols of Art. 150, we have

$$\left.\begin{aligned} \text{A}\frac{d\omega_x}{dt} - \text{F}\frac{d\omega_y}{dt} - \text{E}\frac{d\omega_z}{dt} &= \text{L}', \\ \text{B}\frac{d\omega_y}{dt} - \text{D}\frac{d\omega_z}{dt} - \text{F}\frac{d\omega_x}{dt} &= \text{M}', \\ \text{C}\frac{d\omega_z}{dt} - \text{E}\frac{d\omega_x}{dt} - \text{D}\frac{d\omega_y}{dt} &= \text{N}'; \end{aligned}\right\} \quad (58)$$

$$\text{G}'^2 = \text{L}'^2 + \text{M}'^2 + \text{N}'^2;$$

which are indeed the same expressions as (31), Art. 151, and give the moment of the couple of the expressed angular momentum-increments of the system.

159.] Again, let X'', Y'', Z'' be the axial components of the momentum-increments of all the particles which arise from the centrifugal forces; and let P'' be their resultant. As ω is the angular velocity of the system about the instantaneous axis at the time t, and as r is the perpendicular distance to that axis from (x, y, z) the place of m, $m\omega^2 r$ is the centrifugal force of m, the line of action of which lies along r. Now as r is drawn from (x, y, z) at right angles to (α, β, γ) which passes through the origin, the direction-cosines of r are

$$\frac{x - \cos\alpha(x\cos\alpha + y\cos\beta + z\cos\gamma)}{r}, \quad \frac{y - \cos\beta(x\cos\alpha + y\cos\beta + z\cos\gamma)}{r},$$
$$\frac{z - \cos\gamma(x\cos\alpha + y\cos\beta + z\cos\gamma)}{r};$$

and therefore the axial components of the momentum-increment of m, due to the centrifugal force, are respectively

$$\left.\begin{aligned} &m\omega^2\{x - \cos\alpha(x\cos\alpha + y\cos\beta + z\cos\gamma)\}, \\ &m\omega^2\{y - \cos\beta(x\cos\alpha + y\cos\beta + z\cos\gamma)\}, \\ &m\omega^2\{z - \cos\gamma(x\cos\alpha + y\cos\beta + z\cos\gamma)\}, \end{aligned}\right\} \quad (59)$$

the tendency of these forces being to increase x, y, z as t increases: and, taking the aggregate of these for all the particles of the system,

$$\left.\begin{aligned} \Sigma.m\{\omega^2 x - \omega_x(x\omega_x + y\omega_y + z\omega_z)\} &= \text{X}'', \\ \Sigma.m\{\omega^2 y - \omega_y(x\omega_x + y\omega_y + z\omega_z)\} &= \text{Y}'', \\ \Sigma.m\{\omega^2 z - \omega_z(x\omega_x + y\omega_y + z\omega_z)\} &= \text{Z}''; \end{aligned}\right\} \quad (60)$$

and $$\text{P}''^2 = \text{X}''^2 + \text{Y}''^2 + \text{Z}''^2.$$

If the origin moves, P'' is proportional to the impressed velocity-increment of it, which is due to the centrifugal forces of all the

particles; and if the origin is absolutely fixed, P″ produces a pressure on it.

Let L″, M″, N″ be the axial components of the moments of the couples which arise from these centrifugal forces; and let G″ be the resultant moment of them; then as the axial components of the moment of the couple which arises from the centrifugal force of m are, by reason of (59),

$$\left.\begin{array}{l} m\{(x\omega_x+y\,\omega_y+z\,\omega_z)(z\,\omega_y-y\,\omega_z)\}, \\ m\{(x\omega_x+y\,\omega_y+z\,\omega_z)(x\,\omega_z-z\,\omega_x)\}, \\ m\{(x\omega_x+y\,\omega_y+z\,\omega_z)(y\,\omega_x-x\,\omega_y)\}; \end{array}\right\} \quad (61)$$

so taking the aggregates of these for all the particles of the system, and using the abridging symbols of Art. 150, we have

$$\left.\begin{array}{l} \mathrm{D}\,(\omega_y{}^2-\omega_z{}^2)+\mathrm{E}\,\omega_x\omega_y-\mathrm{F}\,\omega_x\omega_z+(\mathrm{B}-\mathrm{C})\,\omega_y\omega_z = \mathrm{L}'', \\ \mathrm{E}\,(\omega_z{}^2-\omega_x{}^2)+\mathrm{F}\,\omega_y\,\omega_z-\mathrm{D}\,\omega_y\omega_x+(\mathrm{C}-\mathrm{A})\,\omega_z\omega_x = \mathrm{M}'', \\ \mathrm{F}\,(\omega_x{}^2-\omega_y{}^2)+\mathrm{D}\,\omega_z\omega_x-\mathrm{E}\,\omega_z\omega_y+(\mathrm{A}-\mathrm{B})\,\omega_x\omega_y = \mathrm{N}''; \end{array}\right\} \quad (62)$$

and $$\mathrm{G}''^2 = \mathrm{L}''^2+\mathrm{M}''^2+\mathrm{N}''^2.$$

The couple, of which G″ is the moment, is called the centrifugal couple.

I would also observe that (62) may be expressed symmetrically in the following forms

$$\left.\begin{array}{l} \omega_y\,\omega_z\left\{\dfrac{-\mathrm{F}\omega_x+\mathrm{B}\,\omega_y-\mathrm{D}\omega_z}{\omega_y}-\dfrac{-\mathrm{E}\,\omega_x-\mathrm{D}\omega_y+\mathrm{C}\omega_z}{\omega_z}\right\} = \mathrm{L}'', \\ \omega_z\,\omega_x\left\{\dfrac{-\mathrm{E}\,\omega_x-\mathrm{D}\,\omega_y+\mathrm{C}\,\omega_z}{\omega_z}-\dfrac{\mathrm{A}\omega_x-\mathrm{F}\omega_y-\mathrm{E}\omega_z}{\omega_x}\right\} = \mathrm{M}'', \\ \omega_x\,\omega_y\left\{\dfrac{\mathrm{A}\,\omega_x-\mathrm{F}\,\omega_y-\mathrm{E}\,\omega_z}{\omega_x}-\dfrac{-\mathrm{F}\,\omega_x+\mathrm{B}\,\omega_y-\mathrm{D}\,\omega_z}{\omega_y}\right\} = \mathrm{N}''; \end{array}\right\} \quad (63)$$

and introducing the notation of Art. 94, we have

$$\left.\begin{array}{l} h_2\,\omega_z-h_3\,\omega_y = \mathrm{L}'', \\ h_3\,\omega_x-h_1\,\omega_z = \mathrm{M}'', \\ h_1\,\omega_y-h_2\,\omega_x = \mathrm{N}''; \end{array}\right\}$$

$$\therefore\quad h_1\mathrm{L}''+h_2\mathrm{M}''+h_3\mathrm{N}'' = 0;$$

that is, the axis of the centrifugal couple is perpendicular to the axis of the couples of the momenta.

From (60) and (63) it follows that

$$\left.\begin{array}{l} \mathrm{X}''\,\omega_x+\mathrm{Y}''\,\omega_y+\mathrm{Z}''\,\omega_z = 0, \\ \mathrm{L}''\,\omega_x+\mathrm{M}''\,\omega_y+\mathrm{N}''\,\omega_z = 0; \end{array}\right\}$$

so that the instantaneous axis of rotation is perpendicular to both the line of action of the resultant of translation, and to the axis of the couple, which arise from the centrifugal forces of all

the particles, and is therefore perpendicular to the plane which contains the two lines.

160.] Now, as we have already observed, the moment of the couple of the expressed momentum-increment is equal to the moment of the couple of the impressed momentum-increment together with that of the centrifugal couple: and this equality is true relatively to any rotation-axis, so that for the coordinate-axes of x, y, z we have respectively

$$\text{L}' = \text{L} + \text{L}'', \quad \text{M}' = \text{M} + \text{M}'', \quad \text{N}' = \text{N} + \text{N}''; \tag{64}$$

and therefore we have

$$\left.\begin{aligned} \text{A}\frac{d\omega_x}{dt} + (\text{C}-\text{B})\omega_y\omega_z - \text{D}(\omega_y^2-\omega_z^2) - \text{E}\left(\frac{d\omega_z}{dt}+\omega_x\omega_y\right) - \text{F}\left(\frac{d\omega_y}{dt}-\omega_x\omega_z\right) &= \text{L}, \\ \text{B}\frac{d\omega_y}{dt} + (\text{A}-\text{C})\omega_z\omega_x - \text{E}(\omega_z^2-\omega_x^2) - \text{F}\left(\frac{d\omega_x}{dt}+\omega_y\omega_z\right) - \text{D}\left(\frac{d\omega_z}{dt}-\omega_y\omega_x\right) &= \text{M}, \\ \text{C}\frac{d\omega_z}{dt} + (\text{B}-\text{A})\omega_x\omega_y - \text{F}(\omega_x^2-\omega_y^2) - \text{D}\left(\frac{d\omega_y}{dt}+\omega_z\omega_x\right) - \text{E}\left(\frac{d\omega_x}{dt}-\omega_z\omega_y\right) &= \text{N}; \end{aligned}\right\} \tag{65}$$

which are the equations (54). By the preceding process therefore they are, as it were, dissected, and the meaning and origin of the several terms are traced out; and those which arise from the expressed momentum-increments are distinguished from those which arise from the centrifugal forces.

If the system is at rest when the forces begin to act, then at that instant $\omega = 0$, and $\text{L}'' = \text{M}'' = \text{N}'' = 0$; and the equations are reduced to forms identical with (31), Art. 151, where of course $\frac{d\omega_x}{dt}, \frac{d\omega_y}{dt}, \frac{d\omega_z}{dt}$ take the places of $\Omega_x, \Omega_y, \Omega_z$.

161.] Before we leave this part of our subject it is fit to explain the form which the expressions for the axial components of the acceleration take when they are expressed in terms of angular velocity and angular velocity-increments. The origin, through which the rotation-axis passes and relatively to which the velocity of m as expressed in the equations (45), Art. 154, is estimated, is considered not to change its place during the time dt, so that the velocity and velocity-increment of m are due to the rotation only. If we differentiate (45), Art. 154, we have

$$\frac{d^2x}{dt^2} = z\frac{d\omega_y}{dt} - y\frac{d\omega_z}{dt} - \{\omega^2 x - \omega_x(x\omega_x + y\omega_y + z\omega_z)\}, \tag{66}$$

and similar values for $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$; so that the axial components

of the expressed momentum-increments relatively to the origin are

$$\left.\begin{aligned}&\Sigma.m\left\{z\frac{d\omega_y}{dt}-y\frac{d\omega_z}{dt}\right\}-\Sigma.m\left\{\omega^2x-\omega_x(x\omega_x+y\omega_y+z\omega_z)\right\},\\ &\Sigma.m\left\{x\frac{d\omega_z}{dt}-z\frac{d\omega_x}{dt}\right\}-\Sigma.m\left\{\omega^2y-\omega_y(x\omega_x+y\omega_y+z\omega_z)\right\},\\ &\Sigma.m\left\{y\frac{d\omega_x}{dt}-x\frac{d\omega_y}{dt}\right\}-\Sigma.m\left\{\omega^2z-\omega_z(x\omega_x+y\omega_y+z\omega_z)\right\}.\end{aligned}\right\} \quad (67)$$

These expressions are evidently $X'-X''$, $Y'-Y''$, $Z'-Z''$, as determined in Articles 158 and 159, so that the first two terms in each arise from the expressed angular velocity-increments, and the latter terms from the centrifugal forces. Now the excess of the corresponding impressed momentum-increments over these quantities evidently acts at the origin, and produces either a motion of it, if it is capable of motion, or a pressure on it if it is fixed; and the circumstances of each can be determined from the preceding expressions.

162.] Thus if the origin is fixed; if P is the pressure at it, and λ, μ, ν are the direction-angles of its line of action,

$$\left.\begin{aligned}P\cos\lambda &= \Sigma.mX - X' + X'',\\ P\cos\mu &= \Sigma.mY - Y' + Y'',\\ P\cos\nu &= \Sigma.mZ - Z' + Z''.\end{aligned}\right\} \quad (68)$$

If the origin, fixed or moveable, is the mass-centre,

and $$X' = Y' = Z' = 0, \quad X'' = Y'' = Z'' = 0,$$

$$P = \frac{\Sigma.mX}{\cos\lambda} = \frac{\Sigma.mY}{\cos\mu} = \frac{\Sigma.mZ}{\cos\nu} = \{(\Sigma.mX)^2+(\Sigma.mY)^2+(\Sigma.mZ)^2\}^{\frac{1}{2}},$$

so that the pressure at the origin is, as to intensity and line of action, the same as the resultant of the impressed momentum-increments.

163.] In all cases the equations (31) and (54) admit of great simplification. It will have been observed that the equations of translation of a system of material particles, viz. (34) and (37), Art. 73, are much simplified if the mass-centre is taken for the origin, as we have shewn in section 2 of the preceding chapter; equations (60) and (62) in Arts. 82 and 83 are more simple than (34) and (37) of Art. 73. It does not however appear thus far that any simplification is hereby introduced into the forms of the equations of motion of rotation; (61) and (63), in Arts. 82 and 83, are exactly the same in form as (35) and (38)

of Art. 73. Neither does it appear that any change of axes will generally introduce a further simplification into the equations of motion of translation; it may do so in a particular case, because $\Sigma . m\text{X}$, $\Sigma . m\text{Y}$, $\Sigma . m\text{Z}$ may then take simple forms. In the equations of rotation however it is otherwise. Consider the equations (31), which are equivalent to (35), and equations (54), which are equivalent to (38), of Art. 73; they contain the quantities $\Sigma . mx^2$, $\Sigma . my^2$, $\Sigma . mz^2$, $\Sigma . myz$, $\Sigma . mzx$, $\Sigma . mxy$; and these are dependent on the position of the coordinate axes relative to and in the body. They will be determined by the ordinary processes of summation, and of integration if the moving mass is a continuous body. Now thus far the position of the coordinate axes, to which the moving system is referred, has not been determined; it is fixed neither in the body nor in space. Henceforward we shall suppose a system to be fixed in the body and to move with it, and to have a particular position relatively to the body, which we shall determine with the view of simplifying the preceding equations (31), (32), and (54). By this method we shall investigate the angular velocities of the body about three axes fixed in the moving body; and we can thence determine the angular velocities about three axes fixed in space by means of the equations (87), Art. 57; and $\frac{d\theta}{dt}, \frac{d\phi}{dt}, \frac{d\psi}{dt}$ may be determined by means of (123), (124), (125), of Art. 64. By either process the position of the body in space at the time t will be determined.

164.] Let us examine the coefficients in (31) and in (54) of the angular velocities and of their t-differentials; and let us suppose the moving mass to have volume of three dimensions. Whatever is the system of coordinate axes, it is evident that it cannot be such that generally either $\Sigma . mx^2 = 0$, or $\Sigma . my^2 = 0$, or $\Sigma . mz^2 = 0$; because each of these expressions is the sum of the products of the mass-element and of a quantity which is necessarily positive. Thus, A, B, C, A′, B′, C′, defined as they are in (25) and (26) of Art. 150, are always positive quantities for masses whose volume is of three dimensions: in plates of infinitesimal thickness, if the surface of the plate is taken for the plane of (x, y), $z = 0$ for every element; and therefore $\Sigma . mz^2 = 0$: and in straight wires or rods, of which the transverse section is an infinitesimal area, if the axis of x lies along the rod, $y = z = 0$

for every mass-element, and consequently $\Sigma . m y^2 = \Sigma . m z^2 = 0$; in all other cases A, B, C, and A′, B′, C′ are positive quantities. In reference however to D, E, F, which are the symbols for $\Sigma . m y z$, $\Sigma . m z x$, $\Sigma . m x y$ respectively, the coordinate-axes may have a position such that

$$\Sigma . m y z = \Sigma . m z x = \Sigma . m x y = 0, \tag{69}$$

or that one or two of them may be zero; because in the series, the sums of which are represented by these abridging symbols, some of the terms may be negative and others may be positive, and the result of the whole may be zero.

Thus, for instance, let us suppose an elliptical plate of infinitesimal thickness to be referred to the centre as origin, and to a system of rectangular coordinate-axes, of which the axes of x and y are coincident with the major and minor axes of the ellipse, and that of z is perpendicular to the plane of the elliptic plate. Then, as $z = 0$ for all the mass-elements of the plate, it is evident that

$$\Sigma . m y z = \Sigma . m z x = 0;$$

and since for an element of the plate at (x, y) there is always an equal element at $(-x, y)$, it is plain that $\Sigma . m x y = 0$.

This last result may also thus be proved. Let $\text{Y} = \frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$, and let $\rho =$ the density, $\tau =$ the thickness of the plate; then

$$\begin{aligned} \Sigma . m x y &= \rho \tau \int_{-a}^{a} \int_{-\text{Y}}^{\text{Y}} x y \, dy \, dx \\ &= 0. \end{aligned}$$

We will now shew that a system of rectangular coordinate-axes, fulfilling the conditions (69), exists at every point of a body or of system of particles. Such a system is called a system of principal axes relatively to or at that point. The geometrical definition of them is, that they satisfy the conditions (69): in the following sections however several mechanical properties of them will be demonstrated.

SECTION 2.—*Principal axes, and their properties.*

165.] Let us consider a body or a system of particles in reference to a point of it which we take as the origin; and at it let two systems of rectangular coordinates originate; one of which (x, y, z) is fixed absolutely; and the other (ξ, η, ζ) is fixed

in the body; and let the position of the latter be determined, if it is possible, so that

$$\Sigma . m\eta\zeta = 0; \quad \Sigma . m\zeta\xi = 0; \quad \Sigma . m\xi\eta = 0. \tag{70}$$

Let these two systems be related by the direction-cosines of the scheme (1), Art. 2. Then, as the systems are rectangular, the nine direction-cosines are subject to six conditions (4) and (6) or (5) and (7), of Art. 2: and as three other conditions are given in (70), we have sufficient data for the determination of the nine direction-cosines.

Substituting in (70) the values of ξ, η, ζ given in (3), Art. 2, and replacing $\Sigma . mx^2$, $\Sigma . my^2$, $\Sigma . mz^2$, $\Sigma . myz$, $\Sigma . mzx$, $\Sigma . mxy$ by their symbols, we have

$$\begin{aligned}
&\text{A}' b_1 c_1 + \text{B}' b_2 c_2 + \text{C}' b_3 c_3 \\
&\qquad + \text{D}(b_2 c_3 + b_3 c_2) + \text{E}(b_3 c_1 + b_1 c_3) + \text{F}(b_1 c_2 + b_2 c_1) = 0, \\
&\text{A}' c_1 a_1 + \text{B}' c_2 a_2 + \text{C}' c_3 a_3 \\
&\qquad + \text{D}(c_2 a_3 + c_3 a_2) + \text{E}(c_3 a_1 + c_1 a_3) + \text{F}(c_1 a_2 + c_2 a_1) = 0, \\
&\text{A}' a_1 b_1 + \text{B}' a_2 b_2 + \text{C}' a_3 b_3 \\
&\qquad + \text{D}(a_2 b_3 + a_3 b_2) + \text{E}(a_3 b_1 + a_1 b_3) + \text{F}(a_1 b_2 + a_2 b_1) = 0.
\end{aligned}$$

Now these equations are in form identical with (34), Art. 6, and are subject to the same conditions, viz. (5) and (7), Art. 2, so that we have, as in (40), Art. 6,

$$\frac{\text{A}' a_1 + \text{F} a_2 + \text{E} a_3}{a_1} = \frac{\text{F} a_1 + \text{B}' a_2 + \text{D} a_3}{a_2} = \frac{\text{E} a_1 + \text{D} a_2 + \text{C}' a_3}{a_3}; \tag{71}$$

$$\begin{aligned}
&= \frac{\text{A}' a_1^2 + \text{B}' a_2^2 + \text{C}' a_3^2 + 2\text{D} a_2 a_3 + 2\text{E} a_3 a_1 + 2\text{F} a_1 a_2}{a_1^2 + a_2^2 + a_3^2} \\
&= \text{A}' a_1^2 + \text{B}' a_2^2 + \text{C}' a_3^2 + 2\text{D} a_2 a_3 + 2\text{E} a_3 a_1 + 2\text{F} a_1 a_2; \qquad (72) \\
&= \Sigma . m(a_1 x + a_2 y + a_3 z)^2 \\
&= \Sigma . m\xi^2 = \text{A}'' \text{ (say)}. \qquad (73)
\end{aligned}$$

By a similar process we may obtain the following:

$$\frac{\text{A}' b_1 + \text{F} b_2 + \text{E} b_3}{b_1} = \frac{\text{F} b_1 + \text{B}' b_2 + \text{D} b_3}{b_2} = \frac{\text{E} b_1 + \text{D} b_2 + \text{C}' b_3}{b_3} = \Sigma . m\eta^2 = \text{B}'' \text{ (say)}; \tag{74}$$

$$\frac{\text{A}' c_1 + \text{F} c_2 + \text{E} c_3}{c_1} = \frac{\text{F} c_1 + \text{B}' c_2 + \text{D} c_3}{c_2} = \frac{\text{E} c_1 + \text{D} c_2 + \text{C}' c_3}{c_3} = \Sigma . m\zeta^2 = \text{C}'' \text{ (say)}. \tag{75}$$

As these last three equations are of precisely the same form, let us, as in Art. 6, take a type-expression of all; and assume K to be the type of A″, B″, C″; so that the discriminating cubic will take the forms

$$(\text{A}' - \text{K})(\text{B}' - \text{K})(\text{C}' - \text{K}) - \text{D}^2(\text{A}' - \text{K}) - \text{E}^2(\text{B}' - \text{K}) - \text{F}^2(\text{C}' - \text{K}) + 2\text{DEF} = 0; \tag{76}$$

and

$$\frac{\text{EF}}{\text{D}(\text{K}-\text{A}')+\text{EF}}+\frac{\text{FD}}{\text{E}(\text{K}-\text{B}')+\text{FD}}+\frac{\text{DE}}{\text{F}(\text{K}-\text{C}')+\text{DE}}-1=0; \qquad (77)$$

each of which equations has three real roots, viz. A'', B'', C'', as we have shewn in Arts. 7 and 8; these quantities are therefore known functions of $\text{A}', \text{B}', \text{C}', \text{D}, \text{E}, \text{F}$, and will henceforth be treated as such.

166.] Also the direction-cosines of the three principal axes of ξ, η, ζ are given by either of the following formulæ: Let us take the axis of ξ, say; and let $a_1, a_2, a_3, \text{A}''$ correspond to it; then equations (51) and (55), Art. 9, give

$$\frac{a_1^2}{(\text{B}'-\text{A}'')(\text{C}'-\text{A}'')-\text{D}^2}=\frac{a_2^2}{(\text{C}'-\text{A}'')(\text{A}'-\text{A}'')-\text{E}^2}=\frac{a_3^2}{(\text{A}'-\text{A}'')(\text{B}'-\text{A}'')-\text{F}^2}; \qquad ($$

and

$$a_1\{\text{EF}-\text{D}(\text{A}'-\text{A}'')\} = a_2\{\text{FD}-\text{E}(\text{B}'-\text{A}'')\} = a_3\{\text{DE}-\text{F}(\text{C}'-\text{A}'')\}; \qquad (7$$

and similar forms are true for b_1, b_2, b_3, and for c_1, c_2, c_3, in terms of B'' and C'' respectively.

It appears then that at every point of a body, and of a system of material particles, a system of rectangular coordinate-axes in terms of ξ, η, ζ exists, so that, if the body is referred to it,

$$\Sigma.m\eta\zeta = \Sigma.m\zeta\xi = \Sigma.m\xi\eta = 0;$$

and this system at any point is generally unique, and is called the system of principal axes at that point. Thus the term Principal Axis properly belongs to an axis which is one of a system of three rectangular axes. But we shall find it convenient to apply the term to an axis fulfilling any two of the three conditions (70). Thus, if the axis of x is such that $\Sigma.mxy = \Sigma.mxz = 0$, the axis of x is called a principal axis; and if $\Sigma.mzx = \Sigma.mzy = 0$, the axis of z is a principal axis. Hence, if any two of the rectangular coordinate-axes are principal, the third is also principal. Also the three planes which are perpendicular to the three principal axes are called principal planes.

167.] All these results admit of a geometrical interpretation by means of the ellipsoid. From (73), which we will take to be the type of (74) and (75) also, since the form is the same in all, we have

$$\text{A}'a_1^2+\text{B}'a_2^2+\text{C}'a_3^2+2\text{D}a_2a_3+2\text{E}a_3a_1+2\text{F}a_1a_2=\text{A}''. \qquad (80)$$

Along the axis (a_1, a_2, a_3), which is the axis of ξ, say, take a

length r, and let its extremity be (x, y, z); so that, in reference to the system of axes fixed in space,

$$\frac{x}{a_1} = \frac{y}{a_2} = \frac{z}{a_3} = r; \tag{81}$$

and therefore, from (80), we have

$$\text{A}'x^2 + \text{B}'y^2 + \text{C}'z^2 + 2\text{D}yz + 2\text{E}zx + 2\text{F}xy - \text{A}''r^2 = 0. \tag{82}$$

As r is indeterminate thus far, let it be such that $\text{A}''r^2 = \mu'$; where μ' is an undetermined constant, which we may assume to be unity, if such an assumption is convenient; then

$$r^2 = \frac{\mu'}{\text{A}''};$$

and (82) becomes

$$\text{A}'x^2 + \text{B}'y^2 + \text{C}'z^2 + 2\text{D}yz + 2\text{E}zx + 2\text{F}xy - \mu' = 0; \tag{83}$$

which is the equation to a central quadric surface, of which the origin is the centre, and which is an ellipsoid, because $\text{A}', \text{B}', \text{C}'$ are all positive quantities, and their relations to $\text{D}, \text{E}, \text{F}$ are such that all plane sections of the surface are elliptical; also the radius vector, corresponding to the direction-cosines (a_1, a_2, a_3), is that along which the axis of ξ lies; and the length of the corresponding radius vector is $\left(\frac{\mu'}{\text{A}''}\right)^{\frac{1}{2}}$.

By a similar process we may obtain the same equation of an ellipsoid from (74) and (75), if we take the lengths of the central radii vectores, which lie along the axes of η, ζ respectively, equal to $\left(\frac{\mu'}{\text{B}''}\right)^{\frac{1}{2}}$ and $\left(\frac{\mu'}{\text{C}''}\right)^{\frac{1}{2}}$; so that (83) represents an ellipsoid, the lengths of three of whose central radii vectores, which are at right angles to each other, are given: this ellipsoid is called the ellipsoid of principal axes.

And these three radii are the geometrical principal axes of the ellipsoid; for if we apply to (83) the processes for determining the lengths and the position of the principal axes of an ellipsoid, which have been developed in Arts. 6–9, the equations for the direction-cosines, given in (51) and (55) of Art. 9 are the same as (78) and (79), by which the position of our principal axes is determined; and the coefficients of ξ^2, η^2, and ζ^2 in the reduced equation are A'', B'', and C'', which are determined by (73), (74), (75); so that the equation to the ellipsoid, referred to the axes of ξ, η, ζ, is

$$\text{A}''\xi^2 + \text{B}''\eta^2 + \text{C}''\zeta^2 - \mu' = 0. \tag{84}$$

Hence it appears that at every point of a body or system of particles as a centre, an ellipsoid, whose equation is (83), may be described, the principal axes of which are the principal axes of the body relatively to that point. And if for the body, relatively to the centre and the principal axes of the ellipsoid,

$$A'' = \Sigma . m\xi^2, \qquad B'' = \Sigma . m\eta^2, \qquad C'' = \Sigma . m\zeta^2,$$

the principal axes of the ellipsoid are inversely proportional to the square roots of A'', B'', C'' respectively. Thus the form of the ellipsoid, as well as the position of its principal axes, depends on the configuration of the system of particles relatively to the point which is the centre of the ellipsoid.

In our subsequent investigations on this subject we shall assume

$$\left.\begin{array}{l} \Sigma . m\xi^2 > \Sigma . m\eta^2 > \Sigma . m\zeta^2, \\ \text{that is,} \quad A'' > B'' > C''; \end{array}\right\} \qquad (85)$$

so that the ξ-axis, and the ζ-axis, of the ellipsoid are respectively the least and the greatest of all the axes.

168.] Let us shortly examine the particular forms which the ellipsoid (84) and the position of the principal axes take, corresponding to singular values of the roots of the cubic (76) or (77). The analytical criteria of the conditions it is unnecessary to specify, as they are precisely the same as those which have been determined in Art. 12.

(1) Let two roots be equal; say $A'' = B''$; then the equation (84) represents a prolate spheroid whose axis of revolution is the coordinate-axis of ζ; as C'' is definite, c_1, c_2, c_3 are also determinate, and the axis of ζ is a principal axis; the direction-cosines of the axes of ξ and η, which are the other two principal axes, are indeterminate, and any pair of rectangular axes in the plane of (ξ, η) is a pair of principal axes, and with the axis of ζ completes the system. If $B'' = C''$, the ellipsoid becomes an oblate spheroid; the axis of revolution of which is the axis of ξ, and is the determinate principal axis; and any two axes in the plane of (η, ζ), which are perpendicular to each other, will complete the system of principal axes.

(2) Let all the roots of the cubic be equal; then

$$A'' = B'' = C'' = K \text{ (say)};$$

and equation (84) represents a sphere whose equation is

$$\xi^2 + \eta^2 + \zeta^2 = \frac{\mu'}{K}; \qquad (86)$$

and every three axes passing through the given point at right angles to each other will form a system of principal axes. In this case

$$\Sigma . m x^2 = \Sigma . m y^2 = \Sigma . m z^2 = \Sigma . m \xi^2 = \Sigma . m \eta^2 = \Sigma . m \zeta^2 ; \quad (87)$$

and all the nine direction-cosines are indeterminate.

Hence, for every point of a system of particles, there is one set of principal axes; and, for certain points of certain bodies, every system of rectangular axes originating thereat may be a principal system.

The ellipsoid (84) will also assume particular forms if one or two of the quantities A″, B″, C″ vanish, that is, if the body is a plate or a straight wire; but these cases are so evident that it is unnecessary to explain them at length.

169.] In illustration of the preceding theorems, let us determine the principal axes of a cube, with reference to one of its angles. Let each side $= a$; and let the axes of x, y, z lie along the edges of the cube. Let ρ be the density; then

$$A' = \Sigma . m x^2 = \int_0^a \int_0^a \int_0^a \rho x^2 \, dz \, dy \, dx = \frac{\rho a^5}{3} = B' = C',$$

as the symmetry indicates.

$$D = \Sigma . m y z = \int_0^a \int_0^a \int_0^a \rho y z \, dz \, dy \, dx = \frac{\rho a^5}{4} = E = F;$$

thus the cubic equation (76) becomes

$$K^3 - \rho a^5 K^2 + \frac{7}{48} \rho^2 a^{10} K - \frac{5}{864} \rho^3 a^{15} = 0;$$

the roots of which are $\frac{5 \rho a^5}{6}$, $\frac{\rho a^5}{12}$, $\frac{\rho a^5}{12}$; two therefore are equal; let these be B″, C″; so that

$$A'' = \frac{5 \rho a^5}{6}, \qquad B'' = C'' = \frac{\rho a^5}{12}.$$

Hence from (78), $\quad a_1^2 = a_2^2 = a_3^2 = \frac{1}{3},$

$$b_1^2 = b_2^2 = b_3^2 = c_1^2 = c_2^2 = c_3^2 = \frac{0}{0};$$

so that the axis of ξ is the diagonal of the cube; and the position of the other principal axes is indeterminate; and therefore any two lines perpendicular to each other, and in the plane passing through the angle of the cube and perpendicular to the diagonal, will complete a system of principal axes.

Thus the equation to the ellipsoid (83) is

$$\frac{\rho a^5}{3}(x^2+y^2+z^2)+\frac{\rho a^5}{2}(yz+zx+xy)-\mu' = 0,$$

and the equation to the reduced ellipsoid (84) is

$$\frac{5\rho a^5}{6}\xi^2+\frac{\rho a^5}{12}(\eta^2+\zeta^2) = \mu',$$

which represents an oblate spheroid, whose axis of revolution is the axis of ξ.

If it is required to determine the position of the principal axes at the centre of the cube; then, if $2a =$ the length of a side of the cube,

$$\text{A}' = \text{B}' = \text{C}' = \frac{8\rho a^5}{3};$$

$$\text{D} = \text{E} = \text{F} = 0;$$

$$\text{A}'' = \text{B}'' = \text{C}'' = \frac{8\rho a^5}{3};$$

so that the position of each axis is indeterminate; and any rectangular system originating at the centre of the cube is a system of principal axes; and the ellipsoid (83) becomes a sphere.

170.] It is evident from the preceding general investigation that the position of the principal axes of a body, relatively to a given point, depends on the values of the definite integrals which are expressed by the symbols A′, B′, C′, D, E, F; and therefore on the symmetry of the body relatively to the origin and to the axes of x, y, z: thus, for a solid of revolution bounded by a plane perpendicular to the axis of revolution, for any point on the axis of revolution that axis is evidently one of the principal axes, and the other two are indeterminate in the plane perpendicular to the axis of revolution. Of a sphere, relatively to the centre, every system of rectangular axes is principal. Of an ellipsoid, relatively to the centre, the principal system is unique; and the principal axes coincide with the geometrical principal axes of the body. Similarly the principal system can often be inferred by general reasoning from the symmetry of the system.

Thus in the case of a regular polyhedron, with respect to its centre, the ellipsoid of principal axes is evidently a sphere, because the surface is necessarily symmetrical with respect to the perpendiculars from the centre on the several faces, and this symmetry cannot exist in any quadric other than a sphere. Hence any line drawn through the centre is a principal axis.

Also in the case of a thin plate whose boundary line is a regular polygon, in reference to its centre, the ellipsoid of principal axes is a prolate spheroid whose axis of revolution is perpendicular to the plate.

171.] If two principal axes relatively to a given point are given, the third is also given.

If however one principal axis is given, the other two are at right angles to each other in a plane perpendicular to the given principal axis, and may be determined by the following process.

Let the given principal axis be the axis of z; and let h be the distance from the origin of the plane in which the two other principal axes lie: so that

$$\Sigma . mx(z-h) = \Sigma . my(z-h) = 0; \tag{88}$$

$$\therefore \quad h = \frac{\Sigma . mxz}{\Sigma . mx} = \frac{\Sigma . myz}{\Sigma . my}, \tag{89}$$

which assigns a value of h, and thus determines the position of the plane of the two other principal axes; it also gives a condition, viz. that contained in the last two of the equalities (89), which must be satisfied when the axis of z is a principal axis at all, as shewn below.

Let the new axes of ξ and η, which are to be principal, lie in this plane; and let the new axis of ξ be inclined at an angle ϕ to the plane of (z, x), so that

$$\left.\begin{aligned} \xi &= y \sin\phi + x\cos\phi, \\ \eta &= y\cos\phi - x\sin\phi; \end{aligned}\right\} \tag{90}$$

therefore

$$\Sigma . m\xi\eta = \{(\cos\phi)^2 - (\sin\phi)^2\}\, \Sigma . mxy + \sin\phi\cos\phi\, \Sigma . m(y^2 - x^2); \tag{91}$$

and, as this is to vanish, we have

$$\tan 2\phi = \frac{2\Sigma . mxy}{\Sigma . m(x^2 - y^2)} = \frac{2\text{F}}{\text{A}' - \text{B}'} = \frac{2\text{F}}{\text{B} - \text{A}}; \tag{92}$$

and the equations to the two principal axes are

$$(\xi^2 - \eta^2)\, \Sigma . mxy + \xi\eta\, \Sigma . m(y^2 - x^2) = 0. \tag{93}$$

The point where the plane of (ξ, η) cuts the axis of z is called the principal point of that line; and if $z - h = \zeta$, we have also

$$\Sigma . m\zeta\eta = \Sigma . m\zeta\xi = 0.$$

In the case of a plate, any line perpendicular to it is a principal axis, and the point where it cuts the plate is its principal point.

If $\text{A}' = \text{B}'$ and $\text{F} = 0$, $\tan 2\phi$ is indeterminate; and con-

sequently any straight line which is perpendicular to the axis of z at its principal point is a principal axis.

172.] If the axis of z is a principal axis, the centrifugal forces, which arise from rotation about it, either balance each other, or compound into a single resultant. For let (x, y, z) be the place of m, and let ω be the angular velocity of the body about the axis of z: then the components of the centrifugal force along the axes of x and y are respectively $m\omega^2 x$, and $m\omega^2 y$, which act at (x, y, z). Hence, if X$''$, Y$''$, Z$''$, L$''$, M$''$, N$''$ denote the same quantities as in Art. 159,

$$\text{X}'' = \Sigma . m\omega^2 x, \qquad \text{Y}'' = \Sigma . m\omega^2 y, \qquad \text{Z}'' = 0;$$

$$\text{L}'' = -\Sigma . m\omega^2 yz, \quad \text{M}'' = \Sigma . m\omega^2 xz, \qquad \text{N}'' = 0;$$

$$\therefore \quad \text{L}''\text{X}'' + \text{M}''\text{Y}'' + \text{N}''\text{Z}'' = -\Sigma . m\omega^2 x \times \Sigma . m\omega^2 yz + \Sigma . m\omega^2 y \times \Sigma . m\omega^2 xz;$$

and this $= 0$, if $$\frac{\Sigma . mxz}{\Sigma . mx} = \frac{\Sigma . myz}{\Sigma . my},$$

which is the same condition as (89). This, then, is the condition that the axis of z is a principal axis.

173.] The following are examples of the process for determining the position of two principal axes at a given point when the other principal axis and its principal point are given.

Ex. 1. To determine the principal axes of a rectangular plate of infinitesimal thickness relatively to the point of intersection of the two diagonals of the plate.

In this case, as in all problems of plane plates of infinitesimal thickness, an axis which passes through the origin and is perpendicular to the plate is one of the principal axes: and if it is taken to be the axis of z, and the plate to be the plane of (x, y), $\Sigma . mxz = \Sigma . myz = 0$, because $z = 0$ for all the elements of the plate.

Let $2a$ and $2b$ be the sides of the plate, $\tau =$ the thickness, $\rho =$ the density;

$$\Sigma . mxy = \int_{-a}^{a}\int_{-b}^{b} \rho\tau xy\, dy\, dx = 0;$$

$$\Sigma . m(x^2 - y^2) = \int_{-a}^{a}\int_{-b}^{b} \rho\tau (x^2 - y^2)\, dy\, dx$$

$$= \frac{4\rho\tau ab}{3}(a^2 - b^2);$$

therefore $\tan 2\phi = 0$; and $\phi = 0$, $\phi = 90°$; so that the other

two principal axes are parallel to the sides of the plate. If the rectangle is a square, $b = a$; in which case $\tan 2\phi = \frac{0}{0}$; and ϕ is indeterminate, so that every pair of rectangular axes in the plane of the plate, together with the given axis, constitutes a principal system.

Ex. 2. To determine the principal axes of a triangular plate through one of its angles.

Let O, an angle of the triangle, be the origin; $\text{OA}=a$, $\text{OB}=b$; and let the axes of x and y lie along these sides; then the equation to the base is

$$\frac{x}{a}+\frac{y}{b}=1;$$

let $\text{Y} = \frac{b}{a}(a-x)$; and let ω be the angle of the triangle at O;

then
$$\text{F} = \int_0^a \int_0^{\text{Y}} \rho\tau(x+y\cos\omega)\,y\,(\sin\omega)^2\,dy\,dx$$
$$= \frac{\rho\tau(\sin\omega)^2 ab^2}{24}(a+2b\cos\omega);$$

$$\text{A}'-\text{B}' = \int_0^a \int_0^{\text{Y}} \rho\tau\{(x+y\cos\omega)^2-(y\sin\omega)^2\}\,dy\,dx\sin\omega$$
$$= \frac{\rho\tau ab\sin\omega}{12}(a^2+ab\cos\omega+b^2\cos 2\omega);$$

$$\therefore\quad \tan 2\phi = \frac{b\sin\omega\,(a+2b\cos\omega)}{a^2+ab\cos\omega+b^2\cos 2\omega}.$$

If $b = a$, $2\phi = \omega$; in which case the triangle is isosceles, and of the principal axes in its plane one bisects the vertical angle, and the other is perpendicular to the bisecting line.

If $\omega = \frac{\pi}{2}$, $\tan 2\phi = \frac{ab}{a^2-b^2}$.

Ex. 3. To determine the position of the principal axes in the plane of a thin elliptic plate relatively to a point whose place, relatively to the centre and principal axes of the ellipse, is (α, β).

Let the equation to the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$; and let $\text{Y} = \frac{b}{a}(a^2-x^2)^{\frac{1}{2}}$: then, as the ellipse is referred to its mass-centre as origin, and to its principal axes as coordinate-axes,

$$\Sigma.mx = 0, \qquad \Sigma.my = 0; \qquad \Sigma.mxy = 0.$$

Then
$$\text{F} = \int_{-a}^{a}\int_{-\text{Y}}^{\text{Y}} \rho\tau(x-a)(y-\beta)\,dy\,dx$$
$$= \int_{-a}^{a}\int_{-\text{Y}}^{\text{Y}} \rho\tau a\beta\,dy\,dx$$
$$= \pi\rho\tau ab\,a\beta.$$
$$\text{A}'-\text{B}' = \int_{-a}^{a}\int_{-\text{Y}}^{\text{Y}} \rho\tau\{(x-a)^2-(y-\beta)^2\}\,dy\,dx$$
$$= \int_{-a}^{a}\int_{-\text{Y}}^{\text{Y}} \rho\tau\{x^2-y^2+a^2-\beta^2\}\,dy\,dx$$
$$= \pi\rho\tau ab\left\{\frac{a^2-b^2}{4}+a^2-\beta^2\right\};$$
$$\therefore\ \tan 2\phi = \frac{8\,a\beta}{a^2-b^2+4(a^2-\beta^2)};$$

whereby we have the position of the two principal axes in the plane of the plate, and these, with the axis perpendicular to the plate, form the complete system of principal axes.

174.] Many other properties of principal axes will arise incidentally in the following section, and will be there demonstrated. We may therefore now return to the equations of motion, and make those simplifications in them which the theory of principal axes supplies.

Let the rotation of the body or system of particles be referred to a system of axes fixed in the body and moving with it; and let this system be that of principal axes, so that

$$\Sigma.myz = \Sigma.mzx = \Sigma.mxy = 0, \quad \text{or} \quad \text{D} = \text{E} = \text{F} = 0.$$

Firstly, let us take the case of a body, originally at rest, acted on by instantaneous forces; and, for the sake of distinctness, let the axial components of the expressed angular velocities, due to the instantaneous forces about the principal axes, be denoted by Ω_1, Ω_2, Ω_3; and let us reserve Ω_x, Ω_y, Ω_z for the axial components of the expressed angular velocities when the system of axes is not principal; then equations (31), Art. 151, become

$$\Omega_1 = \frac{\text{L}}{\text{A}}, \qquad \Omega_2 = \frac{\text{M}}{\text{B}}, \qquad \Omega_3 = \frac{\text{N}}{\text{C}}; \tag{94}$$

where A, B, C are the moments of inertia of the body about the principal axes; and L, M, N are the moments of the axial components of the couples of the impressed momenta about the corresponding axes.

Secondly, let us take the case of a body rotating under the action of continuous forces; and here again let ω_1, ω_2, ω_3 represent the axial components of the expressed angular velocity relatively to the principal axes; and let ω_x, ω_y, ω_z be the axial components of the expressed angular velocity when the axes are not principal; then, since $\text{D} = \text{E} = \text{F} = 0$, equations (54), Art. 156, become

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{B})\,\omega_2\omega_3 &= \text{L}, \\ \text{B}\frac{d\omega_2}{dt} + (\text{A}-\text{C})\,\omega_3\omega_1 &= \text{M}, \\ \text{C}\frac{d\omega_3}{dt} + (\text{B}-\text{A})\,\omega_1\omega_2 &= \text{N}; \end{aligned}\right\} \qquad (95)$$

which are evidently much simpler than (54), and are equally general; they were investigated first by Euler, and are now commonly called Euler's equations of rotation. No way of integrating them in the form in which they stand is known at present. Particular forms of them will be discussed in the following Chapters; and we shall then employ such artifices of abbreviation and of interpretation as the particular problem suggests.

The origin of the several terms of (95) is at once evident from the analysis given in Arts. 158 and 159. From (58) we have the following values of the axial components of the moment of the couple arising from the expressed angular momentum-increments

$$\text{L}' = \text{A}\frac{d\omega_1}{dt}, \quad \text{M}' = \text{B}\frac{d\omega_2}{dt}, \quad \text{N}' = \text{C}\frac{d\omega_3}{dt}. \qquad (96)$$

Also from (62) we have the following values for the axial components of the moment of the centrifugal couple

$$\left.\begin{aligned} \text{L}'' &= (\text{B}-\text{C})\,\omega_2\omega_3, \\ \text{M}'' &= (\text{C}-\text{A})\,\omega_3\omega_1, \\ \text{N}'' &= (\text{A}-\text{B})\,\omega_1\omega_2; \end{aligned}\right\} \qquad (97)$$

and hence by D'Alembert's Principle the equations of motion (95) follow immediately.

175.] Thus far no condition has been made as to the point at which the coordinate axes for reference of rotation originate; it is entirely arbitrary: let us consider whether any simplification will be introduced into the results if we take it at the mass-centre of the moving system or body; that is, let us suppose the

point which has motion of translation to be the mass-centre and the axis of rotation to pass through it; and let us also suppose the coordinate axes fixed in the body, and originating at the mass-centre, to be principal axes; and these axes we shall henceforth call central principal axes. The geometrical definition of such an origin and such axes is,

$$\left.\begin{array}{l}\Sigma . m x=\Sigma . m y=\Sigma . m z=0, \\ \Sigma . m y z=\Sigma . m z x=\Sigma . m x y=0 .\end{array}\right\}$$

Let us first consider the axial components of the expressed momentum-increments of all the particles which arise from the angular velocity-increments. Then, on referring to Art. 158 wherein the effects have been investigated, from (55) and (58) we have

$$\text{X}'=\text{Y}'=\text{Z}'=0;$$

$$\text{L}'=\text{A}\frac{d\omega_1}{dt}, \quad \text{M}'=\text{B}\frac{d\omega_2}{dt}, \quad \text{N}'=\text{C}\frac{d\omega_3}{dt}; \tag{98}$$

so that the momentum-increments are as to translation in equilibrium at the mass-centre, neither accelerating nor retarding its motion if it is moving, nor producing a pressure if it is fixed; and the moments of the axial components of the couple arising from these expressed angular velocity-increments are respectively the first terms of (95).

Next let us take the axial components which arise from the centrifugal forces which are discussed in Art. 159. From (60) we have

$$\text{X}''=\text{Y}''=\text{Z}''=0; \tag{99}$$

so that all the centrifugal forces as to translation balance at the mass-centre; and from (62)

$$\text{L}''=(\text{B}-\text{C})\omega_2\omega_3, \quad \text{M}''=(\text{C}-\text{A})\omega_3\omega_1, \quad \text{N}''=(\text{A}-\text{B})\omega_1\omega_2.$$

Thus it appears that relatively to the mass-centre as the origin the forces of translation which are due to the angular velocity-increments and to the centrifugal forces are each separately in equilibrium; but that the equations of rotation are the same in form as in the general case.

Suppose moreover that the axis of rotation is a principal axis, say the axis of x; then $\omega_2=\omega_3=0$; and

$$\text{L}'=\text{A}\frac{d\omega_1}{dt}, \quad \text{M}'=0, \quad \text{N}'=0;$$

$$\text{L}''=0, \quad \text{M}''=0, \quad \text{N}''=0;$$

so that the centrifugal forces balance themselves as to rotatory effects on the body; producing no change either of the angular velocity of the body or of the position of the axis of rotation; and, as (99) shew, they balance each other as to effect of translation.

Hence if a body rotates about a central principal axis, the centrifugal forces which are thereby generated are in equilibrium, and thus do not cause any change of rotation or of the position of the rotation-axis.

Hence also, if no force acts on the body, its equation of motion is

$$\text{A}\,\frac{d\,\omega_1}{dt} = 0;$$

$$\therefore \quad \omega_1 = \text{a constant}.$$

Thus the angular velocity is constant, and the position of the rotation-axis is the same throughout the motion. For this reason the central principal axes are called the *permanent axes* of the body or of the system; they are also sometimes called the *natural* axes.

It is evident from what has been said that a permanent axis of a body at a certain point may be defined as that line about which if the body revolves, the centrifugal forces generated by the rotation are either in equilibrium, or have a simple resultant passing through the point.

176.] The central principal axes are also the only axes which possess the property, that the centrifugal forces balance each other about them both as to translation and as to rotation; for this object it is necessary that, see Art. 159,

$$\text{X}'' = \text{Y}'' = \text{Z}'' = 0, \tag{100}$$

$$\text{L}'' = \text{M}'' = \text{N}'' = 0; \tag{101}$$

from these last three we have, from (62), Art. 159, remembering that

$$\text{A} + \text{A}' = \text{B} + \text{B}' = \text{C} + \text{C}',$$

and replacing ω_x, ω_y, ω_z by $\cos\alpha$, $\cos\beta$, $\cos\gamma$, which are proportional to them,

$$\left.\begin{array}{l} (\text{E}\cos\alpha + \text{D}\cos\beta + \text{C}'\cos\gamma)\cos\beta - (\text{F}\cos\alpha + \text{B}'\cos\beta + \text{D}\cos\gamma)\cos\gamma = 0, \\ (\text{A}'\cos\alpha + \text{F}\cos\beta + \text{E}\cos\gamma)\cos\gamma - (\text{E}\cos\alpha + \text{D}\cos\beta + \text{C}'\cos\gamma)\cos\alpha = 0, \\ (\text{F}\cos\alpha + \text{B}'\cos\beta + \text{D}\cos\gamma)\cos\alpha - (\text{A}'\cos\alpha + \text{F}\cos\beta + \text{E}\cos\gamma)\cos\beta = 0; \end{array}\right\} \tag{102}$$

so that

$$\frac{\text{A}'\cos\alpha + \text{F}\cos\beta + \text{E}\cos\gamma}{\cos\alpha} = \frac{\text{F}\cos\alpha + \text{B}'\cos\beta + \text{D}\cos\gamma}{\cos\beta} = \frac{\text{E}\cos\alpha + \text{D}\cos\beta + \text{C}'\cos\gamma}{\cos\gamma}; \tag{103}$$

which are the same equations as (71) by which the principal axes are determined; the principal axes therefore are permanent axes so far as the conditions (101) indicate; that is, corresponding to principal axes the centrifugal forces either are in equilibrium, or have a single resultant passing through the origin.

As to (100) let us replace x, y, z in (60) by $\bar{x}+x'$, $\bar{y}+y'$, $\bar{z}+z'$, wherein the mass-centre is $(\bar{x}, \bar{y}, \bar{z})$; then, since

$$\Sigma.mx' = \Sigma.my' = \Sigma.mz' = 0;$$

(100) become

$$\left.\begin{aligned} \bar{x}-\cos\alpha\,(\bar{x}\cos\alpha+\bar{y}\cos\beta+\bar{z}\cos\gamma) &= 0, \\ \bar{y}-\cos\beta\,(\bar{x}\cos\alpha+\bar{y}\cos\beta+\bar{z}\cos\gamma) &= 0, \\ \bar{z}-\cos\gamma\,(\bar{x}\cos\alpha+\bar{y}\cos\beta+\bar{z}\cos\gamma) &= 0; \end{aligned}\right\} \qquad (104)$$

and therefore

$$\frac{\bar{x}}{\cos\alpha} = \frac{\bar{y}}{\cos\beta} = \frac{\bar{z}}{\cos\gamma}; \qquad (105)$$

so that the rotation-axis, that is, the principal axis at the point, must pass through the mass-centre of the body. Thus (105) are to be true for each principal axis at the point; that is, for each of three different values of (α, β, γ); and this is possible only when $\bar{x} = \bar{y} = \bar{z} = 0$; only when the point is the mass-centre, in which case either of the three central principal axes is a permanent axis. If however a principal axis at a point passes through the mass-centre, that axis is a permanent axis for that point.

177.] A remarkable application of this theory of permanent axes has been made by M. Foucault to the proof of the rotation of the earth about its polar axis. It was presented to the Academy of Sciences in Paris in the month of September, 1852. He has devised a machine which he calls a Gyroscope, and of which a drawing is given in Fig. 21. I will describe it as it is originally in its position of rest. AB A′B′ is a metallic ring suspended by a wire SA from a point S which is fixed to the earth; and at A′ is a small pivot working in a small hole, by which the motion of the ring about the vertical line SAA′ is kept steady but is not retarded: BB′ is the horizontal diameter of the vertical ring; and at B and B′ are small holes capable of receiving small pivots or axles; BCB′ is a horizontal metallic ring, of which BB′ is the diameter, and O is the centre, at B and B′ are pivots which work in the beforementioned small holes, so that BCB′ is capable of rotating about the horizontal diameter BB′. Across the horizontal ring BCB′ another axis COC′ is placed, at right angles to, and bisecting, BOB′,

and capable of rotating about pivots at C and C′; on this axis is fixed a heavy metallic disc DD′, whose centre is at O, and which is consequently capable of rotation about the axis COC′, and the greater part of the matter of the disc is arranged in a ring as near to the circumference as possible, so as to increase the centrifugal force of the disc. This is the arrangement of the several parts of the Gyroscope; and it is evident from the arrangement that the disc is capable of rotation about any axis, so that whatever are the forces which act upon it, it can take the axis which they require, and is indeed in construction identical with Bohnenberger's machine, except that the central mass is a disc instead of a sphere. It is evident also that the centre of gravity of the whole machine is at O, and that all axes of rotation pass through O, and thus gravity does not produce any change of position in the rotation-axis or in the velocity of the disc. Adjustment-screws are placed in various parts of the machine, so that the conditions required may be fulfilled as nearly as possible.

Now the disc DD′ is taken out of the ring BCB′, and has a very rapid rotation given to it by a machine properly contrived for that purpose. While it has this rapid rotation it is replaced in the horizontal ring BCB′; and as the axis COC′ is manifestly one of the principal axes of the ring through its centre of gravity, it is a permanent axis, and as the disc is not under the action of any forces, whether external or centrifugal, whereby its velocity or the position of its rotation-axis may be changed, its axis COC′ keeps an invariable position in space. But what apparent effect is produced by this invariable position? Let us suppose the Gyroscope to be at the north pole: then the earth rotates about the axis SAA′, and to an observer the axis of COC′ will retain its horizontal position, and will have a motion in azimuth, in the same direction as the fixed stars appear to have in passing through 360° in 24 hours, if the rotation of the disc can be kept up as long: if the Gyroscope is at the equator, no such apparent effect will take place, because the axis will have only a parallel displacement of itself in space. In any other latitude the polar axis of the earth will be inclined to the principal axis COC′ of the disc; and as this, being a permanent axis, retains its direction in space, it has an apparent motion about the polar axis; the ring BCB′ will revolve slowly about the axis BB′, and the ring AA′ will also revolve slowly about the vertical axis AA′; and these

rotations may be observed by means of microscopes properly placed for the purpose. Indeed the earth truly rotates about the line coc' which has an invariable position, and that rotation is shewn by the apparent motion of the line. We shall hereafter come to the mathematical calculation of these quantities.

SECTION 3.—*Moments of inertia, and the distribution in space of principal axes.*

178.] In the present section I propose to examine more closely the theory and properties of moments of inertia, of which definitions have been given in Art. 148.

Moment of inertia is relative to an axis, and is (geometrically) the sum of the products of every particle of a body or of a system of particles and the square of its distance from that axis. Thus, if m is a particle of a moving system, and r is the perpendicular distance from the place of m on the axis, $\Sigma . m r^2$ is the moment of inertia, the summation including all the particles of the system, and becoming integration if the system is a continuous body. We shall however find it convenient to use the symbol $\Sigma . m r^2$ in all cases, bearing in mind that this symbol includes integration in the cases wherein the moving system is a continuous mass.

Let us in the first place investigate the moment of inertia in its most general form. Let the origin be taken on the axis, of which relatively to a system of axes fixed in the body let the direction-angles be (α, β, γ); let (x, y, z) be the place of m; and let r be the perpendicular distance from m on the rotation-axis; so that

$$r^2 = (z \cos\beta - y \cos\gamma)^2 + (x \cos\gamma - z \cos\alpha)^2 + (y \cos\alpha - x \cos\beta)^2 ; \quad (106)$$

$$\Sigma . m r^2 = \Sigma . m (y^2 + z^2)(\cos\alpha)^2 + \Sigma . m (z^2 + x^2)(\cos\beta)^2 + \Sigma . m (x^2 + y^2)(\cos\gamma)^2$$
$$- 2\Sigma . m yz \cos\beta\cos\gamma - 2\Sigma . m zx \cos\gamma\cos\alpha - 2\Sigma . m xy \cos\alpha\cos\beta. \quad (107)$$

Let H be the general symbol for the moment of inertia; then, using the symbols of Art. 150,

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2$$
$$- 2\text{D}\cos\beta\cos\gamma - 2\text{E}\cos\gamma\cos\alpha - 2\text{F}\cos\alpha\cos\beta ; \quad (108)$$

and thus if A, B, C, D, E, F are determined for any body and for

a particular system of rectangular axes, the moment of inertia of the body for any axis may be found by means of this equation.

Hence if the moments are given for any six axes passing through the given point, there will be six equations for determining the six quantities A, B, C, D, E, F, and these are sufficient for the purpose; and thus the moment of inertia will be determined for any other lines passing through the point.

If the body is a thin plate lying in the plane of (x, y), $D = E = 0$; and for all axes lying in the plane of the plate, $\cos\gamma = 0$, and $\beta = 90° - \alpha$; so that the preceding equation becomes
$$H = A(\cos\alpha)^2 + B(\sin\alpha)^2 - 2F\sin\alpha\cos\alpha,$$
and therefore if the moments of inertia are given for any three axes lying in the plane of the plate, A, B, F can be determined, and the moment of inertia can be found for any other line, whether being in the plate or not, since $C = A + B$.

179.] It is evident that H, as given in (108), admits of critical values, viz. maxima and minima, total and partial, which may be determined as to position and magnitude by the process of Art. 10. Hence we have the following equations:
$$\frac{A\cos\alpha - F\cos\beta - E\cos\gamma}{\cos\alpha} = \frac{-F\cos\alpha + B\cos\beta - D\cos\gamma}{\cos\beta}$$
$$= \frac{-E\cos\alpha - D\cos\beta + C\cos\gamma}{\cos\gamma} = H;$$
$$\therefore \left.\begin{array}{r} (H-A)\cos\alpha + F\cos\beta + E\cos\gamma = 0, \\ F\cos\alpha + (H-B)\cos\beta + D\cos\gamma = 0, \\ E\cos\alpha + D\cos\beta + (H-C)\cos\gamma = 0; \end{array}\right\}$$
and by cross-multiplication,
$$(H-A)(H-B)(H-C) - D^2(H-A) - E^2(H-B) - F^2(H-C) + 2DEF = 0,$$
which is a cubic equation having three real roots; these are called the principal moments of inertia at the point, and the positions of them are given by the preceding equations.

If the body is a thin plate lying in the plane of (x, y), then $D = E = 0$, and the last equation breaks up into two factors; whence we have
$$(H-A)(H-B) - F^2 = 0,$$
$$H - C = 0,$$
of which the former gives the two principal moments in the plane of the plate, and the latter gives the moment about the axis which is perpendicular to the plate and is equal to C; and

if θ is the angle which the axis of a principal moment in the plane of the plate makes with the x-axis,

$$\tan 2\theta = \frac{2\text{F}}{\text{B}-\text{A}},$$

the same result as that obtained in Art. 171.

If the system of axes to which the body is referred is a principal system at the point, then $\text{D} = \text{E} = \text{F} = 0$; and (108) becomes

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2; \qquad (109)$$

where A, B, C are the moments of inertia of the body relatively to the three principal axes of x, y, z respectively, and are for that reason called the principal moments of inertia.

In terms of A′, B′, C′, (109) becomes

$$\text{H} = \text{A}'(\sin\alpha)^2 + \text{B}'(\sin\beta)^2 + \text{C}'(\sin\gamma)^2. \qquad (110)$$

The student will observe how greatly the expression for moment of inertia is simplified by the use of principal axes.

180.] As an example of (108) let us investigate the moment of inertia of a rectangular parallelepipedon about an axis passing through one of its angles.

Let the sides of the parallelepipedon, which meet at the angle through which the axis passes, be a, b, c; and let the coordinate axes lie along these sides respectively; let ρ be the density of the volume-element at (x, y, z); so that the mass-element $= \rho\, dx\, dy\, dz$. Then

$$\text{A} = \int_0^a \int_0^b \int_0^c \rho\,(y^2+z^2)\,dz\,dy\,dx = \frac{\rho abc\,(b^2+c^2)}{3};$$

$$\text{B} = \frac{\rho abc\,(c^2+a^2)}{3}, \qquad \text{C} = \frac{\rho abc\,(a^2+b^2)}{3};$$

$$\text{D} = \int_0^a \int_0^b \int_0^c \rho yz\,dz\,dy\,dx = \frac{\rho ab^2c^2}{4};$$

$$\text{E} = \frac{\rho bc^2a^2}{4}; \qquad \text{F} = \frac{\rho ca^2b^2}{4};$$

and the moment of inertia about the line (α, β, γ)

$$= \rho abc \left\{ \frac{b^2+c^2}{3}(\cos\alpha)^2 + \frac{c^2+a^2}{3}(\cos\beta)^2 + \frac{a^2+b^2}{3}(\cos\gamma)^2 \right.$$
$$\left. - \frac{bc}{2}\cos\beta\cos\gamma - \frac{ca}{2}\cos\gamma\cos\alpha - \frac{ab}{2}\cos\alpha\cos\beta \right\};$$

and if k is the corresponding radius of gyration,

$$k^2 = \frac{b^2+c^2}{3}(\cos\alpha)^2 + \frac{c^2+a^2}{3}(\cos\beta)^2 + \frac{a^2+b^2}{3}(\cos\gamma)^2$$
$$- \frac{bc}{2}\cos\beta\cos\gamma - \frac{ca}{2}\cos\gamma\cos\alpha - \frac{ab}{2}\cos\alpha\cos\beta.$$

Thus the moment of inertia of a cube about a diagonal $= \frac{\rho a^5}{6}$; and about one of its edges $= \frac{2\rho a^5}{3}$.

Other examples of the determination of moments of inertia will be given in the following section of this chapter.

181.] The following process gives a geometrical interpretation of (108), and consequently of (109), which indicates results of great importance.

Along the rotation-axis from the origin let a length ρ be taken, of which let the end be (x, y, z); then

$$\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma} = \rho; \tag{111}$$

and (108) becomes

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 - 2\text{D}yz - 2\text{E}zx - 2\text{F}xy = \text{H}\rho^2. \tag{112}$$

Let $\quad \text{H}\rho^2 = \mu; \quad$ so that $\text{H} = \frac{\mu}{\rho^2}$, $\quad$ (113)

where μ is a constant quantity at present undetermined, and may be unity if such a value is convenient; then (112) becomes

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 - 2\text{D}yz - 2\text{E}zx - 2\text{F}xy - \mu = 0. \tag{114}$$

Now this is the equation to a central quadric, of which the origin is the centre; and expresses an ellipsoid since A, B, C, are all positive, and their relations to D, E, F are such that all plane sections of the surface are ellipses. It is also such that whatever radius vector is the rotation-axis, the moment of inertia of the body relatively to that axis varies as the square of the reciprocal of the length of that radius vector. For this reason the ellipsoid is called the momental ellipsoid. As μ is at present undetermined the actual size of the ellipsoid is not fixed: however, according as μ varies, all the corresponding ellipsoids are concentric and coaxial, and that corresponding to any particular value of μ will suffice for our present purpose. Let us imagine therefore the ellipsoid, whose equation is (114), to be described with the given point as centre; then that ellipsoid, by means of its central radii vectores, indicates the law of variation of the several moments of inertia of the moving system which correspond to the radii vectores as rotation-axes, the moment of inertia relatively to any one being proportional to the square of the reciprocal of that radius vector. This ellipsoid is the same as that introduced in Art. 152, equation (38).

182.] The momental ellipsoid is evidently concentric with the ellipsoid of principal axes, equation (83); it is also coaxial with

it. For, by Art. 6, the equations for determining the principal axes of (114) are

$$\frac{\text{A}a_1-\text{F}a_2-\text{E}a_3}{a_1}=\frac{-\text{F}a_1+\text{B}a_2-\text{D}a_3}{a_2}=\frac{-\text{E}a_1-\text{D}a_2+\text{C}a_3}{a_3}$$
$$= \text{H}_a, \tag{115}$$

if H_a is the moment of inertia about the axis (a_1, a_2, a_3) of ξ, so that $\text{H}_a = \Sigma . m(\eta^2+\zeta^2)$. Now replacing A, B, C severally by $\text{B}'+\text{C}'$, $\text{C}'+\text{A}'$, $\text{A}'+\text{B}'$, and subtracting each term of the equalities (115) from $\text{A}'+\text{B}'+\text{C}'$, we have

$$\frac{\text{A}'a_1+\text{F}a_2+\text{E}a_3}{a_1}=\frac{\text{F}a_1+\text{B}'a_2+\text{D}a_3}{a_2}=\frac{\text{E}a_1+\text{D}a_2+\text{C}'a_3}{a_3}$$
$$= \text{A}'+\text{B}'+\text{C}'-\text{H}_a$$
$$= \Sigma . m\xi^2; \tag{116}$$

which are identical with (73), whereby the principal axes of the body are determined. By similar processes we might find equations identical with (74) and (75) in terms of H_b and H_c, which are the moments of inertia about the axes of η and ζ respectively. Thus the geometrical principal axes of the momental ellipsoid lie along the principal momental axes of the body at the origin and the two systems are identical; let it be referred to these as axes; it is manifest from (115) that H_a, H_b, H_c are the coefficients of x^2, y^2, z^2 in the reduced equation. Let, however, henceforth A, B, C represent the moments of inertia about the principal axes; then the equation to the momental ellipsoid, referred to the principal axes as coordinate axes of x, y, z, is

$$\text{A}x^2+\text{B}y^2+\text{C}z^2 = \mu. \tag{117}$$

This result might have been inferred directly from (114). As the position of the coordinate axes is undetermined, let the system be the principal system; then $\text{D} = \text{E} = \text{F} = 0$, and (114) becomes (117).

183.] The momental ellipsoid and the ellipsoid of principal axes intersect on the surface of a sphere; for since their equations are respectively

$$\text{A}x^2+\text{B}y^2+\text{C}z^2-2\text{D}yz-2\text{E}zx-2\text{F}xy = \mu,$$
$$\text{A}'x^2+\text{B}'y^2+\text{C}'z^2+2\text{D}yz+2\text{E}zx+2\text{F}xy = \mu',$$

we have by addition,

$$(\text{A}+\text{A}')x^2+(\text{B}+\text{B}')y^2+(\text{C}+\text{C}')z^2 = \mu+\mu',$$

but
$$\text{A}+\text{A}' = \text{B}+\text{B}' = \text{C}+\text{C}' = \Sigma . m(\xi^2+\eta^2+\zeta^2),$$
$$= \text{I (see Art. 150)};$$

$$\therefore \quad x^2+y^2+z^2 = \frac{\mu+\mu'}{\text{I}};$$

which is the equation to a sphere. Hence all the radii vectores drawn to the line of intersection of the two ellipsoids are equal, and the moments of inertia corresponding to them are also all equal. Thus they lie in a cone which is called an equimomental cone, see Art. 190. As μ and μ' may vary, there is a series of such equimomental cones. The cyclic planes of these two ellipsoids are manifestly coincident.

184.] In Art. 167 we assumed $\Sigma . mx^2 > \Sigma . my^2 > \Sigma . mz^2$; therefore

$$\Sigma . m(y^2+z^2) < \Sigma . m(z^2+x^2) < \Sigma . m(x^2+y^2);$$

$$\therefore \quad \text{A} < \text{B} < \text{C};$$

so that the moments of inertia are respectively the greatest and the least about the axes of z and x; and the maximum, mean, and minimum axes of the momental ellipsoid lie along the axes of x, y, z respectively, and correspond to the minimum, mean, and maximum axes of the ellipsoid of principal axes.

One word as to μ; let us give it a value which will make the equations homogeneous; let M be the mass of the moving body, and let a, b, c be the radii of gyration about the axes of x, y, z respectively; so that

$$\text{A} = \text{M}a^2, \qquad \text{B} = \text{M}b^2, \qquad \text{C} = \text{M}c^2; \tag{118}$$

let therefore

$$\mu = \text{M}g^4; \tag{119}$$

we shall hereafter determine the meaning of g; then (117) becomes

$$a^2x^2 + b^2y^2 + c^2z^2 = g^4; \tag{120}$$

so that the maximum, mean, and minimum axes of the momental ellipsoid are respectively $\frac{g^2}{a}, \frac{g^2}{b}, \frac{g^2}{c}$, and the equation is homogeneous. Notwithstanding, however, we shall still find it convenient to employ μ.

185.] The form of (120), and the circumstance of the moment of inertia about any rotation-axis varying inversely as the square of the radius vector of the momental ellipsoid which coincides with it, suggest another geometrical interpretation of the law of variation of the moments of inertia which arises out of the theory of reciprocation as explained in Art. 22. We will apply it here however in a slightly different form.

Let a radius vector of the ellipsoid (120) be r, of which let (x, y, z) be the extremity; along r from the origin take a length

p, and through its extremity draw a plane perpendicular to r; let $pr = g^2$: then the equation to this plane is

$$\xi x + \eta y + \zeta z = g^2, \tag{121}$$

of which the envelope, as shewn in Art. 22, is

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} + \frac{\zeta^2}{c^2} = 1, \tag{122}$$

which is an ellipsoid; and is the reciprocal ellipsoid of (120).

Now (121) is the equation to the tangent plane of this ellipsoid, the perpendicular from the centre being p and being coincident with r the radius vector of the momental ellipsoid. Hence if H is the moment of inertia about r or p, substituting for μ and r^2, we have

$$\text{H} = \frac{\mu}{r^2} = \text{M}g^4 \frac{p^2}{g^4} = \text{M}p^2: \tag{123}$$

and consequently p is the corresponding radius of gyration.

As the principal axes of the ellipsoid (122) are the radii of gyration of the axes as rotation-axes, the ellipsoid is called the ellipsoid of gyration. Hence we have the following theorem.

If at any point the ellipsoid of gyration is described, and a perpendicular is drawn from the centre to any tangent plane, the length of that perpendicular is the radius of gyration of the body relatively to it as the rotation-axis.

Hence this ellipsoid, by the variation in length of the perpendiculars from its centre on its tangent planes, represents the variation of the moments of inertia about these perpendiculars as rotation-axes, as adequately as the momental ellipsoid represents moments of inertia about its radii vectores as rotation-axes varying inversely as the square root of the moment of inertia; and whatever critical values there are of the moments of inertia, corresponding critical values are there in each ellipsoid.

It will be observed that the greatest radius vector in one ellipsoid lies along the least radius vector in the other, and vice versâ; also that the mean axes are coincident. Hence it follows that the momental ellipsoid in its shape and form is a better representation of the body or system of particles than the ellipsoid of gyration. Thus take an ellipsoid in reference to its centre, the minimum moment of inertia is evidently that about the greatest axis, the mean about the mean axis, and the maximum about the least axis: thus $\text{A} < \text{B} < \text{C}$, and consequently in its momental ellipsoid, the x-axis is the greatest, the y-axis is

the mean, and the z-axis is the least; and it consequently accords better with the material ellipsoid which it represents. The same result is true whatever is the distribution of the matter of which the moment of inertia is required.

186.] The result given in (123) may also be derived directly from (122).

If p is the perpendicular from the origin on the tangent plane of (122), l, m, n are the direction-cosines of p, and H is the moment of inertia about p as a rotation-axis, then

$$p^2 = a^2 l^2 + b^2 m^2 + c^2 n^2; \tag{124}$$

and by (109),

$$\begin{aligned} \text{H} &= \text{A} l^2 + \text{B} m^2 + \text{C} n^2 \\ &= \text{M}(a^2 l^2 + b^2 m^2 + c^2 n^2) \\ &= \text{M} p^2; \end{aligned}$$

which is the same result as (123), and expresses the same theorem.

187.] Since p is the length of the perpendicular from the origin on the tangent plane of the ellipsoid of gyration, the locus of its extremity is the central pedal of (122); and the equation to this surface is, as shewn in Art. 20,

$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2:$$

so that the surface is such that each radius vector is the radius of gyration of the body in reference to it as the rotation-axis.

This may also thus be shewn. Along the rotation-axis from the origin let a length r be taken equal to the corresponding radius of gyration: so that

$$\text{H} = \text{M} r^2,$$

then if a, b, c are the principal radii of gyration at the point, and α, β, γ are the direction-angles of r, (109) becomes,

$$r^2 = a^2(\cos\alpha)^2 + b^2(\cos\beta)^2 + c^2(\cos\gamma)^2.$$

Let (x, y, z) be the extremity of r; so that

$$\frac{x}{\cos\alpha} = \frac{y}{\cos\beta} = \frac{z}{\cos\gamma} = r;$$

$$\therefore\quad r^4 = (x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2, \tag{125}$$

which is the central pedal of the ellipsoid of gyration.

188.] We have thus obtained several geometrical expressions of the law of variation of moments of inertia about axes which pass through one and the same point: and all of course indicate critical values of the geometrical quantities which correspond to the critical values of the moments: and every one shews a special system of rectangular coordinate axes which corresponds to these

critical values; and from any one we might have taken our particular conception of them; and thus whereas we have defined principal axes as those in reference to which

$$\Sigma . myz = \Sigma . mzx = \Sigma . mxy = 0,$$

they might have been defined as those axes for which the moments of inertia have critical values. The former conception of them arose first, in the simplification of the equations of the motion, and therefore we pursued it. It is however to be observed that whatever is true of the axes of principal moments of inertia is also true of the principal axes and of the principal planes; and several properties which are true of principal planes, as defined in the last section and which might have been there demonstrated, will be proved in the course of the present section. Henceforth then we shall treat principal axes, and the geometrical principal axes of the momental ellipsoid, as identical in position.

189.] All other moments of inertia relatively to the given point are evidently intermediate to C and A; that is, are less than C, and are greater than A. From (109)

$$\text{H} = \text{A}(\cos \alpha)^2 + \text{B}(\cos \beta)^2 + \text{C}(\cos \gamma)^2;$$

which may be expressed in either of the following forms;

$$\text{H} = \text{A} + (\text{B}-\text{A})(\cos \beta)^2 + (\text{C}-\text{A})(\cos \gamma)^2;$$
$$\text{H} = \text{C} - (\text{C}-\text{A})(\cos \alpha)^2 - (\text{C}-\text{B})(\cos \beta)^2;$$

and as, see Art. 167, $\text{B}-\text{A}$, $\text{C}-\text{A}$, $\text{C}-\text{B}$, are positive quantities, C is the greatest and A is the least of all moments of inertia.

If two principal moments are equal, the momental ellipsoid becomes a spheroid; if $\text{B} = \text{C}$, the spheroid is prolate and the moments for all axes lying in the plane of (y, z) are equal to one another and to C; and other moments are less than C: if $\text{A} = \text{B}$, the momental ellipsoid becomes an oblate spheroid, and the moments for all axes lying in the plane of (x, y) are equal to one another and to A, and the moments for all other axes are greater than A. In the former case the ellipsoid of principal axes becomes an oblate spheroid; and in the latter case a prolate spheroid.

If the three principal moments are equal, $\text{A} = \text{B} = \text{C}$, and the momental ellipsoid becomes a sphere, and the moments of inertia for all axes are equal to one another. In this case also the ellipsoid of principal axes becomes a sphere.

190.] All the rotation-axes passing through a given point, for which the moments of inertia are equal to each other, lie on a cone of the second degree, whose vertex is the origin.

Let H be the moment of inertia to which all are to be equal; then, since $\rho^2 = x^2+y^2+z^2$, from (112) we have

$$\text{A}x^2+\text{B}y^2+\text{C}z^2-2\text{D}yz-2\text{E}zx-2\text{F}xy = \text{H}(x^2+y^2+z^2);$$

$$\therefore\ (\text{H}-\text{A})x^2+(\text{H}-\text{B})y^2+(\text{H}-\text{C})z^2+2\text{D}yz+2\text{E}zx+2\text{F}xy = 0; \quad (126)$$

which is the equation to a cone of the second degree whose vertex is at the origin; and the principal axes are evidently coincident with those of the momental ellipsoid. All rotation-axes therefore lying on the surface of this cone are axes of equal moment, and the cone is consequently called equimomental.

If the coordinate axes are principal axes at the origin,

$$\text{D} = \text{E} = \text{F} = 0,$$

and the equation to the equimomental cone is

$$(\text{H}-\text{A})x^2+(\text{H}-\text{B})y^2+(\text{H}-\text{C})z^2 = 0, \quad (127)$$

where A, B, C are the principal moments of inertia. As we have proved that C is the greatest and A is the least of the moments of inertia for axes passing through the origin, H must be intermediate to C and A; so that necessarily one of the coefficients in (127) is negative; and not more than two can be negative.

Let H be greater than B; then H−A and H−B are positive, and H−C is negative; in which case the axis of z is the internal principal axis, and the axes of x and y are the external principal axes. See Art. 14. And all plane sections parallel to the plane of (x, y) are ellipses.

Let H = B; then

$$(\text{B}-\text{A})^{\frac{1}{2}}x = \pm(\text{C}-\text{B})^{\frac{1}{2}}z; \quad (128)$$

which are the equations to two planes; these are indeed the cyclic planes of the momental ellipsoid. Thus all the rotation-axes at any point for which the moments of inertia are equal to the mean moment of inertia lie in two planes intersecting along the axis of mean moment, and equally inclined to the plane of mean and least moments.

Let H be less than B; then H−B and H−C are negative, and H−A is positive, so that the axis of x is the internal principal axis of the cone, and the axes of y and z are the external prin-

cipal axes; and all plane sections of the cone perpendicular to the axis of x are ellipses.

Thus, according to the configuration which we have chosen, all axes lying within the planes (128) towards the axis of z are rotation-axes of greater moment than the mean; all those lying in the planes (128) are rotation-axes of moment equal to the mean; and all those lying without the planes towards the axis of x are rotation-axes of moment less than the mean. See Fig. 2, in which the cyclic planes are delineated; all axes within the angles UOU′ and VOV′ are of the first kind, and all those within the angles UOV and U′OV′ are of the last kind.

Also the cyclic planes of the equimomental cone (127) are the cyclic planes of the momental ellipsoid; for, by reason of (98), Art. 16, the equations to the cyclic planes of (127) are

$$\{\text{H}-\text{B}-(\text{H}-\text{A})\}^{\frac{1}{2}}x = \pm \{\text{H}-\text{C}-(\text{H}-\text{B})\}^{\frac{1}{2}}z;$$
$$(\text{B}-\text{A})^{\frac{1}{2}}x = \pm (\text{C}-\text{B})^{\frac{1}{2}}z;$$

which are the same equations as (128).

If two principal moments are equal, so that the momental ellipsoid becomes a spheroid, the equimomental cones become cones of revolution.

If the three principal moments are equal, the equimomental cone degenerates into a rotation-axis, the position of which is indeterminate, and consequently all axes are principal.

191.] I propose now to express the moments of inertia and the momental ellipsoid relatively to any point of a body, in terms of the moments of inertia and the momental ellipsoid relatively to the mass-centre. We shall hereby be led to general theorems which will clear up many obscurities as to the distribution of principal axes in space, and will indicate remarkable symmetry as to their arrangement.

The following theorem must be demonstrated in the first place.

The moment of inertia of a body, or of a system of particles, about any axis is equal to the sum of the moment of inertia about a parallel axis passing through the mass-centre, and of the product of the mass and the square of the distance between the axes.

Let H be the moment of inertia about the given rotation-axis, and H′ the moment of inertia about a parallel axis passing through the mass-centre; let h be the perpendicular distance

between these two axes. Let r and r' be the distances of m from the axes of H and H$'$ respectively, and let ϕ be the angle at which r' is inclined to h; so that

$$r^2 = r'^2 - 2r'h\cos\phi + h^2;$$

$$\therefore \quad \Sigma . mr^2 = \Sigma . mr'^2 - 2h\Sigma . mr'\cos\phi + h^2\Sigma . m;$$

but $r'\cos\phi$ is the perpendicular distance from m on the plane which passes through the mass-centre and is perpendicular to h; so that $\Sigma . mr'\cos\phi = 0$. Let the mass of the body or system of particles $=$ M; then, since $\Sigma . mr^2 = \text{H}$, $\Sigma . mr'^2 = \text{H}'$, this becomes

$$\text{H} = \text{H}' + \text{M}h^2; \qquad (129)$$

which is the mathematical expression of the theorem.

Hence, if the moment of inertia about an axis at a given distance from the mass-centre is known, that about a parallel axis through the mass-centre is also known; and also the moments of inertia about all parallel axes are known.

Hence, if k is the radius of gyration relatively to the rotation-axis, and k' is the radius of gyration relatively to the parallel axis through the mass-centre,

$$k^2 = k'^2 + h^2. \qquad (130)$$

Hence also it follows, that if a line is drawn through the mass-centre of a body or system, the moments of inertia are equal for all parallel rotation-axes at equal distances from this line; or, in other words, all axes lying on the surface of a right circular cylinder whose axis passes through the mass-centre of a body or system are rotation-axes of equal moment.

Hence too of all parallel rotation-axes, that for which the moment of inertia is the least passes through the mass-centre of the body.

And of all rotation-axes, that for which the moment of inertia is absolutely the least is the axis passing through the mass-centre for which the moment of inertia is the least. This is the minimum minimorum axis.

192.] As we shall often have occasion to refer to the momental ellipsoid at the mass-centre, it is convenient to give it a distinctive name; I shall call it the Central Ellipsoid; and the principal axes and the principal planes which refer to the mass-centre will be called the central principal axes and the central principal planes: and the principal moments of inertia at the mass-centre will be called the principal central moments.

Let the mass-centre be the origin; and let the equation to the central ellipsoid be

$$\text{A}x^2+\text{B}y^2+\text{C}z^2 = \mu;$$

μ being an arbitrary constant which we may hereafter determine; and A, B, C being the principal central moments of inertia, arranged in order of magnitude as heretofore; viz. $\text{A}<\text{B}<\text{C}$. Let also A′, B′, C′ refer to the central ellipsoid and have the same signification as heretofore; viz.

$$\text{A}' = \Sigma.mx^2, \quad \text{B}' = \Sigma.my^2, \quad \text{C}' = \Sigma.mz^2;$$

also let M be the mass of the body or system of particles.

Let (ξ, η, ζ) be the point at which the principal moments and the position of the principal axes are to be determined; and let (x', y', z') be the place of m relatively to it as an origin, the coordinate axes originating at it being parallel to the central principal axes; so that

$$x' = x-\xi, \qquad y' = y-\eta, \qquad z' = z-\zeta. \tag{131}$$

Now the equation to the momental ellipsoid whose centre is at (ξ, η, ζ) is that given in (114), Art. 181, and in this case is

$$\Sigma.m(y'^2+z'^2)x^2+\Sigma.m(z'^2+x'^2)y^2+\Sigma.m(x'^2+y'^2)z^2$$
$$-2\Sigma.my'z'yz-2\Sigma.mz'x'zx-2\Sigma.mx'y'xy-\mu = 0. \tag{132}$$

Let us express the coefficients of this equation in terms of ξ, η, ζ and the central principal moments. Then, by (131),

$$\begin{aligned}\Sigma.m(y'^2+z'^2) &= \Sigma.m\{(y-\eta)^2+(z-\zeta)^2\}\\ &= \Sigma.m(y^2+z^2)-2\eta\Sigma.my-2\zeta\Sigma.mz+(\eta^2+\zeta^2)\Sigma.m\\ &= \text{A}+\text{M}(\eta^2+\zeta^2).\end{aligned}$$

This is also evident by reason of (129), in the last Article.

Let

$$\xi^2+\eta^2+\zeta^2 = \rho^2; \tag{133}$$

then $\Sigma.m(y'^2+z'^2) = \text{A}+\text{M}\rho^2-\text{M}\xi^2;$

similarly

$$\left.\begin{aligned}\Sigma.m(y'^2+z'^2) &= \text{A}+\text{M}\rho^2-\text{M}\xi^2;\\ \Sigma.m(z'^2+x'^2) &= \text{B}+\text{M}\rho^2-\text{M}\eta^2,\\ \Sigma.m(x'^2+y'^2) &= \text{C}+\text{M}\rho^2-\text{M}\zeta^2.\end{aligned}\right\} \tag{134}$$

Also

$$\begin{aligned}\Sigma.my'z' &= \Sigma.m(y-\eta)(z-\zeta)\\ &= \Sigma.myz-\eta\Sigma.mz-\zeta\Sigma.my+\eta\zeta\Sigma.m;\end{aligned}$$

similarly

$$\left.\begin{aligned}\therefore\quad \Sigma.my'z' &= \text{M}\eta\zeta;\\ \Sigma.mz'x' &= \text{M}\zeta\xi,\\ \Sigma.mx'y' &= \text{M}\xi\eta.\end{aligned}\right\} \tag{135}$$

Thus the equation to the momental ellipsoid, whose centre is at (ξ, η, ζ), which is the origin of x, y, z in this equation, is

$$(\text{A}+\text{M}\rho^2-\text{M}\xi^2)x^2+(\text{B}+\text{M}\rho^2-\text{M}\eta^2)y^2+(\text{C}+\text{M}\rho^2-\text{M}\zeta^2)z^2$$
$$-2\text{M}\eta\zeta yz-2\text{M}\zeta\xi zx-2\text{M}\xi\eta xy-\mu = 0. \tag{136}$$

The equations for determining the position of the principal axes of this ellipsoid are given in Art. 6; and for that principal axis which corresponds to a_1, a_2, a_3 we have

$$\frac{+\text{M}\rho^2-\text{M}\xi^2)a_1-\text{M}\xi\eta a_2-\text{M}\xi\zeta a_3}{a_1}=\frac{-\text{M}\xi\eta a_1+(\text{B}+\text{M}\rho^2-\text{M}\eta^2)a_2-\text{M}\eta\zeta a_3}{a_2}$$

$$=\frac{-\text{M}\xi\zeta a_1-\text{M}\eta\zeta a_2+(\text{C}+\text{M}\rho^2-\text{M}\zeta^2)a_3}{a_3}; \quad (137)$$

$$=(\text{A}+\text{M}\rho^2-\text{M}\xi^2)a_1^2+(\text{B}+\text{M}\rho^2-\text{M}\eta^2)a_2^2+(\text{C}+\text{M}\rho^2-\text{M}\zeta^2)a_3^2$$
$$-2\text{M}\eta\zeta a_2a_3-2\text{M}\zeta\xi a_3a_1-2\text{M}\xi\eta a_1a_2$$
$$=\text{H}_a, \quad (138)$$

by reason of (108), if H_a is the moment of inertia of the body about the axis (a_1, a_2, a_3) at the given point. We have also analogous equations in terms of $b_1, b_2, b_3, \text{H}_b$, and $c_1, c_2, c_3, \text{H}_c$; and from these equations the direction-cosines of the principal axes may be determined as in Arts. 6–9. Let t_1, t_2, t_3 be the direction-cosines of a principal axis, and let H be the type of $\text{H}_a, \text{H}_b, \text{H}_c$; so that from (137) we have

$$+\text{M}\rho^2-\frac{\text{M}\xi(t_1\xi+t_2\eta+t_3\zeta)}{t_1}=\text{B}+\text{M}\rho^2-\frac{\text{M}\eta(t_1\xi+t_2\eta+t_3\zeta)}{t_2}$$

$$=\text{C}+\text{M}\rho^2-\frac{\text{M}\zeta(t_1\xi+t_2\eta+t_3\zeta)}{t_3}=\text{H}; \quad (139)$$

$$(t_1\xi+t_2\eta+t_3\zeta)=\frac{(\text{A}+\text{M}\rho^2-\text{H})t_1}{\xi}=\frac{(\text{B}+\text{M}\rho^2-\text{H})t_2}{\eta}=\frac{(\text{C}+\text{M}\rho^2-\text{H})t_3}{\zeta}; \quad (140)$$

$$=\frac{t_1\xi+t_2\eta+t_3\zeta}{\dfrac{\xi^2}{\text{A}+\text{M}\rho^2-\text{H}}+\dfrac{\eta^2}{\text{B}+\text{M}\rho^2-\text{H}}+\dfrac{\zeta^2}{\text{C}+\text{M}\rho^2-\text{H}}}; \quad (141)$$

whence we have

$$\frac{\xi^2}{\text{A}+\text{M}\rho^2-\text{H}}+\frac{\eta^2}{\text{B}+\text{M}\rho^2-\text{H}}+\frac{\zeta^2}{\text{C}+\text{M}\rho^2-\text{H}}=\frac{1}{\text{M}}; \quad (142)$$

and
$$\frac{t_1}{\dfrac{\xi}{\text{A}+\text{M}\rho^2-\text{H}}}=\frac{t_2}{\dfrac{\eta}{\text{B}+\text{M}\rho^2-\text{H}}}=\frac{t_3}{\dfrac{\zeta}{\text{C}+\text{M}\rho^2-\text{H}}}. \quad (143)$$

Now (142) is a cubic equation in H, of which the three roots are real, as has been demonstrated in Art. 8. These are the three principal moments at the point (ξ, η, ζ); and are respectively less than $\text{A}+\text{M}\rho^2$, greater than $\text{A}+\text{M}\rho^2$ and less than $\text{B}+\text{M}\rho^2$, greater than $\text{B}+\text{M}\rho^2$ and less than $\text{C}+\text{M}\rho^2$.

And (143) determine the positions of the principal axes which correspond to the several values of H.

As all the quantities are given in these equations in terms of the principal central moments, the mass of the material system, and the coordinates of the given point, we shall henceforth consider the principal moments at any point to be known quantities.

Similarly we shall consider the position of the principal axes at any point to be known.

If H_a, H_b, H_c are the three roots of (142),

$$H_a + H_b + H_c = A + B + C + 2M\rho^2.$$

We might also demonstrate, as in Art. 10, that the system of principal axes is a rectangular system.

193.] These equations (142) and (143) admit of the following interpretation. The equation

$$\frac{x^2}{A+M\rho^2-H} + \frac{y^2}{B+M\rho^2-H} + \frac{z^2}{C+M\rho^2-H} = \frac{1}{M} \qquad (144)$$

represents three quadric surfaces; which are an ellipsoid, an hyperboloid of one sheet, and an hyperboloid of two sheets; because, according to the several values of H, the coefficients of x^2, y^2, z^2 (1) are all positive; (2) that of x^2 is negative and the other two are positive; (3) those of x^2 and y^2 are negative, and that of z^2 is positive; and according to our assumption of the order of magnitude of A, B, C, in the ellipsoid the x-, y-, and z-axes are respectively the least, the mean, and the greatest.

And since, as we pass from one point of a body or system of particles to another, ρ^2 and H vary, so in the denominators of the left-hand member $M\rho^2 - H$ varies according as the point changes at which the principal moments are to be determined; and thus the quadric surfaces represented by (144) are confocal with the ellipsoid whose equation is

$$\frac{x^2}{A} + \frac{y^2}{B} + \frac{z^2}{C} = \frac{1}{M}. \qquad (145)$$

Also (142) shews that the point (ξ, η, ζ) is at the point of intersection of these three confocal surfaces. And since the direction-cosines of the normal of either of these surfaces at the common point are proportional to

$$\frac{\xi}{A+M\rho^2-H}, \qquad \frac{\eta}{B+M\rho^2-H}, \qquad \frac{\zeta}{C+M\rho^2-H}, \qquad (146)$$

(143) shew that the principal axes of the body at the point (ξ, η, ζ) lie along the normals to the three confocal surfaces which intersect orthogonally at it. Or if we take one surface only, the

three principal axes are respectively normal to it, and touch the lines of curvature of the surface at the point.

Let a, b, c be the principal central radii of gyration, so that

$$\text{A} = \text{M}a^2, \qquad \text{B} = \text{M}b^2, \qquad \text{C} = \text{M}c^2; \tag{147}$$

then (145) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1; \tag{148}$$

which ellipsoid is called the central ellipsoid of gyration, as its principal axes are the radii of gyration of the body relative severally to them.

Hence we have the following construction for the position of the principal axes at any point in space. Through the given point let three quadric surfaces be drawn which are confocal with the central ellipsoid of gyration, then the normals to these surfaces at their common point are the principal axes at the point; and the principal moments at the point are the three roots of the cubic equation (142).

194.] Again, let us multiply the numerators and denominators of the three last terms of (140) severally by

$$(\text{B}-\text{C})\,t_2 t_3, \qquad (\text{C}-\text{A})\,t_3 t_1, \qquad (\text{A}-\text{B})\,t_1 t_2;$$

and let us add the numerators and the denominators respectively; as the sum of the numerators vanishes, so must also the sum of the denominators; and therefore

$$(\text{B}-\text{C})\,\xi t_2 t_3+(\text{C}-\text{A})\,\eta t_3 t_1+(\text{A}-\text{B})\,\zeta t_1 t_2 = 0. \tag{149}$$

Now as t_1, t_2, t_3 are the direction-cosines of one of the principal axes at the point (ξ, η, ζ) referred to central principal axes, we may replace them by x, y, z; whence we have

$$(\text{B}-\text{C})\,\xi yz+(\text{C}-\text{A})\,\eta zx+(\text{A}-\text{B})\,\zeta xy = 0; \tag{150}$$

which is the equation to a cone of the second degree; on the surface of which therefore lie the three principal axes at the point (ξ, η, ζ), this being the vertex of the cone. Since (150) is satisfied when x, y, z are proportional to ξ, η, ζ it follows that the line drawn from (ξ, η, ζ) to the mass-centre lies on the cone. It is evident also that the three axes of x, y, z lie in the surface of the cone. Hence we have the following geometrical theorem:

The three principal axes at any point of a body, the three lines drawn through that point parallel to the central principal axes, and the line drawn from the point to the mass-centre all lie in the surface of a cone of the second degree.

195.] Now the three surfaces of the second degree, which equation (144) represents, and which are confocal with the central ellipsoid of gyration, intersect orthogonally at not only (ξ, η, ζ), but at seven other points which are situated symmetrically in the other octants, and which correspond to the several combinations of the double signs of ξ, η, and ζ. The equation (142) is the same whatever are the signs of ξ, η, and ζ; so that the principal moments are the same at each of the eight points, which are the angles of a rectangular parallelepipedon whose centre is at the mass-centre, and whose sides are parallel to the central principal axes: and as the equations for determining the position of the principal axes (143) are the same when the signs of ξ, η, ζ are all changed, so the principal axes at (ξ, η, ζ) are parallel to those at $(-\xi, -\eta, -\zeta)$; and similarly the principal axes at the other six points of symmetry are arranged in pairs corresponding to the ends of a diameter.

Thus the body or system of particles is symmetrically arranged as to principal axes, principal moments, and all moments of inertia, relative to the mass-centre, the central axes, and the central principal planes. And as space is divided by the principal central planes into eight portions, so to a point in any octant a point of symmetry corresponds in each of the other seven octants, at which the principal moments are equal, and the momental ellipsoid is similarly situated with respect to the centre and the principal central axes. Therefore whatever is the form of the moving system, be it a continuous body or a system of disconnected particles, however various the distribution of its parts, however unsymmetrical its bounding surface, yet it has a mass-centre, central axes, and a central momental ellipsoid; and the arrangement of all other moments and axes is symmetrical relatively to that point. In discussing therefore the rotation of an irregular mass about an axis passing through a fixed point, we may dismiss from our minds all the irregularities of the mass, and consider in its stead either the regular and symmetrical central ellipsoid, or, the central ellipsoid of gyration; for the properties of either of these surfaces will express all the possible circumstances of motion of the system.

Hence if two material systems have the same mass, the same mass-centre, the same principal axes, and the same principal moments of inertia, moments of inertia of the two systems are

equal about all axes and the systems are dynamically equivalent, for these are the only quantities connected with the moving matter which enter into the equations of motion so far as expressed momenta are concerned. Two such material systems are sometimes said to be equimomental.

196.] The theorem proved in Art. 29 also suggests another construction whereby the position of the principal axes at a given point may be determined.

From the point as a vertex let a cone be described enveloping the central ellipsoid of gyration (148); then, as we have shewn in Art. 29, the principal axes of the cone at its vertex are the normals to the three surfaces of the second degree which intersect at it and are confocal with the enveloped ellipsoid; and as the principal axes lie along these three normals, so they also coincide with the principal axes of the cone at the point.

This construction of the principal axes at a given point is quite as expressive and as fertile in results as that explained in Art. 192. It also exhibits equally well the symmetry of every material system with reference to its mass-centre and the eight octants which meet thereat, which has been pointed out in the preceding Article. We shall hereafter employ one or the other construction as best suits the occasion.

197.] The cone which is reciprocal to this enveloping cone is an equimomental cone: this might be demonstrated directly from the equation to the enveloping cone which is given in (169), Art. 29; for if we determined the equation to its reciprocal cone, it would be identical with (126), Art. 190. The following proof however is more concise. Through the given point let a tangent plane be drawn to the ellipsoid of gyration; this plane being evidently a tangent plane to the enveloping cone. To it let perpendiculars be drawn from the centre of the ellipsoid and from the given point: let the distance between these perpendiculars be q, say; then, since by Art. 185 the moment of inertia of the body about the former line $= \text{M}p^2$, therefore the moment of inertia about the latter line, by reason of Art. 191, is

$$\text{M}p^2 + \text{M}q^2 = \text{M}\rho^2, \tag{151}$$

if ρ is the distance of the given point from the mass-centre: but this latter line is a generating line of the reciprocal cone, and ρ^2 is the same for all generating lines of the cone; and

therefore the moment of inertia is the same for all generating lines of the reciprocal cone; and consequently the reciprocal cone is equimomental.

198.] This is a particular case of a general process by which a series of equimomental cones may be described. Let

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}+\frac{z^2}{c^2+\theta}=1 \tag{152}$$

be the equation to a quadric surface confocal with the central ellipsoid of gyration. Let a tangent plane be drawn to it from the given point, and let the perpendicular to the tangent plane make angles a, β, γ with the central principal axes. Now if H is the moment of inertia about a perpendicular to the tangent plane at the given point, if $p =$ the length of the perpendicular from the centre, and q is the length of the perpendicular distance from the given point on p, then by (129),

$$\begin{aligned} \text{H} &= \text{A}(\cos a)^2+\text{B}(\cos\beta)^2+\text{C}(\cos\gamma)^2+\text{M}q^2, \\ &= \text{M}\{a^2(\cos a)^2+b^2(\cos\beta)^2+c^2(\cos\gamma)^2\}+\text{M}q^2; \end{aligned}$$

but

$$\begin{aligned} p^2 &= (a^2+\theta)(\cos a)^2+(b^2+\theta)(\cos\beta)^2+(c^2+\theta)(\cos\gamma)^2, \\ &= a^2(\cos a)^2+b^2(\cos\beta)^2+c^2(\cos\gamma)^2+\theta: \end{aligned}$$

$$\begin{aligned} \therefore\ \text{H} &= \text{M}(p^2+q^2)-\text{M}\theta, \\ &= \text{M}\rho^2-\text{M}\theta, \end{aligned} \tag{153}$$

if ρ is the distance of the given point from the mass-centre: and as this quantity is independent of the position of the tangent plane, and depends only on the position of the given point, it is the same for all generating lines of the reciprocal cone, and the reciprocal cone is consequently an equimomental cone.

And as the number of quadric surfaces confocal with the ellipsoid of gyration is unlimited, and as from a given point an enveloping cone may be drawn to each, so may reciprocal cones be drawn which will be equimomental cones. All the cones of this system will be coaxial, confocal, and concyclic.

If the three confocal quadrics pass through the given point, the enveloping cones becomes planes, which are the tangent planes to the quadrics; and the three principal axes become the normals to the surfaces; the same result as we have before arrived at. Thus the two constructions, (1) by the normals of the three confocal conics and (2) by the axes of enveloping cones, become identical. This result also follows from Art. 27.

As the focal conics, see Art. 26, are particular and degenerate

forms of quadrics which are confocal with the central ellipsoid of gyration, they may be taken as directors of the cones whose geometrical axes are the principal axes of the system at the vertex of the cone. Their equations are

$$\left.\begin{aligned} x=0, \quad \frac{y^2}{b^2-a^2}+\frac{z^2}{c^2-a^2} &= 1\,; \\ y=0, \quad \frac{z^2}{c^2-b^2}+\frac{x^2}{a^2-b^2} &= 1\,; \\ z=0, \quad \frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2} &= 1\,: \end{aligned}\right\}$$

as found by replacing θ in (152) successively by $-a^2$, $-b^2$, $-c^2$: these express an ellipse in the plane of (y, z), a hyperbola in the plane of (z, x), and an impossible curve in the plane of (x, y).

199.] The three quadrics, confocal with the central ellipsoid of gyration, which pass through a given point not only by their normals determine the position of the three principal axes at the point, but also by their parameters give the values of the principal moments and the principal radii of gyration.

Let (152) be the equation to the confocal quadric passing through (ξ, η, ζ), so that we have

$$\frac{\xi^2}{a^2+\theta}+\frac{\eta^2}{b^2+\theta}+\frac{\zeta^2}{c^2+\theta} = 1; \tag{154}$$

and let θ_1, θ_2, θ_3 be the three values of θ, which are the roots of this equation; of which let θ_1 be the greatest, θ_2 the mean, and θ_3 the least; so that θ_1, θ_2, θ_3 are respectively the parameters of the ellipsoid, of the hyperboloid of one sheet, and of the hyperboloid of two sheets; then we have the following arrangement of quantities in ascending order of magnitude,

$$-c^2,\ \theta_3,\ -b^2,\ \theta_2,\ -a^2,\ \theta_1,\ +\infty\,;$$

being the same order as that shewn in Art. 27.

Now on comparing the above equation with (142), Art. 192, since they are identical, we have

$$\theta = \frac{\mathrm{M}\rho^2-\mathrm{H}}{\mathrm{M}}\,;$$

$$\therefore\ \mathrm{H} = \mathrm{M}(\rho^2-\theta)\,;$$

and since there are three values of θ, viz. θ_1, θ_2, θ_3, there are three corresponding values of H, viz. H_a, H_b, H_c; which lie along the normals to the ellipsoid, the hyperboloid of one sheet, and the hyperboloid of two sheets respectively; so that

$$\mathrm{H}_a = \mathrm{M}(\rho^2-\theta_1), \quad \mathrm{H}_b = \mathrm{M}(\rho^2-\theta_2), \quad \mathrm{H}_c = \mathrm{M}(\rho^2-\theta_3).$$

The moment of inertia is therefore the least for the normal to the ellipsoid, mean for the normal to the hyperboloid of one sheet, and the greatest for the normal to the hyperboloid of two sheets.

This result is the same as that expressed in (153), and found by a different process.

200.] Let us now consider the forms which these constructions and expressions take in certain special circumstances; and we will first take the case where the point at which the position of the central principal axes is to be determined is in one of the central principal planes, say in that of (x, y): then $\zeta = 0$. Now the section of the central ellipsoid of gyration by the plane of (x, y) is the ellipse whose equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1; \tag{155}$$

and we have hereby the following construction of the principal axes. Through the point (ξ, η) describe the two conics, viz. the ellipse and the hyperbola, which are confocal with (155): draw the tangents to the two conics at their point of intersection; these are two of the principal axes at the point, and the other is of course perpendicular to them, and normal to the plane of (x, y) which contains them. Hence if O is the origin, which is the mass-centre of the system, and P is the point in the plane of (x, y) at which the principal axes are to be determined, and S and H are the foci of the ellipse (155) which are on the y-axis, since b is greater than a, and distances $\pm (b^2-a^2)^{\frac{1}{2}}$ from O; then if SP and HP be drawn, and PT and PG are the external and internal bisectors of the angle SPH, these lines are respectively the tangents to the two confocal conics which intersect at P, and are the principal axes at P which lie in the plane of the conics.

If H_a, H_b, H_c are the three principal moments at P, of which H_c is parallel to the z-axis, $\mathrm{H}_c = \mathrm{C}+\mathrm{M}\rho^2$, and H_a, H_b are the roots of the quadratic equation

$$\mathrm{H}^2-\mathrm{MH}(a^2+b^2+\rho^2)+\mathrm{M}^2(a^2b^2+a^2\xi^2+b^2\eta^2)=0, \tag{156}$$

which is the value of (142) when $\zeta=0$.

These however may thus be found.

Let the equation to the conics passing through (ξ, η) and confocal with (155) be

$$\frac{x^2}{a^2+\theta}+\frac{y^2}{b^2+\theta}=1; \tag{157}$$

and let H_a and H_b be the moments about the tangents to these

conics at the point (ξ, η); then by a process exactly similar to that of Art. 198, for the ellipse we have

$$\begin{aligned}\text{H}_a &= \text{B}(\sin\alpha)^2 + \text{A}(\cos\alpha)^2 + \text{M}\{(b^2+\theta)(\cos\alpha)^2 + (a^2+\theta)(\sin\alpha)^2\}\\ &= \text{M}\,b^2 + \text{M}\,a^2 + \text{M}\,\theta\\ &= \text{M}\,a^2 + \text{M}\left(\frac{\text{SP}+\text{PH}}{2}\right)^2\\ &= \text{A} + \frac{\text{M}}{4}(\text{SP}+\text{PH})^2; \qquad (158)\end{aligned}$$

since
$$\text{SP} + \text{PH} = 2(b^2+\theta)^{\frac{1}{2}}.$$

Similarly for the hyperbola,

$$\begin{aligned}\text{H}_b &= \text{M}\,a^2 + \text{M}(b^2+\theta)\\ &= \text{A} + \frac{\text{M}}{4}(\text{SP}-\text{PH})^2. \qquad (159)\end{aligned}$$

Which values satisfy the equation (156).

Equation (158) shews that the moments of inertia are the same for all tangents to the ellipse (157) and are also the same for all tangents to the hyperbola, as shewn by (159). Similar results are of course true for points in each of the other central principal planes.

If the point is in a central principal axis, say that of ξ, at a distance ξ from the centre; then $\eta = \zeta = 0$, and from (142) we have

$$\left.\begin{aligned}\text{H}_a &= \text{A},\\ \text{H}_b &= \text{B} + \text{M}\,\xi^2,\\ \text{H}_c &= \text{C} + \text{M}\,\xi^2;\end{aligned}\right\} \qquad (160)$$

and similar results are true for points on the other axes.

At the points S and H on the axis of η at distances $\pm(b^2-a^2)^{\frac{1}{2}}$ from the mass-centre

$$\left.\begin{aligned}\text{H}_a &= \text{A} + \text{M}(b^2-a^2)\\ &= \text{B},\\ \text{H}_b &= \text{B},\\ \text{H}_c &= \text{C} + \text{M}(b^2-a^2)\\ &= \text{C}+\text{B}-\text{A};\end{aligned}\right\} \qquad (161)$$

so that $\text{H}_a = \text{H}_b$, and the moments of inertia about all axes in the plane of (x, y) are equal and the momental ellipsoids at these two points become spheroids.

Similar results occur at two points on the z-axis at distances $\pm(c^2-a^2)^{\frac{1}{2}}$ from the centre of gravity,

$$\text{H}_a = \text{C}, \quad \text{H}_b = \text{B}+\text{C}-\text{A}, \quad \text{H}_c = \text{C}, \qquad (162)$$

and the momental ellipsoids again become spheroids.

This subject will receive considerable illustration from our subsequent enquiry into the character of principal axes.

201.] Let us in the next place investigate the positions of those points at which (1) two of the three principal moments, (2) all three principal moments, are equal.

If (ξ, η, ζ) is a point at which two principal moments of inertia are equal, two of the three values of H in (142) are equal to one another; and consequently (142) and its H-differential are simultaneously true. Now the H-differential may be put into the form

$$\frac{\xi^2}{(\mathrm{A}+\mathrm{M}\rho^2-\mathrm{H})^2}+\frac{\eta^2}{(\mathrm{B}+\mathrm{M}\rho^2-\mathrm{H})^2}+\frac{\zeta^2}{(\mathrm{C}+\mathrm{M}\rho^2-\mathrm{H})^2}=0; \quad (163)$$

or $\xi^2(\mathrm{B}+\mathrm{M}\rho^2-\mathrm{H})^2(\mathrm{C}+\mathrm{M}\rho^2-\mathrm{H})^2+\eta^2(\mathrm{C}+\mathrm{M}\rho^2-\mathrm{H})^2(\mathrm{A}+\mathrm{M}\rho^2-\mathrm{H})^2$

$$+\zeta^2(\mathrm{A}+\mathrm{M}\rho^2-\mathrm{H})^2(\mathrm{B}+\mathrm{M}\rho^2-\mathrm{H})^2=0. \quad (164)$$

All the roots of this equation as it stands are imaginary; and as the reality of the roots of (142) has been demonstrated, (164) must be satisfied identically: this may be done as follows:

(1) Let $\xi=0$, and $\mathrm{H}=\mathrm{A}+\mathrm{M}\rho^2$; in which case (142) becomes

$$\frac{\eta^2}{\mathrm{B}-\mathrm{A}}+\frac{\zeta^2}{\mathrm{C}-\mathrm{A}}=\frac{1}{\mathrm{M}}; \quad (165)$$

and since $\mathrm{A}=\mathrm{M}a^2$, $\mathrm{B}=\mathrm{M}b^2$, $\mathrm{C}=\mathrm{M}c^2$, this becomes

$$\frac{\eta^2}{b^2-a^2}+\frac{\zeta^2}{c^2-a^2}=1; \quad (166)$$

and, according to our assumption, $c>b>a$; thus (166) represents an ellipse in the plane of (η, ζ), and is in that plane the focal conic of the ellipsoid of gyration. Each of two of the principal moments of inertia $=\mathrm{A}+\mathrm{M}\rho^2$; and the third $=\mathrm{B}+\mathrm{C}-\mathrm{A}$, by reason of equation (142).

(2) Let $\eta=0$, and $\mathrm{H}=\mathrm{B}+\mathrm{M}\rho^2$; in which case (142) becomes

$$\frac{\xi^2}{a^2-b^2}+\frac{\zeta^2}{c^2-b^2}=1; \quad (167)$$

which represents an hyperbola in the plane of (ζ, ξ), whose real and imaginary axes lie respectively along the axes of ζ and ξ, and is another focal conic of the ellipsoid of gyration. Two of the principal moments of inertia $=\mathrm{B}+\mathrm{M}\rho^2$; and the third principal moment $=\mathrm{C}+\mathrm{A}-\mathrm{B}$.

(3) Let $\zeta=0$; and $\mathrm{H}=\mathrm{C}+\mathrm{M}\rho^2$; then (142) becomes

$$\frac{\xi^2}{a^2-c^2}+\frac{\eta^2}{b^2-c^2}=1; \quad (168)$$

which is the other focal conic of the ellipsoid of gyration, and is imaginary.

At all points therefore of the real focal conics of the ellipsoid of gyration, two roots of (142) are equal; and two principal moments are equal: the tangent line to the focal conic is the axis of the unequal principal moment, and the normal plane to the focal conic is the plane which contains the axes of equal moments. All axes therefore in this plane which pass through the point of contact are axes of equal moment: so that the number of axes of equal moment is infinite. Indeed the other two quadric surfaces, which with the focal conic are confocal with the ellipsoid of gyration, become flat, and infinitesimally thin; so that any plane which passes through the tangent of the focal conic is a tangent plane to one of these surfaces, and the perpendiculars to these planes at the point of contact are principal axes.

This result is also evident from the construction of principal axes which is given by the enveloping cone of the ellipsoid of gyration: the enveloping quadric is a cone of revolution if its vertex is on a focal conic; the tangent of the conic is the internal axis of the cone; and any two lines in the plane through the vertex of the cone, which is perpendicular to the internal axis, are external axes. In this construction however it is to be observed that the enveloping cones may be imaginary.

Hence we have two distinct conics, one in the plane of (η, ζ) and the other in the plane of (ζ, ξ), which are respectively an ellipse and hyperbola, at every point of which the position of only one principal axis of the body is determinate; but as the moments corresponding to the other two principal axes are equal, the position of these axes is indeterminate.

At every point on a focal conic, the momental ellipsoid becomes a spheroid, whose axis of revolution is the tangent to the focal conic.

202.] If (ξ, η, ζ) is a point at which all the principal moments are equal, the three roots of (142) are equal; and (136) represents a sphere; so that

$$\eta\zeta = \zeta\xi = \xi\eta = 0; \qquad (169)$$

$$\text{A} + \text{M}(\eta^2 + \zeta^2) = \text{B} + \text{M}(\zeta^2 + \xi^2) = \text{C} + \text{M}(\xi^2 + \eta^2). \qquad (170)$$

From (169) it follows that of the three quantities ξ, η, ζ two

must be equal to zero; and as C is $>$ B $>$ A, the only possible supposition is

$$\xi = \eta = 0;$$

therefore

$$\text{A} + \text{M}\zeta^2 = \text{B} + \text{M}\zeta^2 = \text{C};$$

$$\therefore \quad \text{A} = \text{B}; \tag{171}$$

$$\zeta = \pm\left(\frac{\text{C}-\text{A}}{\text{M}}\right)^{\frac{1}{2}} = \pm\,(c^2-a^2)^{\frac{1}{2}}; \tag{172}$$

therefore two of the central principal moments, viz. the mean and the least, must be equal to each other; and thus the central ellipsoid must be a prolate spheroid; in which case, on the axis of greatest central moment there are two points, viz. the foci of the spheroid of gyration, equally distant from the mass-centre at which all the principal moments are equal, and therefore all axes are principal axes. At each of these points the momental ellipsoid becomes a sphere.

If C = A, that is, if all the central principal moments are equal, from (172), $\zeta = 0$; and at no other point in the body, except the mass-centre, are all the principal moments equal.

These results might have been arrived at from considerations founded on the properties of the focal conics: the three principal moments can be equal only when the focal conics of the ellipsoid of gyration have a common point; and as (166) and (167) can have a common point only on the central axis of ζ, $\xi = \eta = 0$; in which case

$$\zeta = \pm\,(c^2-a^2)^{\frac{1}{2}} = \pm(c^2-b^2)^{\frac{1}{2}}; \tag{173}$$

$$\therefore \quad a = b, \quad \text{or} \quad \text{A} = \text{B}. \tag{174}$$

203.] I propose in the next place to inquire into the locus-surface of these points (ξ, η, ζ) at which *one* of the principal moments has a given value. Let k be the radius of gyration corresponding to this given value of the principal moment of inertia; so that if H is the principal moment, H $= \text{M}k^2$: also let a, b, c be the principal central radii of gyration; then (142) becomes

$$\frac{x^2}{r^2+a^2-k^2} + \frac{y^2}{r^2+b^2-k^2} + \frac{z^2}{r^2+c^2-k^2} = 1; \tag{175}$$

where

$$r^2 = x^2+y^2+z^2;$$

and if we replace 1 in the right-hand member by $\dfrac{x^2+y^2+z^2}{r^2}$, we have

$$\frac{x^2(a^2-k^2)}{r^2+a^2-k^2} + \frac{y^2(b^2-k^2)}{r^2+b^2-k^2} + \frac{z^2(c^2-k^2)}{r^2+c^2-k^2} = 0. \tag{176}$$

The surfaces which these equations represent have been named

by Sir William Thomson Equimomental Surfaces. As generally at every point k has three different values, so will three equimomental surfaces pass through every point.

By giving different values to k we have different equimomental surfaces. According to our hypothesis a is the least radius of gyration for all axes passing through the mass-centre; it is therefore absolutely the least of all radii of gyration, but they have no superior limit; so that k may have all values from a to $+\infty$. Now (176) will express surfaces different in form according as k is greater than c; lies between c and b; is equal to b; and lies between b and a. If k is greater than c, the equimomental surface is the same as the wave surface in biaxial crystals *.

The equation to the equimomental surface given in (175) is, as it will be observed, the same as that of an apsidal of a quadric which has been found in Art. 19, a^2, b^2, c^2 in (124) of that Article, having been replaced by k^2-a^2, k^2-b^2, k^2-c^2.

204.] The principal axis at a given point (x, y, z) of the equimomental surface lies in the tangent plane at that point; and passes through the point where a perpendicular from the origin on the tangent plane meets it.

Let us replace A, B, C, H in (143), severally by Ma^2, Mb^2, Mc^2, Mk^2; then the direction-cosines of a principal axis at the point (x, y, z) are respectively proportional to

$$\frac{x}{r^2+a^2-k^2}, \quad \frac{y}{r^2+b^2-k^2}, \quad \frac{z}{r^2+c^2-k^2}. \tag{177}$$

Now let l, m, n be the direction-cosines of the line drawn through (x, y, z) to the point of intersection of the tangent plane of (175) with the perpendicular on it from the centre. Then, if $F(x, y, z) = 0$ is the equation to the surface, the direction-cosines of this line are easily shewn to be proportional to

$$x\Big\{\Big(\frac{dF}{dx}\Big)^2+\Big(\frac{dF}{dy}\Big)^2+\Big(\frac{dF}{dz}\Big)^2\Big\}-\Big(\frac{dF}{dx}\Big)\Big\{x\Big(\frac{dF}{dx}\Big)+y\Big(\frac{dF}{dy}\Big)+z\Big(\frac{dF}{dz}\Big)\Big\},$$

$$y\Big\{\Big(\frac{dF}{dx}\Big)^2+\Big(\frac{dF}{dy}\Big)^2+\Big(\frac{dF}{dz}\Big)^2\Big\}-\Big(\frac{dF}{dy}\Big)\Big\{x\Big(\frac{dF}{dx}\Big)+y\Big(\frac{dF}{dy}\Big)+z\Big(\frac{dF}{dz}\Big)\Big\},$$

$$z\Big\{\Big(\frac{dF}{dx}\Big)^2+\Big(\frac{dF}{dy}\Big)^2+\Big(\frac{dF}{dz}\Big)^2\Big\}-\Big(\frac{dF}{dz}\Big)\Big\{x\Big(\frac{dF}{dx}\Big)+y\Big(\frac{dF}{dy}\Big)+z\Big(\frac{dF}{dz}\Big)\Big\}.$$

* A full discussion of this surface will be found in a Thèse de Mécanique, by M. Peslin. Mallet-Bachelier, Paris, 1858.

Let $$\frac{x^2}{(r^2+a^2-k^2)^2}+\frac{y^2}{(r^2+b^2-k^2)^2}+\frac{z^2}{(r^2+c^2-k^2)^2} = s; \quad (178)$$

then from (175),

$$\left.\begin{aligned}\left(\frac{d\text{F}}{dx}\right) &= \frac{2x}{r^2+a^2-k^2}-2xs,\\ \left(\frac{d\text{F}}{dy}\right) &= \frac{2y}{r^2+b^2-k^2}-2ys,\\ \left(\frac{d\text{F}}{dz}\right) &= \frac{2z}{r^2+c^2-k^2}-2zs;\end{aligned}\right\} \quad (179)$$

so that l, m, n are proportional to

$$\frac{x}{r^2+a^2-k^2}, \quad \frac{y}{r^2+b^2-k^2}, \quad \frac{z}{r^2+c^2-k^2}; \quad (180)$$

which are the same as (177): and thus the theorem as enunciated is proved.

205.] The investigations of the preceding Articles, and the methods given for the construction of principal axes, shew that an axis taken arbitrarily in a body may not be a principal axis at any point on it; because those axes alone are principal which are normal to a quadric which is confocal with the central ellipsoid of gyration.

If an axis of a body is a principal axis, the point at which it is principal is called its principal point; let us call the plane which is perpendicular to it at its principal point, and which contains the other two principal axes, its principal plane; so that a principal plane of an axis is a plane tangent to a quadric, confocal with the central ellipsoid of gyration, at the point where that axis cuts that surface.

In considering therefore the axes of a body, we may distinguish (1) those which are principal at every point along them; (2) those which are principal at one point; (3) those which are not principal at all. Let us consider them in order.

The three principal central axes cut at right angles all the quadrics which are confocal with the central ellipsoid of gyration; and as the number of such surfaces is infinite, so every point on a central principal axis is principal; and as the other two confocal surfaces at these points degenerate into the coordinate planes, the other two principal axes are always parallel to the central principal axes.

The three central principal axes are the only lines which have

the property of being principal at every point on them: in this respect then, as in others, they form an unique system.

Some special cases of axes which are principal at a particular point deserve consideration.

The sphere whose centre is at the mass-centre, and whose radius is infinitely great, is a surface confocal with the central ellipsoid of gyration; and as all lines drawn through the mass-centre are normal to this sphere at the infinity point, so every line drawn through the mass-centre is a principal axis at a point which is at an infinite distance along it.

If three confocal quadric surfaces pass through a point in one of the central principal planes, one of the confocal surfaces becomes flat, and the normal to this surface is a perpendicular to the central plane; so that one of the principal axes at that point is always normal to the principal central plane; and thus all axes parallel to a central principal axis are principal at the points where they intersect a central principal plane: the other two principal axes are the tangent and the normal respectively to a conic passing through the point which is confocal with the trace of the central ellipsoid of gyration in that plane.

206.] What however are the analytical conditions which are to be satisfied when a line is a principal axis at one of its points? also let us find its principal point, and the equation to its principal plane.

Let the mass-centre be the origin, and let the central principal axes be the coordinate axes; let the equations to a certain line be

$$\frac{x-x_0}{l} = \frac{y-y_0}{m} = \frac{z-z_0}{n} = s, \text{ (say)}; \tag{181}$$

and let the equation to a plane perpendicular to it be

$$lx+my+nz-p = 0. \tag{182}$$

Now if (181) is a principal axis, and (182) is its principal plane, (181) is the normal to, and (182) is the tangent plane at the same point to, a quadric which is confocal with the central ellipsoid of gyration.

Let the equation to this confocal surface be

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{c^2+\theta} = 1; \tag{183}$$

as (181) is to be normal to this surface we have

$$\frac{x}{l(a^2+\theta)} = \frac{y}{m(b^2+\theta)} = \frac{z}{n(c^2+\theta)}; \tag{184}$$

$$\therefore \quad \frac{b^2-c^2}{l}x + \frac{c^2-a^2}{m}y + \frac{a^2-b^2}{n}z = 0. \tag{185}$$

Also, since from (181),

$$x = x_0 + ls, \qquad y = y_0 + ms, \qquad z = z_0 + ns,$$

we have

$$\frac{b^2-c^2}{l}x_0 + \frac{c^2-a^2}{m}y_0 + \frac{a^2-b^2}{n}z_0 = 0, \tag{186}$$

or $$mn(b^2-c^2)x_0 + nl(c^2-a^2)y_0 + lm(a^2-b^2)z_0 = 0; \tag{187}$$

which is a relation between the elements of the line (181) and the central principal radii of gyration, which must be satisfied, when (181) is a principal axis at one of its points.

Now (185) or (186) is the equation to the plane which contains any point in the ellipsoid of gyration and the perpendicular from the centre on the plane which touches the ellipsoid at that point; and the same is equally true, if we read for the ellipsoid of gyration any quadric confocal with that ellipsoid: hence the first condition is, that the line which is to be a principal axis must be in this plane. Hence we have the following construction; if a line is drawn through the mass-centre parallel to the given line, and planes are drawn perpendicular to this line and touching quadrics confocal with the ellipsoid of gyration, then all lines parallel to the given line and passing through the several points of contact are principal axes; and all these lines are in the same plane, viz. (185); so that of a system of parallel straight lines, only those can be principal axes which are in this plane.

This construction also follows from the investigation of Art. 28: the equation (165) of which is identical with (185) or (186); so that if O is the centre of the ellipsoid, and OK (l, m, n) is the perpendicular to the plane which touches the ellipsoid at P, OKP is the plane in which all lines parallel to OK are principal axes.

Hence if a circular cylinder is described about OK as its axis of revolution, all its generators are equimomental axes, but only those which lie in the plane OKP are principal axes. And hence we have the following theorem:

On every circular cylinder whose axis of revolution passes through the mass-centre of a system, there are two, and only

two generators which are principal as well as equimomental, and these are diametrically opposite and lie in the plane OKP.

207.] Now when the line (181) fulfils the condition of being in the plane OKP, the point at which it is normal to a confocal quadric is its principal point, and its coordinates may thus be found.

Let (x, y, z) be the principal point; and let OK, $= k$, be such that

$$k^2 = a^2 l^2 + b^2 m^2 + c^2 n^2; \tag{188}$$

then as p is the perpendicular on the parallel plane which touches the confocal quadric (183),

$$\begin{aligned} p^2 &= (a^2+\theta)\, l^2 + (b^2+\theta)\, m^2 + (c^2+\theta)\, n^2, \\ &= k^2 + \theta. \end{aligned} \tag{189}$$

Then, as (x, y, z) is the point where (181) intersects (183) at right angles,

$$\begin{aligned} \frac{x}{l(a^2+\theta)} = \frac{y}{m(b^2+\theta)} &= \frac{z}{n(c^2+\theta)} = \frac{1}{p} \\ &= \frac{lx+my+nz}{a^2l^2+b^2m^2+c^2n^2+\theta} = \frac{p}{k^2+\theta}; \end{aligned} \tag{190}$$

and substituting the values of x, y, z given in (181), and eliminating s and θ, we have

$$p = \frac{mn(b^2-c^2)}{ny_0 - mz_0} = \frac{nl(c^2-a^2)}{lz_0 - nx_0} = \frac{lm(a^2-b^2)}{mx_0 - ly_0}, \tag{191}$$

whereby p is given; and thus θ and the particular confocal quadric become determined. Also from (190),

$$\left.\begin{aligned} x &= \frac{l(a^2+\theta)}{p} = \frac{l}{p}(a^2+p^2-k^2), \\ y &= \frac{m(b^2+\theta)}{p} = \frac{m}{p}(b^2+p^2-k^2), \\ z &= \frac{n(c^2+\theta)}{p} = \frac{n}{p}(c^2+p^2-k^2); \end{aligned}\right\} \tag{192}$$

which are the coordinates to the principal point of the line (181) when it is a principal axis.

It is also evident from Art. 28, that the principal points corresponding to a series of parallel principal axes lie in a rectangular hyperbola, one of whose asymptotes is the line OK.

From (192) we have

$$(b^2-c^2)\, xmn + (c^2-a^2)\, ynl + (a^2-b^2)\, zlm = 0:$$

now if we take the point (x, y, z), being the principal point of the line (l, m, n), to be an origin, and on the line (l, m, n) take

a point (ξ, η, ζ), then we may replace l, m, n by ξ, η, ζ respectively, and the equation becomes

$$(b^2-c^2)\, x\eta\zeta + (c^2-a^2)\, y\zeta\xi + (a^2-b^2)\, z\xi\eta = 0; \qquad (193)$$

which is the equation to a quadric cone, and is the same as that already found in (150), Art. 194, by another process. Hence, if a line is a principal axis and (x, y, z) is its principal point, the axis lies on the surface of this cone, and as it also lies on the plane (187), which is a tangent plane to this cone, the axis is that along which the plane touches this cone.

208.] As the relation (187) must be satisfied by the elements of a line capable of being a principal axis at some one of its points, it is evident that a line taken arbitrarily may not be a principal axis at any point.

Let us, however, consider certain special cases in which (187) is satisfied.

(1) Let $x_0 = y_0 = z_0$: then (187) is satisfied; and consequently all straight lines passing through the mass-centre are principal axes at some point on them: and (192) shew that generally this point is at infinity. Also (187) is satisfied if

$$\frac{x_0}{l} = \frac{y_0}{m} = \frac{z_0}{n};$$

that is, if (x_0, y_0, z_0) is on a line passing through the mass-centre, in which case, from (191), $p = \infty$, and consequently the principal point and the corresponding principal plane are at infinity.

(2) Next, if (x_0, y_0, z_0) is on a central principal axis, say on that of x, so that $y_0 = z_0 = 0$: then (187) is satisfied if either $m = 0$, or $n = 0$; that is, if the axis is perpendicular to the axis of y, or to the axis of z. Hence every point on a central principal axis may be a principal point, and the other principal axes are parallel to the other two central axes.

(3) Let (x_0, y_0, z_0) be in a central principal plane: say, let $z_0 = 0$: then (187) is satisfied if $n = 0$, that is, if the line lies in the plane of (x, y). Consequently every point in a central principal plane may be a principal point to a line lying in the plane; in other words, all lines lying in a central principal plane are principal axes at such point. (187) is also satisfied when $l = m = 0$, that is for the line parallel to the axis of z. Hence for every point in a central principal plane one principal axis is perpendicular to that plane.

(4) If $a = b = c$, every line in space is a principal axis at some

point on it, the principal point being the intersection of the line with the perpendicular on it from the mass-centre.

209.] That every line in a central principal plane is a principal axis at some point may be proved by the following process. Whatever is the position of the line in the plane (say) of (x, y), a conic, confocal with the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

can always be drawn to which this line shall be a normal.

Thus, let the equation to the line be

$$lx + my = p, \tag{194}$$

where $l^2 + m^2 = 1$. And let the equation to the conic, confocal with the conic of gyration, be

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} = 1; \tag{195}$$

where θ is to be determined so that (194) may be a normal to (195); whence we have

$$\frac{x}{l(a^2+\theta)} = \frac{y}{m(b^2+\theta)} = \frac{p}{a^2l^2+b^2m^2+\theta} = \frac{1}{p}; \tag{196}$$

therefore

$$\theta = p^2 - a^2l^2 - b^2m^2; \tag{197}$$

and therefore from (196)

$$x = \frac{l}{p}\{a^2+p^2-a^2l^2-b^2m^2\}; \tag{198}$$

$$y = \frac{m}{p}\{b^2+p^2-a^2l^2-b^2m^2\}; \tag{199}$$

which determine the principal point in the line (194); and shew that whatever that line is, it is always a principal axis, and has consequently a principal point.

If $p = 0$, $x = y = \infty$; that is, if a line in a principal central plane passes through the mass-centre, the principal point of that line is at an infinite distance; a theorem which has been stated before. If $p = 0$, and $l = 0$, $x = \frac{0}{0}$, $y = 0$; thus on the central axis of x every point is a principal point. A similar theorem is true of the axis of y.

210.] It is unnecessary to say more as to lines in space which may not be principal axes at all; the criterion of such lines is, that their equations do not satisfy the condition (187). I will however again observe that the fact is evident from this consideration. Let the given straight line be produced to meet one

of the principal central planes, and let the polar of that point be drawn relatively to the focal conic in that plane; it is evident that the trace of a plane perpendicular to the given line need not be parallel with that line. If it is parallel, the original line has a principal point, and is a principal axis. The condition of parallelism is expressed by (187).

211.] Although a line in space may not be a principal axis at all, yet every plane is a principal plane for some point in it, because, whatever is the position of the plane, it is a tangent plane to some quadric which is confocal with the central ellipsoid of gyration. And the principal axis may be found in the following way. Let us consider the trace of the plane on one of those central principal planes in which the focal conic is real; and let this trace be considered a polar relatively to that conic; let the corresponding pole be then determined, and from it let a perpendicular be drawn to the given plane; that perpendicular is the principal axis of the plane, and the point of intersection of it with the plane is the principal point of the plane. We may investigate these results by the process similar to that of Art. 207.

Let the equation to the plane be

$$lx+my+nz=p, \tag{200}$$

where $l^2+m^2+n^2=1$; and let (ξ, η, ζ) be the point where this plane touches the surface

$$\frac{\xi^2}{a^2+\theta}+\frac{\eta^2}{b^2+\theta}+\frac{\zeta^2}{c^2+\theta}=1; \tag{201}$$

of which the tangent plane at (ξ, η, ζ) is

$$\frac{\xi x}{a^2+\theta}+\frac{\eta y}{b^2+\theta}+\frac{\zeta z}{c^2+\theta}=1; \tag{202}$$

and as (200) and (202) are identical, we have

$$\frac{l(a^2+\theta)}{\xi}=\frac{m(b^2+\theta)}{\eta}=\frac{n(c^2+\theta)}{\zeta}=p \tag{203}$$

$$=\frac{\{l^2(a^2+\theta)+m^2(b^2+\theta)+n^2(c^2+\theta)\}}{p}; \tag{204}$$

$$\therefore \quad \theta=p^2-(a^2l^2+b^2m^2+c^2n^2); \tag{205}$$

$$\left.\begin{aligned} \therefore \quad \xi &= \frac{l}{p}\{p^2+a^2-(a^2l^2+b^2m^2+c^2n^2)\}, \\ \eta &= \frac{m}{p}\{p^2+b^2-(a^2l^2+b^2m^2+c^2n^2)\}, \\ \zeta &= \frac{n}{p}\{p^2+c^2-(a^2l^2+b^2m^2+c^2n^2)\}; \end{aligned}\right\} \tag{206}$$

so that (205) gives the particular ellipsoid which is confocal with the ellipsoid of gyration; and (206) assign the principal point in the plane, and is that point at which the plane touches the surface (201).

Hence it appears that if a plane is given, a confocal surface can be assigned which shall be touched by that plane; and also the point of contact can be determined, and this is the principal point of the plane.

212.] The equations to the principal axis whose principal plane is (200) may thus be found.

Let the trace of (200) be taken on the plane of (x, y); then, if it is considered a polar relatively to the focal conic in that plane, the pole is $\left(\frac{a^2-c^2}{p}l, \frac{b^2-c^2}{p}m\right)$; and therefore the equations to the axis, which is principal to the plane (200), are

$$\frac{\xi - \frac{a^2-c^2}{p}l}{l} = \frac{\eta - \frac{b^2-c^2}{p}m}{m} = \frac{\zeta}{n};$$

or, as they may be expressed,

$$\frac{\xi - \frac{a^2 l}{p}}{l} = \frac{\eta - \frac{b^2 m}{p}}{m} = \frac{\zeta - \frac{c^2 n}{p}}{n}. \qquad (207)$$

Hence every plane is a principal plane at some one point of it. A central principal plane is a principal plane for every point of it, because every axis which is perpendicular to a central principal plane is a principal axis with its principal point in the central principal plane.

Also the plane at an infinite distance is a principal plane at every point of it; and all the corresponding principal axes pass through the mass-centre. The three central principal planes and the plane at infinity alone have this property, that every point in them is a principal point. In this respect the system is unique.

213.] It will have been observed that the theory of principal axes and of principal moments of inertia might be investigated on either one or the other of two principles; either as a part of the geometry of masses, or on a purely dynamical principle in respect of centrifugal forces. We have chosen the former, as it is the main line along which our enquiry has proceeded, and we

have shewn that in reference to a given point the principal axes, as defined in Article 179, are also those about which as rotation-axes the moments of inertia have critical values, being severally a maximum, a mean, and a minimum, with their special forms. If we had taken a dynamical standpoint, principal axes would have been defined as those axes about which the moment of the centrifugal couple vanishes, or about which the centrifugal forces equilibrate, or compound into a single resultant of translation passing through the origin, and thus produce a motion of translation of the origin or a pressure at it; in other words, the principal axes would have been the axes relative to which, see Art. 159,

$$\text{L}'' = \text{M}'' = \text{N}'' = 0; \tag{208}$$

and consequently, see (63) of that Article,

$$\frac{\text{A}\,\omega_x - \text{F}\,\omega_y - \text{E}\,\omega_z}{\omega_x} = \frac{-\text{F}\,\omega_x + \text{B}\,\omega_y - \text{D}\,\omega_z}{\omega_y} = \frac{-\text{E}\,\omega_x - \text{D}\,\omega_y + \text{C}\,\omega_z}{\omega_z}. \tag{209}$$

Now as $\omega_x, \omega_y, \omega_z$ are proportional to the direction-cosines of the rotation-axis fulfilling that condition, on comparing these equations with those given in (115), Art. 182, it will be seen that they are identical; and that consequently the lines coincide. Hence it follows that the principal axes, as already defined, are those about which the centrifugal forces either equilibrate or compound into a single resultant; for in both cases we have the condition

$$\text{L}''\text{X}'' + \text{M}''\text{Y}'' + \text{N}''\text{Z}'' = 0;$$

and consequently principal axes might have been defined on either of these principles.

214.] If the centrifugal forces compound into a single resultant, the components of that resultant are X'', Y'', Z'', as given in (60), Art. 159, and if $(\bar{x}, \bar{y}, \bar{z})$ is the mass-centre of the system, we have, if M is the mass of the moving particles,

$$\left.\begin{aligned} \text{X}'' &= \text{M}\{\omega^2\bar{x} - \omega_x(\bar{x}\,\omega_x + \bar{y}\,\omega_y + \bar{z}\,\omega_z)\}, \\ \text{Y}'' &= \text{M}\{\omega^2\bar{y} - \omega_y(\bar{x}\,\omega_x + \bar{y}\,\omega_y + \bar{z}\,\omega_z)\}, \\ \text{Z}'' &= \text{M}\{\omega^2\bar{z} - \omega_z(\bar{x}\,\omega_x + \bar{y}\,\omega_y + \bar{z}\,\omega_z)\}; \end{aligned}\right\} \tag{210}$$

each of these vanishes, if the mass-centre is the origin. Consequently in reference to central principal axes the centrifugal forces are in equilibrium. If these do not vanish, they compound into a single resultant.

If A = B = C, that is, if the three principal moments at any point are equal, then $\text{L}'' = \text{M}'' = \text{N}'' = 0$, and the moment of the centrifugal couple vanishes.

215.] The axis, the moment, and the direction of the centrifugal couple may be aptly represented by means of the momental ellipsoid and also of the ellipsoid of gyration in the following manner.

From (97), Art. 174, we have

$$\left.\begin{aligned} L''\omega_1 + M''\omega_2 + N''\omega_3 &= 0, \\ AL''\omega_1 + BM''\omega_2 + CN''\omega_3 &= 0; \end{aligned}\right\} \qquad (211)$$

whence it appears that the axis of the centrifugal couple is perpendicular to the rotation-axis, and also to the line whose direction-cosines are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$, and consequently to the plane which contains these two lines. Now if the rotation-axis coincides with a radius vector of the momental ellipsoid whose equation is (117), Art. 182, the line whose direction-cosines are proportional to $A\omega_1$, $B\omega_2$, $C\omega_3$ is the perpendicular to the plane which touches the ellipsoid at the point where the rotation-axis cuts it; hence the axis of the centrifugal couple is at right angles to the plane which contains the rotation-axis and the perpendicular to the corresponding tangent plane, and, as by Article 174, this perpendicular is the axis of the effective couple, that is of a couple by the action of which the body would be brought to rest, it follows that the resultant of the centrifugal forces acts in the plane which contains the axis of the resultant angular velocity and the axis of the resultant effective couple.

216.] Also in reference to the ellipsoid of gyration whose equation is (148), Art. 193, if M denotes the mass of the moving system of particles

$$\left.\begin{aligned} L'' &= M\omega^2(b^2-c^2)\cos\beta\cos\gamma, \\ M'' &= M\omega^2(c^2-a^2)\cos\gamma\cos\alpha, \\ N'' &= M\omega^2(a^2-b^2)\cos\alpha\cos\beta; \end{aligned}\right\} \qquad (212)$$

$$\left.\begin{aligned} \therefore \quad L''\cos\alpha + M''\cos\beta + N''\cos\gamma &= 0, \\ L''a^2\cos\alpha + M''b^2\cos\beta + N''c^2\cos\gamma &= 0. \end{aligned}\right\} \qquad (213)$$

Let O be the centre of the ellipsoid, OK the rotation-axis whose direction-angles are α, β, γ, and let P be the point (x, y, z) on the ellipsoid where the plane perpendicular to OK touches it; let $OK, = k$, be the perpendicular on this plane, k being the radius of gyration of the system about OK. Hence we have

$$\cos\alpha = \frac{kx}{a^2}, \quad \cos\beta = \frac{ky}{b^2}, \quad \cos\gamma = \frac{kz}{c^2}; \qquad (214)$$

and therefore the axis of the centrifugal couple is perpendicular

to OK and to OP, and consequently to the plane OKP; this plane therefore is the plane of the centrifugal couple.

217.] Also the moment of the couple is proportional to the area of the triangle OKP. Let the area of the triangle be denoted by ∇, and let ∇_x, ∇_y, ∇_z denote the projections of this triangle on the planes of (y, z), (z, x), (x, y) respectively; then as the three angular points of the triangle are the origin, the point (x, y, z), and the point $(k \cos a, k \cos \beta, k \cos \gamma)$ respectively,

$$2\nabla_x = k(y \cos \gamma - z \cos \beta)$$
$$= (b^2 - c^2) \cos \beta \cos \gamma; \qquad (215)$$

with similar values for ∇_y and ∇_z: hence we have

$$L'' = 2M\omega^2 \nabla_x, \quad M'' = 2M\omega^2 \nabla_y, \quad N'' = 2M\omega^2 \nabla_z; \qquad (216)$$
$$\therefore \quad G'' = 2M\omega^2 \nabla;$$

and consequently the moment of the centrifugal couple varies as the triangle OKP.

Hence the moment of the centrifugal couple vanishes, whenever the area of the triangle OKP vanishes; that is when the radius vector and the corresponding perpendicular coincide; or, in other words, when the rotation-axis is a principal axis.

The effect of the centrifugal couple is to change the position of the rotation-axis, the new rotation-axis being in the plane which contains OK and the axis of the centrifugal couple, so far as the centrifugal couple affects its position.

218.] One net result of the preceding investigation is to shew that in discussing the most general motion of a body or of a system of material particles we may omit all consideration of the individual constituent particles as to their motions, and attend only to the motion of one point, to the mass, to the principal axes and the principal moments referred thereto, for these are the only quantities which are involved in the equations of motion; and it is convenient to take the mass-centre as the point whose motion of translation we consider, because the equations of motion of translation become thereby much simplified. We are hereby authorised to omit all consideration of irregularities of form and of want of uniformity and symmetry in the distribution of particles; we have to consider only the mass, the mass-centre, the principal central axes and the principal central moments. This method of enquiry introduces great simplification, and brings within our grasp many problems which would otherwise be beyond it. It gives a simplifica-

tion, an order, and an elegance of form to many problems and expressions which in their original structure are too complicated and too confused to admit of treatment. In illustration of this remark may be cited the theory of the central ellipsoid of gyration and its allied system of confocal quadrics. How marvelously does it bring into order lines which otherwise seem scattered promiscuously in space, and shew the character which they must exhibit if they are principal axes at any point on them!

219.] In connection with these observations we may remark on the simplification in the expressions for the kinetic energy and for the moment of momentum of a system in reference to a given point and to axes passing through that point, when the coordinate axes at the point are principal axes. Taking the value of the kinetic energy which is given in (108), Art. 111, we have

$$2\text{T} = \text{A}\omega_x^2 + \text{B}\omega_y^2 + \text{C}\omega_z^2 - 2\text{D}\omega_y\omega_z - 2\text{E}\omega_z\omega_x - 2\text{F}\omega_x\omega_y,$$

so that if the axes are principal,

$$2\text{T} = \text{A}\omega_1^2 + \text{B}\omega_2^2 + \text{C}\omega_3^2.$$

Also taking the values of h_1, h_2, h_3 which are given in (85), Art. 94, we have

$$h_1 = \quad \text{A}\omega_x - \text{F}\omega_y - \text{E}\omega_z = \left(\frac{d\text{T}}{d\omega_x}\right),$$

$$h_2 = -\text{F}\omega_x + \text{B}\omega_y - \text{D}\omega_z = \left(\frac{d\text{T}}{d\omega_y}\right),$$

$$h_3 = -\text{E}\omega_x - \text{D}\omega_y + \text{C}\omega_z = \left(\frac{d\text{T}}{d\omega_z}\right);$$

these become, if the axes are principal,

$$h_1 = \text{A}\omega_1 = \left(\frac{d\text{T}}{d\omega_1}\right), \quad h_2 = \text{B}\omega_2 = \left(\frac{d\text{T}}{d\omega_2}\right), \quad h_3 = \text{C}\omega_3 = \left(\frac{d\text{T}}{d\omega_3}\right),$$

$$h^2 = \text{A}^2\omega_1^2 + \text{B}^2\omega_2^2 + \text{C}^2\omega_3^2;$$

and if α, β, γ are the direction-angles of the axis of h, in reference to the principal axes,

$$\cos\alpha = \frac{h_1}{h} = \frac{\text{A}\omega_1}{h}, \quad \cos\beta = \frac{h_2}{h} = \frac{\text{B}\omega_2}{h}, \quad \cos\gamma = \frac{h_3}{h} = \frac{\text{C}\omega_3}{h}.$$

Hence, if ϕ is the angle between the instantaneous and the invariable axes,

$$\cos\phi = \frac{h_1\omega_1 + h_2\omega_2 + h_3\omega_3}{h\omega} = \frac{2\text{T}}{h\omega}.$$

$$\therefore \quad \omega\cos\phi = \frac{2\text{T}}{h},$$

which expresses the component of the instantaneous angular velocity about the axis of h.

Section 4.—*Examples of moments of inertia.*

220.] In this section I propose to apply the general formulæ of the preceding section to the calculation of moments of inertia and radii of gyration, relatively to certain given axes, of material lines or wires, of thin plates and curved shells, and of solid bodies. It will be found most convenient to make the calculations with reference to certain axes to which the bodies are geometrically related, and which yield the most simple forms of integration. And by means of them, and the theorems of the preceding section, to investigate the moments of inertia about other axes. The following theorems are most useful for the purpose.

(1) If at any point of a body, A, B, C are the principal moments of inertia, and H is the moment of inertia about the axis (α, β, γ) passing through that point, then

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + \text{C}(\cos\gamma)^2; \tag{217}$$

and if $\quad \text{A}' = \Sigma . mx^2, \quad \text{B}' = \Sigma . my^2, \quad \text{C}' = \Sigma . mz^2,$

$$\text{H} = \text{A}'(\sin\alpha)^2 + \text{B}'(\sin\beta)^2 + \text{C}'(\sin\gamma)^2. \tag{218}$$

(2) If H and H′ are the moments of inertia of the mass M about two parallel axes, one of which passes through a given point, and the other passes through the mass-centre; and if h is the distance between these axes, then

$$\text{H} = \text{H}' + \text{M}h^2; \tag{219}$$

and therefore if k and k' are the radii of gyration about the axis through the given point, and the parallel axis through the mass-centre respectively, then $\text{H} = \text{M}k^2$, $\text{H}' = \text{M}k'^2$;

$$\therefore \quad k^2 = k'^2 + h^2. \tag{220}$$

221.] The moments of inertia of material lines or wires.

Ex. 1. The moment of inertia of a straight wire of uniform thickness and density.

Let the length of the wire be $2a$, $\rho =$ its density, $\omega =$ the area of a transverse section; and let it lie along the axis of x.

(1) Let the rotation-axis be perpendicular to its length, and pass through its middle point; then

$$\text{the moment of inertia} = \int_{-a}^{a} \rho\,\omega\, x^2\, dx = \frac{2}{3}\rho\,\omega\, a^3.$$

(2) Let the rotation-axis be perpendicular to the wire, and at a distance c from the middle point which is its mass-centre; then, by (210), since the mass of the wire $= 2\rho\omega a$,

$$\text{the moment of inertia} = \frac{2}{3}\rho\omega a^3 + 2\rho\omega a c^2.$$

Hence, if an equilateral triangle is formed of a wire whose length is $6a$, the moment of inertia relatively to an axis passing through the mass-centre of the triangle and perpendicular to its plane is $4\rho\omega a^3$.

(3) Let the rotation-axis be perpendicular to the wire and pass through one of its ends; then

$$\text{the moment of inertia} = \int_0^{2a} \rho\omega x^2 dx = \frac{8\rho\omega a^3}{3}.$$

(4) Let the rotation-axis intersect the wire in its middle point at an angle α; then

$$\text{the moment of inertia} = \frac{2}{3}\rho\omega a^3 (\sin\alpha)^2.$$

Ex. 2. The moment of inertia of a wire of uniform thickness and density whose form is a circular arc.

Let ρ = the density, ω = the area of a transverse section, a = the radius of the circle, 2α = the angle subtended by the arc at the centre of the circle.

(1) Let the rotation-axis pass through the centre and be perpendicular to the plane of the arc; then

$$\text{the moment of inertia} = \int_0^{2\alpha} \rho\omega a^3 d\theta = 2\rho\omega a^3 \alpha;$$

and therefore the moment of inertia of a complete circular wire about an axis which passes through its centre and is perpendicular to its plane is $2\pi\rho\omega a^3$.

(2) Let the rotation-axis be perpendicular to the plane of the wire and pass through its middle point; then

$$y^2 = 2ax - x^2;$$

$$\therefore \quad \frac{dy}{a-x} = \frac{dx}{y} = \frac{ds}{a}:$$

$$\therefore \quad \text{the moment of inertia} = \int \rho\omega (x^2+y^2) ds$$

$$= 4\rho\omega a^3 \{\alpha - \sin\alpha\};$$

and the moment of inertia of a complete circle $= 4\pi\rho\omega a^3$.

(3) Let the rotation-axis be in the plane of the wire and pass through the centre and its middle point; then

$$\text{the moment of inertia} = \int_{-a}^{a} \rho\,\omega\,a^3 (\sin\theta)^2 d\theta$$
$$= \rho\,\omega\,a^3 \{a - \sin a \cos a\};$$

and therefore the moment of inertia of a complete circular wire about its diameter is $\pi\rho\,\omega\,a^3$.

(4) Let the rotation-axis pass through the centre of a complete circular ring, and be inclined at an angle γ to the plane of the circle; then, by (217),

$$\text{the moment of inertia} = \pi\rho\,\omega\,a^3 (\cos\gamma)^2 + 2\pi\rho\,\omega\,a^3 (\sin\gamma)^2$$
$$= \pi\rho\,\omega\,a^3 \{1 + (\sin\gamma)^2\}.$$

Ex. 3. A wire of uniform thickness and density, whose length is a, is bent into the form of a complete cycloidal arc: the moment of inertia of it about a rotation-axis which joins its two ends is $\frac{\rho\,\omega\,a^3}{30}$.

Ex. 4. To find the plane curve of given length of which the moment of inertia about an axis perpendicular to the plane is a minimum. If ρ is the radius of curvature, p is the perpendicular on the tangent, and r is the radius vector, $2\rho p = r^2 + k^2$.

In each of the preceding examples the mass of the wire can be easily found: and as the square of the radius of gyration is the moment of inertia divided by the mass, so the radius of gyration can be found without difficulty.

If the wire lies wholly in one plane, say in the plane of (x, y), that plane is a principal plane of it; because in this case $z = 0$ for all elements of it; and therefore $\Sigma.mzx = \Sigma.myz = 0$, and the axis of z is a principal axis. The other two principal axes must be found by the process of Art. 171.

222.] The moment of inertia of thin plates and of curved shells.

In all cases we shall assume the thickness of the plates and shells to be infinitesimal, and to be represented by the symbol τ; and thus, if it is convenient, we shall take the plate-plane to be the plane of (x, y); in this case, as $z = 0$ for all elements of the plate, $\Sigma.mzx = \Sigma.myz = 0$, and the plane of (x, y) is a principal plane and the axis of z is a principal axis. The other principal axes will be found by the method of Art. 171; and the principal moments of inertia having been determined, the moment of inertia about any other axis may be determined by means of the theorems given in (217) and (219).

Also, since $z = 0$, the moments of inertia about the axes of x and y are respectively $\Sigma . my^2$ and $\Sigma . mx^2$; and as $\Sigma . m(x^2+y^2)$ is the moment of inertia about the axis of z, it follows that the moment of inertia about an axis perpendicular to the plate is equal to the sum of the moments of inertia about any two axes at right angles to each other in the plate.

Also, since $\Sigma . m(x^2+y^2)$ is the same for any two rectangular axes in the plane of the plate, if $\Sigma . my^2 = \text{A}$ is a maximum, $\Sigma . mx^2 = \text{B}$ is a minimum, and vice versâ.

If the axes of coordinates are principal axes, from (217) we have

$$\begin{aligned} \text{H} &= \text{A}(\cos\alpha)^2 + \text{B}(\cos\beta)^2 + (\text{A}+\text{B})(\cos\gamma)^2 \\ &= \text{A}\{(\cos\alpha)^2 + (\cos\gamma)^2\} + \text{B}\{(\cos\beta)^2 + (\cos\gamma)^2\} \\ &= \text{A}(\sin\beta)^2 + \text{B}(\sin\alpha)^2; \qquad (221) \end{aligned}$$

and if the rotation-axis is in the plane of (x, y), $\sin\beta = \cos\alpha$; and

$$\text{H} = \text{A}(\cos\alpha)^2 + \text{B}(\sin\alpha)^2. \qquad (222)$$

Ex. 1. The moment of inertia of a square plate.

Let $a =$ the side of the plate, $\rho =$ the density at the point (x, y).

(1) Let the rotation-axis pass through the centre of the plate and be perpendicular to its plane; then

$$\text{the moment of inertia} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho\tau(x^2+y^2)\,dy\,dx = \frac{\rho\tau a^4}{6}.$$

(2) Let the rotation-axis be the line joining the middle points of two opposite sides; then

$$\text{the moment of inertia} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho\tau y^2\,dy\,dx = \frac{\rho\tau a^4}{12}.$$

(3) Let the rotation-axis pass through an angular point of the plate, and be perpendicular to its plane; then

$$\text{the moment of inertia} = \frac{2\rho\tau a^4}{3}.$$

(4) Let the rotation-axis pass through the centre of the plate; and let its direction-angles, with reference to two lines bisecting the opposite sides of the plate and the perpendicular through its centre, be α, β, γ; then, as these lines are the principal axes of the plate,

$$\begin{aligned} \text{the mom. of in.} &= \frac{\rho\tau a^4}{12}\{(\cos\alpha)^2 + (\cos\beta)^2\} + \frac{\rho\tau a^4}{6}(\cos\gamma)^2 \\ &= \frac{\rho\tau a^4}{12}(\sin\gamma)^2 + \frac{\rho\tau a^4}{6}(\cos\gamma)^2. \end{aligned}$$

Also, if the rotation-axis is the diagonal of the plate, $\gamma = 90°$, and

$$\text{the moment of inertia} = \frac{\rho \tau a^4}{12}.$$

As this is a case in which the two principal moments of inertia in the plane of (x, y) are equal, and the third is greater than each of them, two points on the axis of z, which are at distances from the origin, which is the mass-centre of the plate (see Art. 202), equal to

$$\pm \left(\frac{\text{C}-\text{A}}{\text{M}}\right)^{\frac{1}{2}} = \pm \frac{a}{6} 3^{\frac{1}{2}},$$

are such that at them the principal moments of inertia, and therefore all the moments of inertia, are equal. At these points the momental ellipsoid becomes a sphere.

Ex. 2. The moment of inertia of a triangular plate.

(1) Let the triangular plate be isosceles; and let the rotation-axis pass through its vertex and be perpendicular to its plane; let $a =$ the altitude, $2b =$ the base; then

$$\text{the moment of inertia} = 2\int_0^a \int_0^{\frac{bx}{a}} \rho \tau (x^2+y^2)\, dy\, dx$$

$$= \frac{\rho \tau a b}{6}(3a^2+b^2).$$

(2) Let the triangular plate be isosceles; and let the rotation-axis be the line which passes through the vertex and bisects the base; then

$$\text{the moment of inertia} = 2\int_0^a \int_0^{\frac{bx}{a}} \rho \tau y^2\, dy\, dx = \frac{\rho \tau a b^3}{6}.$$

(3) Let the triangular plate be that whose sides and angles are a, b, c, A, B, C; and let the rotation-axis pass through C and be perpendicular to the plane of the plate; let C be the origin, and let the lines lying along the sides a and b respectively be the axes of x and y; so that the equation to the side c is

$$\frac{x}{a} + \frac{y}{b} = 1;$$

and let

$$\text{Y} = \frac{b}{a}(a-x);$$

$$\text{the mom. of in.} = \int_0^a \int_0^{\text{Y}} \rho \tau (x^2 + 2xy \cos \text{C} + y^2)\, dy\, dx \sin \text{C}$$

$$= \frac{\rho \tau a b}{24}(3a^2+3b^2-c^2) \sin \text{C}.$$

(4) Let the triangular plate be that of the preceding case; and let the rotation-axis pass through the mass-centre of the plate and be perpendicular to its plane; then, if G is the mass-centre,

$$\text{C G}^2 = \frac{2a^2+2b^2-c^2}{9}; \text{ and the mass of the plate} = \frac{\rho\tau a b \sin \text{C}}{2};$$

therefore, by reason of (219),

$$\text{mom. of in.} = \frac{\rho\tau a b}{24}(3a^2+3b^2-c^2)\sin \text{C} - \frac{\rho\tau a b}{18}(2a^2+2b^2-c^2)\sin \text{C}$$

$$= \frac{\rho\tau a b}{72}(a^2+b^2+c^2)\sin \text{C};$$

and therefore, if k is the radius of gyration relative to a rotation-axis passing through the mass-centre of a triangular plate and perpendicular to its plane,

$$k^2 = \frac{a^2+b^2+c^2}{36}.$$

(5) Hence if M is the mass of the triangular plate, the moments of inertia relative to three axes perpendicular to the plate and passing through the three angles A, B, C are respectively

$$\frac{\text{M}}{4}\left(b^2+c^2-\frac{a^2}{3}\right), \quad \frac{\text{M}}{4}\left(c^2+a^2-\frac{b^2}{3}\right), \quad \frac{\text{M}}{4}\left(a^2+b^2-\frac{c^2}{3}\right).$$

(6) If α, β, γ are the perpendiculars from the angles A, B, C on the opposite sides, the moments of inertia of the plate about the three sides are respectively

$$\frac{\text{M}\alpha^2}{6}, \quad \frac{\text{M}\beta^2}{6}, \quad \frac{\text{M}\gamma^2}{6}.$$

(7) Also, if h, k, l are the lengths of the lines drawn from the angles to the middle points of the opposite sides, the moments of inertia about these lines vary inversely as the squares of the lines.

(8) Also if M is the mass of a triangular plate, three particles, of each of which the mass is $\frac{\text{M}}{12}$, placed at the angular points of the triangle, together with a particle of mass $\frac{3\text{M}}{4}$ placed at the mass-centre of the triangle, form a system equimomental with the triangular plate.

Ex. 3. The moment of inertia of a circular plate, and of a circular annulus.

Let the radius of the plate $= a$; and let ρ and τ express the same quantities as heretofore.

(1) Let the rotation-axis pass through the centre and be perpendicular to the plane of the plate; then

$$\text{the moment of inertia} = \int_0^{2\pi}\int_0^a \rho\tau r^3 dr\,d\theta = \frac{\pi\rho\tau a^4}{2}.$$

(2) Let the rotation-axis pass through the circumference and be perpendicular to the plate; then, by (219),

$$\text{the moment of inertia} = \frac{\pi\rho\tau a^4}{2} + \pi\rho\tau a^4 = \frac{3\pi\rho\tau a^4}{2}.$$

(3) Let the rotation-axis be the diameter of the plate; then

$$\text{the moment of inertia} = \int_0^{2\pi}\int_0^a \rho\tau r^3(\sin\theta)^2 dr\,d\theta = \frac{\pi\rho\tau a^4}{4}.$$

(4) Let the rotation-axis be a tangent to the plate; then, by (219),

$$\text{the moment of inertia} = \frac{5\pi\rho\tau a^4}{4}.$$

(5) Let the interior of the circular plate be removed, so that the remainder is a circular annulus, the radii of the exterior and interior bounding circles of which are a and b: then the moment of inertia relative to a rotation-axis passing through the centre of the annulus and perpendicular to its plane is

$$\frac{\pi\rho\tau(a^4-b^4)}{2}.$$

Also the moment of inertia of the annulus relative to its diameter is

$$\frac{\pi\rho\tau(a^4-b^4)}{4}.$$

Ex. 4. The moment of inertia of an elliptical plate.

Let the equation to the bounding ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

and let

$$\mathrm{Y} = \frac{b}{a}(a^2-x^2)^{\frac{1}{2}}.$$

(1) Let the rotation-axis be the major axis of the ellipse; then

$$\begin{aligned}\text{the moment of inertia} &= 4\rho\tau\int_0^a\int_0^{\mathrm{Y}} y^2\,dy\,dx\\ &= \frac{4\rho\tau b^3}{3a^3}\int_0^a (a^2-x^2)^{\frac{3}{2}}dx\\ &= \frac{\pi\rho\tau a b^3}{4}.\end{aligned}$$

(2) Let the rotation-axis be the minor axis of the ellipse; then

$$\text{the moment of inertia} = 4\rho\tau\int_0^a\int_0^{\text{Y}} x^2\,dy\,dx = \frac{\pi\rho\tau a^3 b}{4}.$$

(3) Let the rotation-axis be a line perpendicular to the plane of the plate and passing through its centre; then

$$\begin{aligned}\text{the moment of inertia} &= 4\rho\tau\int_0^a\int_0^{\text{Y}}(x^2+y^2)\,dy\,dx \\ &= \frac{\pi\rho\tau ab}{4}(a^2+b^2).\end{aligned}$$

(4) Let the rotation-axis pass through the centre of the plate and make angles, α, β, γ severally with the major axis, the minor axis, and the perpendicular to the plate through its centre; then, as these are the principal axes of the plate, we have, by (217),

$$\begin{aligned}\text{the mom. of in.} &= \frac{\pi\rho\tau ab}{4}\{b^2(\cos\alpha)^2+a^2(\cos\beta)^2+(a^2+b^2)(\cos\gamma)^2\} \\ &= \frac{\pi\rho\tau ab}{4}\{a^2(\sin\alpha)^2+b^2(\sin\beta)^2\}.\end{aligned}$$

(5) Let the rotation-axis be a central radius vector r of the plate, making an angle α with the major-axis; then, from the last result, as $\alpha+\beta = 90°$, we have

$$\text{the moment of inertia} = \frac{\pi\rho\tau ab}{4}\{a^2(\sin\alpha)^2+b^2(\cos\alpha)^2\};$$

but by the equation to the ellipse

$$a^2(\sin\alpha)^2+b^2(\cos\alpha)^2 = \frac{a^2b^2}{r^2};$$

$$\therefore \quad \text{the moment of inertia} = \frac{\pi\rho\tau a^3b^3}{4r^2}.$$

(6) If M is the mass of an elliptical plate, whose semi-axes are a and b, the system of particles, $\frac{\text{M}}{2}$ placed at the mass-centre, and $\frac{\text{M}}{8}$ placed at the extremities of the principal axes, is equimomental with the plate.

(7) In respect of a triangular plate, the ellipse which touches the triangle at the middle points of its sides is a momental ellipse of the plate.

Ex. 5. The moment of inertia of a spherical shell of radius a and thickness τ about its diameter $= \frac{8}{3}\pi\rho\tau a^4$.

223.] The moment of inertia of a solid body bounded by a

surface of revolution relative to its geometrical axis as its rotation-axis.

Let the axis be that of x; and let the equation to the curve, by the revolution of which about the axis of x the bounding surface is formed, be $y = f(x)$.

Let the solid be divided into a series of circular plates by planes at an infinitesimal distance apart and perpendicular to the axis of revolution; let the density be uniform and be ρ; then, at the distance x from the origin, y is the radius of a circular plate whose thickness is dx; and therefore, by Ex. 3, Art. 222, the moment of inertia of this circular slice, relative to an axis passing through its centre and perpendicular to its plane, is

$$= \frac{\pi \rho y^4 dx}{2} = \frac{\pi \rho}{2} \{f(x)\}^4 dx;$$

and therefore, if x_n and x_0 are the limits of x,

$$\text{the moment of inertia} = \frac{\pi \rho}{2} \int_{x_0}^{x_n} \{f(x)\}^4 dx.$$

Ex. 1. The moment of inertia of a cylinder.

Let the altitude of the cylinder $= a$, and the radius of the base $= b$; therefore

$$\text{the moment of inertia} = \frac{\pi \rho a b^4}{2}.$$

Ex. 2. The moment of inertia of a cone; let the altitude $= a$, and the radius of the base $= b$; then

$$\text{the moment of inertia} = \frac{\pi \rho b^4}{2 a^4} \int_0^a x^4 dx = \frac{\pi \rho a b^4}{10}.$$

Ex. 3. If $a =$ the altitude, and $b =$ the radius of the base of a paraboloid, then

$$\text{the moment of inertia} = \frac{\pi \rho a b^4}{6}.$$

Ex. 4. If $a =$ the radius of a sphere, then relatively to a diameter as the rotation-axis,

$$\text{the moment of inertia} = \frac{8 \pi \rho a^5}{15}.$$

Hence the moment of inertia of a spherical shell contained between two concentric spheres whose radii are a and b respectively, relatively to the diameter as the rotation-axis,

$$= \frac{8 \pi \rho (a^5 - b^5)}{15}.$$

Ex. 5. The moment of inertia of a prolate spheroid relatively to its axis as the rotation-axis $= \dfrac{8\pi\rho a b^4}{15}$.

Ex. 6. The moment of inertia of an oblate spheroid, whose axis is the rotation-axis $= \dfrac{8\pi\rho a^4 b}{15}$.

Ex. 7. If the radius of each surface of an equiconvex lens is a, and the thickness of the lens is $2t$, then the moment of inertia of the lens relative to its axis as the rotation-axis

$$= \pi\rho \int_0^t (2ax - x^2)^2 dx$$

$$= \frac{\pi\rho t^3}{15}(20a^2 - 15at + 3t^2).$$

224.] The moment of inertia of a solid body bounded by a surface of revolution relative to an axis intersecting its geometrical axis at right angles.

Let the point in which the rotation-axis intersects the axis of revolution be the origin; and let $y = f(x)$ be the equation of the generating curve of the bounding surface; then, using the notation of the preceding Article, and applying the result of Ex. 3, Art. 222, the moment of inertia of the type-slice relative to its own diameter

$$= \frac{\pi\rho y^4 dx}{4};$$

and therefore by (219) the moment of inertia of this slice about the rotation-axis is

$$= \frac{\pi\rho y^4 dx}{4} + \pi\rho y^2 x^2 dx;$$

and if x_n and x_0 are the limits of the x-integration,

$$\text{the moment of inertia} = \pi\rho \int_{x_0}^{x_n} \left(\frac{y^4}{4} + y^2 x^2\right) dx.$$

Ex. 1. The moment of inertia of a right cone relative to a rotation-axis passing through its vertex and perpendicular to its own axis.

Let the altitude of the cone $= a$; let the radius of the base $= b$; then

$$\text{the moment of inertia} = \pi\rho \int_0^a \left(\frac{b^4}{4a^4} + \frac{b^2}{a^2}\right) x^4 dx$$

$$= \frac{\pi\rho a b^2}{20}(4a^2 + b^2).$$

It is evident that relative to the vertex of a right cone the principal axes are the axis of the cone and any two lines perpendicular to each other and to the axis of the cone. So that the moment of inertia relative to a rotation-axis passing through the vertex of the cone and inclined at an angle α to the axis

$$= \frac{\pi \rho a b^2}{20}(4a^2+b^2)(\sin\alpha)^2 + \frac{\pi\rho a b^4}{10}(\cos\alpha)^2.$$

Ex. 2. The moment of inertia of a cone of which the altitude $= a$, and the radius of whose base $= b$, relative to a rotation-axis passing through its mass-centre and perpendicular to its own axis,

$$= \frac{\pi\rho a b^2}{80}(a^2+4b^2).$$

Ex. 3. If the altitude of a paraboloid of revolution is a, and the radius of the base $= b$, the moment of inertia relative to a rotation-axis passing through its vertex and perpendicular to its own axis

$$= \frac{\pi\rho a b^2}{12}(b^2+3a^2).$$

Ex. 4. If the altitude of a cylinder is a, and the radius of its base $= b$; and if the rotation-axis is perpendicular to the axis, and at a distance c from its end, then

$$\text{the moment of inertia} = \int_c^{a+c}\Big(\frac{\pi\rho b^4}{4} + \pi\rho b^2 x^2\Big)\,dx$$

$$= \frac{\pi\rho a b^4}{4} + \frac{\pi\rho a b^2}{3}(a^2+3ac+3c^2).$$

Hence, if the rotation-axis passes through the end of the axis,

$$\text{the moment of inertia} = \frac{\pi\rho a b^2}{12}(3b^2+4a^2);$$

and if the rotation-axis passes through the middle point of the axis of the cylinder,

$$\text{the moment of inertia} = \int_{-\frac{a}{2}}^{\frac{a}{2}}\Big(\frac{\pi\rho b^4}{4} + \pi\rho b^2 x^2\Big)\,dx$$

$$= \frac{\pi\rho a b^2}{12}(a^2+3b^2).$$

225.] The moment of inertia of various solid bodies.

Ex. 1. The moment of inertia of a rectangular parallelepipedon about an edge.

Let the edges be a, b, c; and let the lines which coincide with the edges be the axes of x, y, z respectively; let the density $= \rho$; then the moment of inertia relative to the edge a

$$= \int_0^a \int_0^b \int_0^c \rho\,(y^2+z^2)\,dz\,dy\,dx = \frac{\rho\,a\,b\,c}{3}(b^2+c^2);$$

and symmetrical values are of course true for the moments of inertia relative to the edges b and c.

Thus the moment of inertia of a cube whose side is a, relative to one of its edges as a rotation-axis, $= \dfrac{2\rho\,a^5}{3}$.

Ex. 2. The moment of inertia of a cube relative to a diagonal.

Let the side of the cube be a; and let the centre of the cube be the origin, and let the three lines which pass through the centres of the opposite sides be the coordinate axes; these lines are evidently principal axes; and relatively to either of them

$$\text{the moment of inertia} = \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} \rho(y^2+z^2)\,dz\,dy\,dx = \frac{\rho\,a^5}{6};$$

and as the moment of inertia is the same for each of these principal axes, it is the same for every axis passing through this point; thus, the central ellipsoid is a sphere, and all its radii vectores are equal; and therefore relative to the diagonal of the cube,

$$\text{the moment of inertia} = \frac{\rho\,a^5}{6}.$$

Ex. 3. The moment of inertia of an ellipsoid.

Let the equation to the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

The axes of the ellipsoid being the principal axes of the body, when the moments of inertia relative to them are determined, that about any other axis may be found from (217).

$$\text{Let} \quad c\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}^{\frac{1}{2}} = \mathrm{Z}, \qquad b\left\{1-\frac{x^2}{a^2}\right\}^{\frac{1}{2}} = \mathrm{Y}.$$

$$\text{Now} \quad \Sigma.mx^2 = 8\int_0^a \int_0^{\mathrm{Y}} \int_0^{\mathrm{Z}} \rho\,x^2\,dz\,dy\,dx = \frac{4\pi\rho\,a^3\,b\,c}{15};$$

$$\text{similarly} \quad \Sigma.my^2 = \frac{4\pi\rho\,a\,b^3\,c}{15}, \quad \Sigma.mz^2 = \frac{4\pi\rho\,a\,b\,c^3}{15};$$

$$\therefore \quad \mathrm{A} = \Sigma.m\,(y^2+z^2) = \frac{4\pi\rho\,a\,b\,c}{15}(b^2+c^2),$$

$$\mathrm{B} = \Sigma.m\,(z^2+x^2) = \frac{4\pi\rho\,a\,b\,c}{15}(c^2+a^2),$$

$$\mathrm{C} = \Sigma.m\,(x^2+y^2) = \frac{4\pi\rho\,a\,b\,c}{15}(a^2+b^2);$$

and therefore the moment of inertia relative to the axis (α, β, γ)

$$= \frac{4\pi\rho abc}{15}\{(b^2+c^2)(\cos\alpha)^2+(c^2+a^2)(\cos\beta)^2+(a^2+b^2)(\cos\gamma)^2\}$$

$$= \frac{4\pi\rho abc}{15}\{a^2(\sin\alpha)^2+b^2(\sin\beta)^2+c^2(\sin\gamma)^2\}.$$

Ex. 4. If in the preceding example $a = b$, and a is $> c$, the ellipsoid becomes an oblate spheroid; and

$$\text{A} = \text{B} = \frac{4\pi\rho a^2 c}{15}(a^2+c^2), \quad \text{C} = \frac{8\pi\rho a^4 c}{15}.$$

Therefore, by Art. 171, at two points on the axis of z all the moments of inertia are equal, and at them the momental ellipsoid becomes a sphere: the distances of them from the centre

$$= \pm\left\{\frac{\text{C}-\text{A}}{\text{M}}\right\}^{\frac{1}{2}} = \pm\left\{\frac{a^2-c^2}{5}\right\}^{\frac{1}{2}};$$

and if these points are at the poles of the spheroid, $a^2 = 6c^2$.

Ex. 5. The centre of a sphere of radius c moves in a circle of radius a and generates thereby a solid ring, as an anchor ring; prove that the moment of inertia of this ring about an axis passing through the centre of the director-circle and perpendicular to its plane is $\frac{\pi^2\rho ac^2}{4}(4a^2+3c^2)$.

226.] From the preceding results the moments of inertia of many curved shells and of systems of thin plates may be deduced.

For if the equation of the bounding surface of the solid contains a single parameter, by the infinitesimal variation of that parameter, the content of the solid will receive an infinitesimal variation in the form of a thin shell, the thickness of which will be the variation of the parameter. Thus, if the radius of a sphere is increased by an infinitesimal variation, say dr, the content will be increased by a spherical shell of thickness dr. Similarly, if a solid is increased by the variation of the parameter on which the bounding surface depends, the moment of inertia of that increase is the increase of the moment of inertia of the solid; and the former is generally a thin shell or a system of thin plates, so that the moment of inertia of this may be determined by the variation of the moment of inertia of the solid.

Thus by the preceding Article the moment of inertia of a cube about a diagonal is $\frac{\rho a^5}{6}$; let the edge of the cube be in-

creased by da; then all the sides of the cube receive increments in the form of thin plates, the thickness of which $= da = \tau$, say; and therefore the moment of inertia of the hollow box, formed by these six plates relative to a diagonal $= \frac{5\rho a^4 \tau}{6}$.

Similarly, by reason of Ex. 1 in the preceding Article, the moment of inertia of the box relative to an edge $= \frac{10\rho a^4 \tau}{3}$.

As the moment of inertia of a sphere relative to a diameter is $\frac{8\pi\rho a^5}{15}$, so that of a spherical shell of thickness τ, relatively to the same rotation-axis, $= \frac{8\pi\rho\tau a^4}{3}$.

As the moment of inertia of a cylinder, relative to its own axis as rotation-axis, is $\frac{\pi\rho a b^4}{2}$, so the moment of inertia of a cylindrical shell whose thickness is $db = \tau$, is, relatively to its own axis, $2\pi\rho\tau a b^3$.

In all the preceding examples we have calculated moments of inertia; and as the masses of the rotating bodies may be found in all the cases, the corresponding radii of gyration can be determined without difficulty.

CHAPTER V.

THE ROTATION OF A BODY ABOUT A FIXED AXIS.

SECTION 1.—*The rotation of a rigid body about a fixed axis under the action of instantaneous forces.*

227.] In the last two sections of the preceding Chapter we have mainly considered that part of our subject which has been called the Geometry of Masses. We come now to the consideration of the most simple case of the dynamics proper of a system of material particles; that, namely, in which a rigid body under the action of given forces revolves about an axis fixed in it and in space. Every particle of the body thus moves in a circle, the plane of which is perpendicular to the rotation-axis, and the centre of which is in that axis.

We shall suppose the form, matter, and density of every part of the moving body or system to be given; and we shall suppose the body to be capable of an unfettered rotation about the axis. This axis may be fixed at many points, provided that all are in the same straight line, or, in the language of machinery, may have many bearings; we shall however suppose that it has only two fixed points; because these are sufficient to fix the axis; and if there are more, the pressures become indeterminate at them both in intensity and in line of action. We shall indeed find, even in the case of two points, the components of the pressures on them along the rotation-axis to be indeterminate. We have already had a similar instance in Art. 112, Vol. III.

Let us in the first place consider the circumstances of rotation of the body, when it is acted on by instantaneous or impulsive forces; that is, we shall investigate the resulting angular velocity of the body, the pressures on the fixed points, and their incidents, which are due to one or more blows impressed at given points of the body. To simplify the formulæ, we shall generally assume the body to be at rest when the impulsive force acts, although the results will be equally applicable if the body is moving with a given angular velocity.

228.] Let the rotation-axis, on which are the two fixed points, be the axis of z; and let the two fixed points be at distances z_1, z_2 from the origin; let the pressure at these two points be P_1, P_2, and let the lines along which they respectively act be $(\alpha_1, \beta_1, \gamma_1)$,

$(\alpha_2, \beta_2, \gamma_2)$. Let m be the type-particle, and let (x, y, z) be its place at the instant, when the instantaneous force acts; let Q be the momentum impressed by this force, of which let the axial components be X, Y, Z; let V_x, V_y, V_z be the components of the actual velocity (or increase of velocity) with which m moves in consequence of this instantaneous force; all these being type-expressions, and therefore applicable to each particle and to each acting force. Thus the equations of motion, (34) and (35), Art. 73, become

$$\left.\begin{aligned} \Sigma.(X - mV_x) - P_1 \cos\alpha_1 - P_2 \cos\alpha_2 &= 0, \\ \Sigma.(Y - mV_y) - P_1 \cos\beta_1 - P_2 \cos\beta_2 &= 0, \\ \Sigma.(Z - mV_z) - P_1 \cos\gamma_1 - P_2 \cos\gamma_2 &= 0; \end{aligned}\right\} \quad (1)$$

$$\left.\begin{aligned} \Sigma.\{y(Z - mV_z) - z(Y - mV_y)\} + z_1 P_1 \cos\beta_1 + z_2 P_2 \cos\beta_2 &= 0, \\ \Sigma.\{z(X - mV_x) - x(Z - mV_z)\} - z_1 P_1 \cos\alpha_1 - z_2 P_2 \cos\alpha_2 &= 0, \\ \Sigma.\{x(Y - mV_y) - y(X - mV_x)\} &= 0. \end{aligned}\right\} \quad (2)$$

Let us express these equations, as in Art. 147, in terms of angular velocities. Let Ω be the angular velocity which results from the instantaneous forces; then, as its rotation-axis is the axis of z, and as there is no motion parallel to the axis of z,

$$V_x = -\Omega y, \qquad V_y = \Omega x, \qquad V_z = 0. \quad (3)$$

Let the moments of the axial components of the couple of the impressed momenta be L, M, N; then (1) and (2) become

$$\left.\begin{aligned} \Sigma.X + \Omega\Sigma.my - P_1 \cos\alpha_1 - P_2 \cos\alpha_2 &= 0, \\ \Sigma.Y - \Omega\Sigma.mx - P_1 \cos\beta_1 - P_2 \cos\beta_2 &= 0, \\ \Sigma.Z \qquad\qquad - P_1 \cos\gamma_1 - P_2 \cos\gamma_2 &= 0; \end{aligned}\right\} \quad (4)$$

$$\left.\begin{aligned} L + \Omega\Sigma.mzx + z_1 P_1 \cos\beta_1 + z_2 P_2 \cos\beta_2 &= 0, \\ M + \Omega\Sigma.myz - z_1 P_1 \cos\alpha_1 - z_2 P_2 \cos\alpha_2 &= 0, \\ N - \Omega\Sigma.m(x^2 + y^2) &= 0; \end{aligned}\right\} \quad (5)$$

which six equations assign all the incidents of motion, and the pressures on the two fixed points.

These equations admit of dissection by means of first principles, in a manner similar to that which has been employed in Arts. 158 and 159. As Ω is the expressed angular velocity about the axis of z, Ωr is the expressed velocity of m at a distance r from that axis; and $m\Omega r$ is the expressed momentum; the x- and y-axial components of which are $-m\Omega y$ and $m\Omega x$. Let us introduce pairs of momenta equal and opposite to these at the origin and in the plane of (x, y) at the foot of the z-ordinate of m; then the momentum $m\Omega r$ of m at the point (x, y, z) is equi-

valent to (1) a momentum $-m\Omega y$ acting at the origin and along the axis of x; (2) a momentum $m\Omega x$ also acting at the origin along the axis of y; (3) three couples $-m\Omega zx$, $-m\Omega yz$, $m\Omega(x^2+y^2)$ whose axes are respectively the coordinate axes of x, y, and z; and a similar result is true for every element of the body. Now, by D'Alembert's principle, the sum of all these expressed momenta, together with the pressures at the fixed points, are in equilibrium with the impressed momenta; and the conditions requisite for the equilibrium are evidently the six equations (4) and (5). We have hereby an intelligible meaning of their several terms. We proceed to deduce from them the value of the angular velocity which results from the impressed forces, and the pressures on the fixed points.

229.] The angular velocity is given by the last equation of (5), and we have

$$\Omega = \frac{\text{N}}{\Sigma . m(x^2+y^2)} = \frac{\text{N}}{\Sigma . mr^2}$$
$$= \frac{\text{The moment of the impressed momenta}}{\text{The moment of inertia}}; \qquad (6)$$

which is the same result as (16), Art. 147. It appears therefore that the resulting angular velocity does not depend on the pressures at the fixed points, or on the distance between them, but only on the moment of the impressed momenta, and on the moment of inertia of the body or system. It is also the same whatever is the number of the bearings. And if no force external to the system acts, the system continues to rotate uniformly with this angular velocity.

Now let us suppose a body capable of rotating about a fixed axis to be at rest, and let us suppose it to be struck by a blow of given momentum at a given point and in a determinate line: we must first resolve the blow into two parts, of one of which the line of action shall be parallel to the rotation-axis, so that the angular velocity will not be affected thereby, for it will only produce pressures at the fixed points along the rotation-axis; of the other, the line of action will be in a plane perpendicular to the rotation-axis; let this plane be the plane of (x, y); let the momentum of this component be Q, and let a be the perpendicular distance from the axis on its line of action; then (6) becomes

$$\Omega = \frac{\text{Q}a}{\text{The moment of inertia}}. \qquad (7)$$

230.] The following are examples of this equation.

Ex. 1. A body M at rest, and capable of moving about a fixed rotation-axis, is simultaneously struck by several masses m_1, m_2, ..., m_n, moving with velocities $v_1, v_2, \ldots\ldots, v_n$ in planes perpendicular to the fixed axis; the masses adhere to the body: it is required to find the angular velocity of the body.

Let the distances of the points of impact of the masses severally from the rotation-axis be $l_1, l_2, \ldots\ldots, l_n$; and let $p_1, p_2, \ldots\ldots, p_n$ be the perpendiculars from the rotation-axis on the lines of the velocities $v_1, v_2, \ldots\ldots, v_n$; then, if k is the radius of gyration of the body relative to the rotation-axis,

$$\Omega = \frac{m_1 v_1 p_1 + m_2 v_2 p_2 + \ldots\ldots + m_n v_n p_n}{\text{M} k^2 + m_1 l_1^2 + m_2 l_2^2 + \ldots\ldots + m_n l_n^2}$$

$$= \frac{\Sigma . m v p}{\text{M} k^2 + \Sigma . m l^2}.$$

Ex. 2. A body M revolving about a fixed axis with an angular velocity Ω, is struck by a particle m, moving with a velocity v in a line perpendicular to the plane containing the rotation-axis and the point of impact; it is required to determine the resulting angular velocity of the rotating body, the velocity of rebound of the striking particle, and the place of percussion when the velocity of rebound is a maximum, the elasticity of the body and particle being e.

Let $\text{M}k^2$ be the moment of inertia of the body relative to the rotation-axis; $p =$ the distance of the point of impact from the axis; $\Omega' =$ the angular velocity of the body after collision; $\text{V} =$ the velocity of m after rebound; and let us suppose Ω and v to be such that the motion of m and of the point of impact may be in the same direction at the instant of collision.

Let v' be the velocity of that point of M at which the impact takes place; so that

$$v' = \Omega p;$$

and let m' be the mass of a particle which, moving with the velocity v', would produce the same circumstances of velocity &c. in m after impact on m', as the rotating body M; so that $m'v'$ is the momentum with which M would strike a body at the point of impact of m, and in the line of m's motion: therefore by (6)

$$m'v' = \frac{\text{M}k^2 \Omega}{p}; \quad \text{and therefore} \quad m' = \frac{\text{M}k^2}{p^2}.$$

Let $\text{V}' = p\,\Omega'$ be the velocity of the point of impact after collision has ceased; then, by (8) and (9), Art. 215, Vol. III,

$$\text{V} = \frac{m p^2 v + \Omega p \,\text{M} k^2 - e\,\text{M} k^2 (v - p\,\Omega)}{m p^2 + \text{M} k^2},$$

$$\text{V}' = \frac{m p^2 v + \Omega p \,\text{M} k^2 + m e p^2 (v - p\,\Omega)}{m p^2 + \text{M} k^2};$$

$$\Omega' = \frac{m p v + \text{M} k^2 \Omega + m e p (v - p\,\Omega)}{m p^2 + \text{M} k^2};$$

$$\Omega' - \Omega = \frac{m p (1+e)(v - p\,\Omega)}{m p^2 + \text{M} k^2},$$

$$v - \text{V} = \frac{\text{M} k^2 (1+e)(v - p\,\Omega)}{m p^2 + \text{M} k^2};$$

whereby we know the velocity of m after collision and the angular velocity of M.

Thus, let M be a cricket bat, and m a ball; let us suppose the ball to meet the bat; then the sign of v must be changed; and if V = the velocity of rebound of the ball,

$$\text{V} = \frac{\text{M} k^2 \Omega p - m p^2 v + e\,\text{M} k^2 (v + \Omega p)}{m p^2 + \text{M} k^2};$$

and to determine the point of impact so that V may be a maximum, the p-differential of V must be equated to zero; whereby we have

$$p = -\frac{v}{\Omega} \pm \left\{\frac{v^2}{\Omega^2} + \frac{\text{M} k^2}{m}\right\}^{\frac{1}{2}}.$$

If m is at rest when it is struck by M, $v = 0$, and

$$p = k\left(\frac{\text{M}}{m}\right)^{\frac{1}{2}}.$$

Ex. 3. Again, let M be a rectangular plate whose sides are a and b, and let the rotation-axis lie along the side a: let us suppose it to be at rest and to be struck by m at a point on the side opposite to the rotation-axis; then $\text{M} k^2 = \dfrac{\text{M} b^2}{3}$; and

$$\Omega' = \frac{3 m v (1+e)}{(3m + \text{M})\, b}.$$

231.] In the next place we have to consider the pressures or the stresses on the two fixed points of the axis; the x- and y-components of P_1 and P_2 can be determined from the first two of (4), and from the first two of (5); and we have

$$\text{P}_1 \cos \alpha_1 = \frac{-\text{M} + z_2 \Sigma . \text{X} + \Omega (z_2 \Sigma . my - \Sigma . myz)}{z_2 - z_1}, \tag{8}$$

$$\text{P}_1 \cos \beta_1 = \frac{\text{L} + z_2 \Sigma . \text{Y} + \Omega (-z_2 \Sigma . mx + \Sigma . mxz)}{z_2 - z_1}, \tag{9}$$

$$\text{P}_2 \cos \alpha_2 = \frac{\text{M} - z_1 \Sigma . \text{X} + \Omega (-z_1 \Sigma . my + \Sigma . myz)}{z_2 - z_1}, \tag{10}$$

$$\text{P}_2 \cos \beta_2 = \frac{-\text{L} - z_1 \Sigma . \text{Y} + \Omega (z_1 \Sigma . mx - \Sigma . mxz)}{z_2 - z_1}. \tag{11}$$

The z-components of P_1 and P_2 enter into only the third equation of (4), and we have

$$\text{P}_1 \cos \gamma_1 + \text{P}_2 \cos \gamma_2 = \Sigma . \text{Z}; \tag{12}$$

therefore the sum of these z-components of the pressures is equal to the sum of the z-components of the impressed momenta; but as the sum only is given, each is indeterminate. An explanation of this indeterminateness has been already made in Art. 112, Vol. III: this is the dynamical case, which is therein alluded to. Thus although we can determine the parts of the pressures at each point which are perpendicular to the rotation-axis, yet we cannot determine the whole pressure at each, as the component along the axis is indeterminate.

232.] In illustration of these equations let us take a particular case and suppose the body to be struck by a single blow along a line perpendicular to the rotation-axis; let Q be the momentum of the blow, and a the shortest distance between the rotation-axis and the line of the blow, being at right angles to both these lines. Let the rotation-axis be the axis of z, and let the line along which a lies be the axis of x, so that the axis of y is parallel to the line of the blow; and we will suppose Q to cause positive rotation about the axis; then the equations (4) and (5) take the following forms:

$$\left.\begin{aligned} \Omega \Sigma . my - \text{P}_1 \cos \alpha_1 - \text{P}_2 \cos \alpha_2 &= 0, \\ \text{Q} - \Omega \Sigma . mx - \text{P}_1 \cos \beta_1 - \text{P}_2 \cos \beta_2 &= 0, \\ -\text{P}_1 \cos \gamma_1 - \text{P}_2 \cos \gamma_2 &= 0; \end{aligned}\right\} \tag{13}$$

$$\left.\begin{aligned} \Omega \Sigma . mzx + z_1 \text{P}_1 \cos \beta_1 + z_2 \text{P}_2 \cos \beta_2 &= 0, \\ \Omega \Sigma . mzy - z_1 \text{P}_1 \cos \alpha_1 - z_2 \text{P}_2 \cos \alpha_2 &= 0, \\ \text{Q} a - \Omega \Sigma . m (x^2 + y^2) &= 0. \end{aligned}\right\} \tag{14}$$

Let M be the mass of the body or system of particles, and let k be the radius of gyration about the rotation-axis; then from the last of (14),

$$\Omega = \frac{\text{Q} a}{\text{M} k^2}. \tag{15}$$

If the mass-centre of the system is at the point $(\bar{x}, \bar{y}, \bar{z})$, when the blow is struck, then at that instant

$$\Sigma . mx = M\bar{x}, \quad \Sigma . my = M\bar{y}.$$

The following are examples in which these expressions are applied.

Ex. 1. A thin rod of length a and mass M, capable of revolving about an axis at one end perpendicular to its length, is struck at the other end by a blow Q perpendicularly to the rod and to the axis. Find the pressures at the fixed points on the axis.

Let the fixed points on the axis be at equal distances c on opposite sides of the rod, so that $z_1 = -z_2 = c$; also $k^2 = \frac{a^2}{3}$; $\Sigma . mx = M\frac{a}{2}$, $\Sigma . my = 0$, $\Sigma . mzx = \Sigma . myz = 0$; then

$$\Omega = \frac{3\,Q}{a\,M};$$

$$P_1 \cos a_1 = P_2 \cos a_2 = 0; \quad P_1 \cos \beta_1 = P_2 \cos \beta_2 = -\frac{Q}{4},$$

so that the only pressures at the fixed points are at right-angles to the rod and each is equal to one-fourth of the blow, and acts in a direction opposite to that of the blow. If c is infinitesimal, so that the two fixed points are close together, and the end of the rod is practically the fixed point, the pressure at it is the sum of the two pressures, and is equal to $\frac{Q}{2}$.

Ex. 2. The following is another form of the same problem. The rod is rotating with an angular velocity Ω about one end in one plane, and meets with an obstacle at the other end. With what momentum does it strike on the obstacle?

Let Q be the momentum; then $Q = \frac{a M \Omega}{3}$, and the pressures on the fixed points are each equal to $\frac{a M \Omega}{12}$, so that the whole pressure on the axis $= \frac{a M \Omega}{6}$.

Ex. 3. An elliptical plate, whose major axis is $2a$ and eccentricity is ·5, is capable of rotation about a latus rectum which has the two points fixed at which it meets the bounding ellipse. The plate is struck at the further focus by a blow Q in a line perpendicular to the plate, and rotates with an angular velocity Ω;

prove that $\Omega = \frac{2\text{Q}}{a\text{M}}$, and that there is no pressure at the fixed points on the axis.

Ex. 4. A circular plate of radius a and mass M revolving with an angular velocity Ω about an axis passing through its centre and fixed at the extremities of the diameter, is struck with a blow Q at right angles to its plane, at a point in the diameter perpendicular to the rotation-axis at a distance c from the centre and is brought to rest; find the pressures on the fixed points of the rotation-axis.

233.] The pressures at the fixed points, as given by (13) and (14), will compound into a single resultant when

$$-\Omega \text{M} \bar{y}\, \Sigma . mxz + (\text{Q} - \Omega \text{M} \bar{x})\, \Sigma . myz = 0,$$

that is, when
$$-a\bar{y}\, \Sigma . mxz + (k^2 - a\bar{x})\, \Sigma . myz = 0; \qquad (16)$$

and this condition is satisfied when the rotation-axis is a principal axis, and the line of action of the blow is in its principal plane; and if R is the single pressure,

$$\text{R} = \frac{\text{Q}}{k^2}\{k^4 - 2a\bar{x}k^2 + a^2(\bar{x}^2 + \bar{y}^2)\}^{\frac{1}{2}};$$

when (16) is satisfied, one point is sufficient to fix the axis.

Hence, if the axis of rotation is a central principal axis, $\text{R} = \text{Q}$; and evidently acts at the mass-centre.

234.] Also in the more general case of forces as exhibited in equations (4) and (5), if the pressures at the fixed points are reducible to a single resultant, we may take the point, at which its action-line cuts the rotation-axis, to be the origin and to be a fixed point; then, using the notation of Art. 151, we have from (5),
$$\text{L} = -\Omega \text{E}, \quad \text{M} = -\Omega \text{D}, \quad \text{N} = \Omega \text{C}; \qquad (17)$$

so that the plane of the couple of the blow is

$$-\text{E}x - \text{D}y + \text{C}z = 0. \qquad (18)$$

Now the momental ellipsoid of the body at the origin is, see Art. 181,

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 - 2\text{D}yz - 2\text{E}zx - 2\text{F}xy = \mu, \qquad (19)$$

of which (18) is the z-derived function, and consequently (18) is the equation to the plane conjugate to the axis of z; and as this axis is in this investigation quite arbitrary, except that it passes through the origin, which is a fixed point, we have the following general theorem:

If one point of a body is fixed and the body is acted on by a blow, the instantaneous axis of rotation is the radius vector of the momental ellipsoid at the fixed point which is conjugate to the plane of the couple of the blow.

Hence there are generally at a point only three planes, viz. the principal planes at the point, in which the line of blow can lie and produce rotation about an axis perpendicular to it. In Article 232 are instances of a blow causing rotation about a line perpendicular to its line of action.

235.] Another case which requires consideration is that in which a body capable of rotation about an axis passing through two fixed points is struck by a blow along a given line, and the effects on the axis are a single pressure along it and no pressure at right angles to it, so that the axis may slide along itself, if such a motion is possible, but may not bear any twist. Let the axis of z be the rotation-axis, and let the line perpendicular to it and passing through the mass-centre at the instant when the blow is struck be the axis of x, and h be the perpendicular distance from the mass-centre on the rotation-axis; let Q be the momentum of the blow, and let it be applied at the point (ξ, η, ζ), along a line whose direction-angles are λ, μ, ν; then we have

$$\text{P}_1 \cos a_1 = \text{P}_1 \cos \beta_1 = 0; \quad \text{P}_2 \cos a_2 = \text{P}_2 \cos \beta_2 = 0; \quad \gamma_1 = \gamma_2 = 0;$$

$$\Sigma . m x = \text{M} h, \quad \Sigma . m y = 0;$$

and equations (4) and (5) become

$$\left.\begin{aligned} \text{Q} \cos \lambda \qquad\quad &= 0, \\ \text{Q} \cos \mu - \Omega \text{M} h \;\; &= 0, \\ \text{Q} \cos \nu - \text{P}_1 - \text{P}_2 &= 0; \end{aligned}\right\} \tag{20}$$

$$\left.\begin{aligned} \text{Q}(\eta \cos \nu - \zeta \cos \mu) + \Omega \Sigma . m z x \qquad\quad &= 0, \\ \text{Q}(\zeta \cos \lambda - \xi \cos \nu) + \Omega \Sigma . m y z \qquad\quad &= 0, \\ \text{Q}(\xi \cos \mu - \eta \cos \lambda) - \Omega \Sigma . m (x^2 + y^2) &= 0. \end{aligned}\right\} \tag{21}$$

From the first of (20), $\cos \lambda = 0$; so that the line of blow must be in a plane perpendicular to the line drawn through the mass-centre at right angles to the rotation-axis.

And if k is the radius of gyration of the body, relatively to the rotation-axis, from the last of (21) and the second of (20) we have

$$\xi = \frac{k^2}{h}; \tag{22}$$

which gives the perpendicular distance from the rotation-axis on

the plane which is parallel to it and contains the line of the blow.

Also if, as heretofore, $D = \Sigma . myz$, $E = \Sigma . mzx$, from the first two of (21), since $\cos\nu = \sin\mu$, we have

$$M k^2 \zeta - D\eta - \frac{E k^2}{h} = 0; \tag{23}$$

which is the equation to the line of the blow in the plane given by (22); this makes with the plane of (x, y) an angle $\tan^{-1}\frac{D}{M k^2}$; so that (22) and (23) determine the line of the blow. The line just determined is called the Axis of Percussion, and is such that if a blow is given along it, it causes a sliding motion along the axis, but no twist of the axis.

If $D = E = 0$, that is, if the rotation-axis is a principal axis, of which the origin is the principal point, $\zeta = 0$, and the axis of percussion lies in the principal plane of the rotation-axis; and its intersection with the line which passes through the mass-centre and is perpendicular to the rotation-axis is called the Centre of Percussion. Its distance from the rotation-axis is given in (22). In this case however $\cos\nu = 0$, and there are no pressures at the fixed points on the axis. These circumstances are considered fully in Art. 237.

236.] This axis of percussion may also be arrived at by the following process. At all points on the rotation-axis let the momental ellipsoids be described, and let the planes be drawn which are conjugate to the rotation-axis; these planes shall all intersect in the same straight line; and that line is the axis of percussion.

The equation to the momental ellipsoid at the origin is

$$A\xi^2 + B\eta^2 + C\zeta^2 - 2D\eta\zeta - 2E\zeta\xi - 2F\xi\eta - \mu = 0;$$

and the plane conjugate to the z-axis is

$$-E\xi - D\eta + C\zeta = 0;$$

so that for the momental ellipsoid, whose centre is at a distance λ from the origin, the equation to the plane conjugate to the axis of z is

$$-(E - \lambda\Sigma . mx)\xi - (D - \lambda\Sigma . my)\eta + C(\zeta - \lambda) = 0;$$

or $$-E\xi - D\eta + C\zeta - \lambda\{-\Sigma . my\,\eta - \Sigma . mx\,\xi + C\} = 0. \tag{24}$$

If we take the notation and coordinate-system of the preceding Article, $C = M k^2$, $\Sigma . my = 0$, $\Sigma . mx = M h$;

so that (24) becomes

$$-\text{E}\xi-\text{D}\eta+\text{M}k^2\zeta-\lambda\text{M}\{k^2-h\xi\}=0;$$

which is the equation to a plane, and contains the indeterminate quantity λ; it therefore represents a series of planes, all of which pass through the straight line which is the intersection of the two planes,

$$\left.\begin{aligned}-\text{E}\xi-\text{D}\eta+\text{M}k^2\zeta &= 0,\\ k^2-h\xi &= 0;\end{aligned}\right\}$$

the latter of which is a plane parallel to the plane of (y, z); and by substitution from the latter in the former we have

$$\text{M}k^2\zeta-\text{D}\eta-\frac{\text{E}k^2}{h}=0;$$

which is a plane perpendicular to the plane of (y, z), and inclined to the plane of (x, y), at an angle whose tangent is $\frac{\text{D}}{\text{M}k^2}$. These results are the same as those found in the preceding Article.

As the z-axis and the origin are, relatively to the body, arbitrary, this theorem is true for all lines which traverse the body; and therefore,

If at all points of a straight line which traverses a body the momental ellipsoids are described, the planes of these ellipsoids, which are conjugate to the given line, all pass through one and the same straight line.

Hence also we have the following theorem:

If a body is capable of rotation about a certain fixed axis, and at all points of the axis the momental ellipsoids are described, and the planes of them, conjugate to the axis, are drawn; then all these pass through the same straight line; and that straight line is the direction of a blow which will produce no transverse pressure on the axis. If the axis is principal at one of its points, this line of blow lies in the corresponding principal plane, and is perpendicular to the plane containing the rotation-axis and the mass-centre, and there will be no pressure at all on the axis. But if the rotation-axis is not principal at any one of its points, the direction of the blow will be oblique to the plane containing the axis and the mass-centre, and there will always be a pressure acting on the axis in the direction of its length.

237.] It still remains to consider generally the circumstances under which a body may be struck by a blow, so that if it is

possible no pressure be thereby caused at the fixed points of the axis. To simplify the expressions we will suppose the body to be struck at a point (ξ, η, ζ) by a single blow whose momentum is Q along a line (λ, μ, ν); then since $P_1 = P_2 = 0$, (4) become

$$\left.\begin{aligned} Q\cos\lambda + \Omega\Sigma.my &= 0, \\ Q\cos\mu - \Omega\Sigma.mx &= 0, \\ Q\cos\nu &= 0; \end{aligned}\right\} \quad (25)$$

the last of which shews that $\cos\nu = 0$; and therefore the line of the blow must lie in a plane which is perpendicular to the rotation-axis. Thus (5) become

$$\left.\begin{aligned} -\zeta Q\cos\mu + \Omega\Sigma.mzx &= 0, \\ \zeta Q\cos\lambda + \Omega\Sigma.myz &= 0, \\ Q(\xi\cos\mu - \eta\cos\lambda) - \Omega\Sigma.m(x^2+y^2) &= 0. \end{aligned}\right\} \quad (26)$$

From the first two of (25) and of (26) we have

$$\zeta = \frac{\Sigma.myz}{\Sigma.my} = \frac{\Sigma.mzx}{\Sigma.mx}; \quad (27)$$

so that

$$\Sigma.my(z-\zeta) = 0,$$
$$\Sigma.mx(z-\zeta) = 0,$$

and consequently the axis of z is a principal axis, of which the principal point is at the distance ζ from the origin. Whence we have the condition

$$\Sigma.mx\,\Sigma.myz - \Sigma.my\,\Sigma.mzx = 0. \quad (28)$$

Hence we have this first condition. If the fixed points of the rotation-axis are free from pressure, that axis must be a principal axis of the body, and the line of the blow must lie in its principal plane.

Also from the last of (26), in combination with the first two of (25), we have

$$\xi\Sigma.mx + \eta\Sigma.my - \Sigma.mr^2 = 0; \quad (29)$$

so that if $(\bar{x}, \bar{y}, \bar{z})$ is the mass-centre, and k is the radius of gyration relative to the rotation-axis, (29) becomes

$$\bar{x}\xi + \bar{y}\eta - k^2 = 0; \quad (30)$$

which is the equation to the line of action of the blow, in the plane parallel to, and at a distance ζ from, the plane of (x,y); (30) is evidently perpendicular to the line joining the mass-centre and the rotation-axis; and if h is the distance of the mass-centre from the axis, and l is the perpendicular distance from the axis on the line of the blow, or the line of percussion, from (30) we have

$$l = \frac{k^2}{h}; \quad (31)$$

hence the line of the blow must be at right angles to the perpendicular from the mass-centre on the rotation-axis, and at that distance l from the rotation-axis which is given in (31).

This value of l has already been found in Art. 235.

Also in this case we have from (6), if $M =$ the whole moving mass,

$$\Omega = \frac{Q l}{M k^2} = \frac{Q}{M h}. \tag{32}$$

Certain special forms of the preceding equations deserve remark.

If the plane of (x, y) is the principal plane of the axis of z, which is the rotation-axis, (28) is satisfied identically; and $\zeta = 0$.

Equation (28) is also satisfied identically if $\Sigma.mx = \Sigma.my = 0$; that is, if the rotation-axis passes through the mass-centre; but in this case $\zeta = \infty$, and therefore $Q = 0$. So that if a body capable of rotation about an axis, passing through the mass-centre, is struck by a blow, whatever is the direction and the intensity of the blow, certain pressures are always produced at the fixed points of the axis. This result obviously depends on the fact that generally the principal point of an axis passing through the mass-centre of a body is at an infinite distance.

If at the time when the blow is given the coordinate planes are so chosen that that of (x, z) contains the mass-centre; then $\Sigma.my = 0$; but as $\Sigma.myz$ evidently vanishes also, ζ has a determinate value.

It appears then that if a body capable of rotation about a fixed axis is struck by a blow and rotates thereby, so that no pressure is produced on those points at which the axis is fixed, it is necessary that (1) the rotation-axis should be a principal axis of the body; (2) the line of the blow should be in the principal plane of this axis, and perpendicular to the plane containing the rotation-axis and the mass-centre, and at a distance from the axis equal to l, as defined by (31).

A representation of these circumstances is given in Fig. 22; $O P_1 P_2$ is the fixed rotation-axis, and is the z-axis; P_1, P_2 are the two fixed points which determine it; O is its principal point, and is the origin, so that in this figure $\zeta = 0$; and the plane (x, y) is its principal plane. G is the mass-centre of the body which is taken to be in the plane of (x, z), so that the line of the blow is parallel to the y-axis. $OK = k$, the radius of gyration of the

system relatively to the rotation-axis. OL $= l$; NG $=$ OM $= h$; so that by (30) OL is a third proportional to OM and OK.

If k' is the radius of gyration of the body relatively to MG, by (130), Art. 191, $$k^2 = h^2 + k'^2;$$

so that $$l = \frac{h^2+k'^2}{h} = h + \frac{k'^2}{h};$$

$$\therefore \quad h(l-h) = k'^2; \tag{33}$$

$$\therefore \quad \text{OM} \times \text{ML} = \text{a constant.}$$

238.] The point L, which has been determined in the preceding Article, is called the Centre of Percussion of the body relative to the given rotation-axis. It determines the line along which a blow must be struck on a body capable of rotation about a principal axis, when the axis receives no pressure thereby; and conversely, if a body rotates about an axis free from all constraint, or if constrained, free from pressure at its bearings, the centre of percussion determines the line in which a blow must be given to the body to reduce it to rest without causing pressure on the bearings; or, in another sense, it determines the positions in which a fixed obstacle may be placed, on which if the body impinges and is brought to rest, the bearings of the axis will suffer no pressure.

It is also evident that as the axis $\text{O}\text{P}_1\text{P}_2$ is free from pressure at its bearings, it is that axis about which the body continues to rotate; it is therefore a permanent axis. We have hereby then arrived at another property of a permanent axis, and have shewn it to be identical with a principal axis.

It is also evident that if the body is free from all constraint, so that it is capable of translation as well as of rotation, the effect of a blow at L along LQ will cause a rotation about $\text{O}\text{P}_1\text{P}_2$; for this reason the axis $\text{O}\text{P}_1\text{P}_2$ is called the Spontaneous Axis of the body relative to the point L. This subject however we shall consider at length in Chapter VII.

239.] I propose now to apply the preceding theory to certain examples, and to exhibit the practical meaning of the results. For this purpose it is often more convenient to express (31) in the following form;

$$l = \frac{\text{M}k^2}{\text{M}h} = \frac{\text{The moment of inertia}}{\text{M}h}. \tag{34}$$

Ex. 1. Find the centre of percussion of a circular plate, capable of rotation about an axis which touches it.

It is evident that the rotation-axis is a principal axis, having its principal point at the point of contact with the plate; hence, with the usual notation,

$$\text{the moment of inertia} = \frac{5\pi\rho\tau a^4}{4}, \quad \text{and} \quad l = \frac{5a}{4}.$$

Also the line of the blow Q must be perpendicular to the plane of the plate; therefore, by (32), if M = the mass of the plate,

$$\Omega = \frac{Q}{\pi\rho\tau a^3} = \frac{Q}{a\,M}.$$

Ex. 2. If an elliptical plate, the eccentricity of whose bounding line is ·5, is capable of rotation about a latus rectum, prove that the further focus is the centre of percussion.

Ex. 3. Find the centre of percussion of a rectangular cube whose rotation-axis is parallel to four parallel edges of the cube, and which is equidistant from the two nearer, as well as from the two farther edges.

Here it is evident that the rotation-axis is a principal axis, and the line drawn through the centre of the cube perpendicular to it cuts it in its principal point. Let $2a$ be a side of the cube, and let c be the distance of the rotation-axis from its mass-centre; then

$$k^2 = c^2 + \frac{2a^2}{3}, \quad \text{and} \quad h = c;$$

$$\therefore \; l = c + \frac{2a^2}{3c}; \qquad \Omega = \frac{Q}{8\rho a^3 c} = \frac{Q}{c\,M}.$$

Ex. 4. A cylinder is capable of revolving about the diameter of one of its circular ends: find the centre of percussion.

Let a = the length of the cylinder, b = the radius of its circular transverse section. It is evident that the rotation-axis is a principal axis; and that the centre of the circular end is its principal point.

$$l = \frac{3b^2 + 4a^2}{6a}; \qquad \Omega = \frac{2Q}{\pi\rho a^2 b^2}.$$

Hence the centre of percussion will be at the end of the cylinder if $3b^2 = 2a^2$. If b is very small in comparison of a, $l = \frac{2a}{3}$; thus, if a straight rod of small transverse section is held by one end in the hand, l gives the point at which it may be struck when the hand will receive no jar.

Ex. 5. Find the centre of percussion of a sphere revolving about an axis, which touches its surface.

This axis is evidently a principal axis, and the point of contact is its principal point; and we find

$$l = \frac{7a}{5}, \qquad \Omega = \frac{3\text{Q}}{4\pi\rho a^4} = \frac{\text{Q}}{a\text{M}}.$$

SECTION 2.—*Rotation of a body about a fixed axis under the action of finite accelerating forces.*

240.] I proceed now to the case of a rigid body rotating about a fixed axis under the action of finite accelerating forces, whereby momenta are continuously impressed. To this case equations (37) and (38), Art. 73, are to be applied.

To simplify the expressions, without any loss of generality, let us take, as in the preceding Section, the rotation-axis to be the z-axis; and suppose it to be fixed at two points whose distances from the origin are respectively z_1 and z_2; let the pressures at these points at the time t be respectively P_1 and P_2; and let the lines of action of these pressures be $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$.

Let $\Sigma . \text{P} \cos\alpha \ldots\ldots \Sigma . \text{P}z \cos\alpha \ldots\ldots$ be abridging symbols of the axial components of these pressures, and of their moments relative to the axes; and let L, M, N, as in (40), Art. 74, be the moments of the axial components of the couples of the impressed momentum-increments at the time t; then the equations of motion are

$$\left.\begin{aligned} \Sigma . m\left(\text{X} - \frac{d^2x}{dt^2}\right) - \Sigma . \text{P}\cos\alpha &= 0, \\ \Sigma . m\left(\text{Y} - \frac{d^2y}{dt^2}\right) - \Sigma . \text{P}\cos\beta &= 0, \\ \Sigma . m\left(\text{Z} - \frac{d^2z}{dt^2}\right) - \Sigma . \text{P}\cos\gamma &= 0; \end{aligned}\right\} \qquad (35)$$

$$\left.\begin{aligned} \Sigma . m\left(y\frac{d^2z}{dt^2} - z\frac{d^2y}{dt^2}\right) - \Sigma . \text{P}z\cos\beta &= \text{L}, \\ \Sigma . m\left(z\frac{d^2x}{dt^2} - x\frac{d^2z}{dt^2}\right) + \Sigma . \text{P}z\cos\alpha &= \text{M}, \\ \Sigma . m\left(x\frac{d^2y}{dt^2} - y\frac{d^2x}{dt^2}\right) \qquad &= \text{N}. \end{aligned}\right\} \qquad (36)$$

241.] In the first place, I propose to consider the motion of

rotation about the fixed axis apart from the pressures at the fixed points of the axis, reserving the latter for consideration hereafter; with this view let us transform the last of (36) into its equivalent in terms of angular velocity about that fixed axis.

Let r be the distance from the rotation-axis of m, whose place at the time t is (x, y, z); and let θ be the angle between r and the plane of (x, z), which plane is assumed to be fixed in space; let ω be the angular velocity about the fixed z-axis; so that

$$\omega = \frac{d\theta}{dt}, \qquad \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{37}$$

Hence we have, $x = r\cos\theta, \qquad y = r\sin\theta;$

$$\therefore \quad \frac{d^2x}{dt^2} = -\omega^2 r\cos\theta - r\sin\theta\frac{d\omega}{dt},$$

$$\frac{d^2y}{dt^2} = -\omega^2 r\sin\theta + r\cos\theta\frac{d\omega}{dt}.$$

so that the last of (36) becomes

$$\Sigma . mr^2\frac{d\omega}{dt} = \text{N};$$

and as $\frac{d\omega}{dt}$ is the same for all the particles of the system, it may be placed outside the sign of summation, so that we have

$$\frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \frac{\text{N}}{\Sigma . mr^2}$$

$$= \frac{\text{The moment of the impressed momentum-increments}}{\text{The moment of inertia}}; \tag{38}$$

each of these quantities being estimated relatively to the fixed rotation-axis. The form which this equation takes shews that it is independent of the particular system of coordinate axes which has been taken. It is indeed identical with (53), Art. 155. By it the angular velocity-increment about the rotation-axis is given; and therefore by integration the angular velocity, and by a subsequent integration the angle described in a given time may be found. Thus the motion of the body about a fixed axis will be determined.

Before however we proceed to examples of this motion, let us shew that (38) may be derived immediately from first principles; for this process will remove any obscurity which may attach to its meaning.

Let m be a type-particle of the body or system; let r be its

distance from the rotation-axis of z, so that if θ is the angle between r and the fixed plane of (x, z), the linear velocity of m is $r\frac{d\theta}{dt}$, and the linear velocity-increment is $r\frac{d^2\theta}{dt^2}$; and therefore the moment of the expressed momentum-increment of m is $mr^2\frac{d^2\theta}{dt^2}$; so that relatively to the axis of z the moment of the whole expressed momentum-increment is $\Sigma . mr^2\frac{d^2\theta}{dt^2}$; and therefore if N is the moment relatively to the same axis of the whole impressed momentum-increment in an unit of time, by D'Alembert's principle we have

$$\Sigma . mr^2\frac{d^2\theta}{dt^2} = \text{N};$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{d\omega}{dt} = \frac{\text{N}}{\Sigma . mr^2}.$$

242.] With respect to this equation, it is to be observed, that if the lines of action of all the impressed forces are parallel to the axis of z, which is the rotation-axis, $\text{N} = 0$; and that

$$\frac{d^2\theta}{dt^2} = 0; \qquad \frac{d\theta}{dt} = \Omega,$$

if Ω is the angular velocity at the time under consideration; so that the system moves with a constant angular velocity. Hence also

$$\theta - a = \Omega t,$$

if a is the value of θ, when $t = 0$; so that equal angles are described in equal times. This is a particular case of the principle of conservation of moments of momenta; see Art. 88. Thus, if a heavy body rotates about a vertical axis, the force of gravity has no effect on the angular velocity.

243.] But one of the most important applications of this theorem is the motion of a heavy body rotating about a fixed horizontal axis. Let us take the system of axes delineated in Fig. 23; let the axis of z be vertical downwards; let the y-axis be the rotation-axis, and let θ be the angle at which the perpendicular from m to the y-axis is inclined to the vertical plane of (y, z); thus the line of action of gravity, which is the only force acting on m, is parallel to the z-axis. Let G be the centre of gravity of the body, and let the plane passing through it and perpendicular to the axis of y be the plane of (x, z); so that as the body rotates about the axis of y, the line OG moves in the plane of (x, z).

Let M be the mass of the body; $\text{OG} = h =$ the distance of the centre of gravity from the rotation-axis; and let $\text{M}k^2$ be the moment of inertia about the rotation-axis; let $\text{GO}z = \theta$, so that as θ increases the body rotates about the axis of y from the z-axis to the x-axis: also $\frac{d\theta}{dt} = \omega$ is the angular velocity; and is the same for all particles of the body. Now the moment of the impressed momentum on m at (x, y, z) at the time t relatively to the rotation-axis is mgx, and tends to diminish θ; so that the moment of the momenta impressed on all the particles at the time t

$$= -\Sigma . mgx = -\text{M}gh \sin\theta;$$

and the moment of the impressed momentum is the same as if the whole mass were collected at its centre of gravity; so from (38) we have

$$\frac{d^2\theta}{dt^2} = -\frac{\text{M}gh\sin\theta}{\text{M}k^2} = -\frac{gh}{k^2}\sin\theta; \tag{39}$$

which equation gives the angular acceleration about the rotation-axis.

Let us multiply both sides of (39) by $d\theta$; and suppose the body to be at rest when $\theta = \alpha$; then, integrating (39), we have

$$\frac{d\theta^2}{dt^2} = \frac{2gh}{k^2}(\cos\theta - \cos\alpha); \tag{40}$$

which assigns the angular velocity in terms of θ. From this equation it appears that $\frac{d\theta}{dt} = 0$; that is, that the angular velocity vanishes, and the body is at rest, when

$$\theta = \alpha, \quad \theta = -\alpha, \quad \theta = 2\pi + \alpha, \quad \theta = 2\pi - \alpha, \ldots \theta = 2m\pi \pm \alpha;$$

so that as (40) expresses the circumstances of the body, the motion of it is oscillatory, the arc of vibration being double of that between the vertical line and the initial position of the line through the axis and the centre of gravity; this latter being the vertical line when the body is at rest. Hence we have the following circumstances of motion of a heavy body capable of oscillation about a horizontal axis. When the body is at rest, the perpendicular from its centre of gravity to the rotation-axis is vertical; let this line be moved through an angle α, and let the body be left to itself; it will oscillate through an angle 2α, the centre of gravity ascending to equal heights on both sides of the

lowest point. Such an oscillating body is called a compound pendulum.

If the body is moving with an angular velocity Ω when $\theta = a$, the equation of the angular velocity becomes

$$\frac{d\theta^2}{dt^2} - \Omega^2 = \frac{2gh}{k^2}(\cos\theta - \cos a); \tag{41}$$

but as this is of the same form as (40), so far as integration is concerned, we shall inquire into the properties of only (40).

244.] Equations (40) and (41) are evidently those of vis viva and of work; for in this case equation (99), Art. 108, takes the form

$$\frac{1}{2}\Sigma . mr^2\left\{\frac{d\theta^2}{dt^2} - \Omega^2\right\} = \text{M}gh(\cos\theta - \cos a); \tag{42}$$

and as $\Sigma . mr^2 = \text{M}k^2$, we have

$$\frac{d\theta^2}{dt^2} - \Omega^2 = \frac{2gh}{k^2}(\cos\theta - \cos a);$$

as there are many cases in which this equation of vis viva can be expressed immediately, we shall not hesitate to make it the starting point of a problem; and if the angular acceleration is required it can be immediately deduced from it by differentiation. This is the case whenever the pressures at the fixed points are to be determined, as second time-differentials are involved in them.

245.] From (40) we have

$$dt = -\frac{k}{(2gh)^{\frac{1}{2}}}\frac{d\theta}{(\cos\theta - \cos a)^{\frac{1}{2}}}; \tag{43}$$

whence, by integration, the time may be found in terms of θ, and the whole time of an oscillation may be determined.

This equation however in its present form is an elliptic function, and therefore t can only be expressed as an elliptic integral. If however the displacement of the body is slight, so that a and θ are both small, then we may expand $\cos\theta$ and $\cos a$, and neglect powers of θ and a above the second; whereby we have

$$\cos\theta = 1 - \frac{\theta^2}{2}, \qquad \cos a = 1 - \frac{a^2}{2};$$

and (43) becomes

$$dt = -\frac{k}{(gh)^{\frac{1}{2}}}\frac{d\theta}{(a^2 - \theta^2)^{\frac{1}{2}}};$$

$$\therefore \quad t = \frac{k}{(gh)^{\frac{1}{2}}}\cos^{-1}\frac{\theta}{a}; \tag{44}$$

if $t = 0$, when $\theta = a$; and therefore if T is the time of a small

oscillation of a heavy body about a horizontal axis, $t = \text{T}$ when $\theta = -\alpha$, and

$$\text{T} = \frac{\pi k}{(gh)^{\frac{1}{2}}}. \qquad (45)$$

246.] Now if we consider a heavy particle of infinitesimal dimensions attached to the end of a rigid imponderable rod of length l, and without weight, and vibrating about a horizontal axis perpendicular to its length, to be a perfect pendulum, then, as we have shewn in Art. 428, Vol. III, if T is the time of small oscillation of such a pendulum,

$$\text{T} = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}};$$

and the time of the heavy oscillating body is identical with this, if

$$l = \frac{k^2}{h}. \qquad (46)$$

Thus, the compound pendulum is isochronous with a perfect pendulum of the length l, which is given in (46); and l is called the length of the simple isochronous pendulum.

The agreement however in motion between the compound and the simple isochronous pendulum is greater than the preceding investigations lead to. For the general equation of a heavy particle attached to the end of a rigid and imponderable rod of length l, and rotating in a vertical circle, is, see Art. 429, Vol. III,

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta;$$

and this equation is identical with (39), which determines the rotation of the heavy body, if

$$l = \frac{k^2}{h};$$

and hence we conclude that if the whole mass of the rotating body is condensed into a particle at a distance l from the rotation-axis along the line which passes through the centre of gravity, the circumstances of equilibrium and of motion of this particle would be identical with the similar circumstances of the body. And if the body is slightly displaced from its position of stable equilibrium, and oscillates through a small angle, the time of an oscillation

$$= \pi \left(\frac{l}{g}\right)^{\frac{1}{2}}, \quad \text{where} \quad l = \frac{k^2}{h};$$

k being the radius of gyration of the body about the horizontal

rotation-axis, and h being the distance of the centre of gravity from the same axis.

The point O, Fig. 23, in which the horizontal rotation-axis pierces the vertical plane containing the centre of gravity, is called the Centre of Suspension; and if OG is produced to C, so that $OC = l$, C is called the Centre of Oscillation, and OC or l is the length of the simple pendulum isochronous with the body; that is, if the whole mass is collected into a particle at C, the circumstances of rotation of the particle thus condensed will be the same as those of the body.

247.] Let k' be the radius of gyration of the body relative to an axis through G, and parallel to the rotation-axis; then, since by (130), Art. 191, $k^2 = k'^2 + h^2$;

$$\therefore \quad l = \frac{k'^2+h^2}{h} = h + \frac{k'^2}{h}; \tag{47}$$

$$\therefore \quad (l-h)\,h = k'^2; \tag{48}$$

and replacing these by the geometrical quantities,

$$\begin{aligned} \text{CG} \times \text{GO} &= k'^2 \\ &= \text{a constant.} \end{aligned} \tag{49}$$

Now this equation would be unaltered if the places of O and C were interchanged; whence we infer that if C is the centre of oscillation for an axis Oy through O, O would be the centre of oscillation for a parallel axis through C. This theorem, as it is commonly stated, asserts the convertibility of the centres of suspension and oscillation. As the length of the simple isochronous pendulum is the same whether C or O is the centre of oscillation, so the time of oscillation is the same for both parallel axes.

It will be observed that we have the same expressions for the determination of the centre of oscillation and the centre of percussion relative to a given rotation-axis, see (31), Art. 237: in the latter case, however, it is necessary that the rotation-axis should be a principal axis at some point on it, and the centre of percussion should be in its principal plane; here no such restriction as to the nature of the axis is necessary. We have hereby a method by which the centre of percussion may be practically determined. Let the body of which the centre of percussion is to be found be suspended by, and made to vibrate about, the relative rotation-axis; let the number of vibrations in a given time be noted; let, say, n vibrations take place in t; then

$$\frac{t}{n} = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}};$$

$$\therefore \quad l = \frac{g t^2}{\pi^2 n^2};$$

thus, if t and n are carefully observed, as the other quantities are known, l is also known ; and this is the distance of the centre of percussion from the rotation-axis.

248.] Before we enter on other investigations connected with the time, &c. of oscillation of bodies, we will determine l in certain cases ; and for this purpose we shall generally find the last of the following forms the more convenient ;

$$l = \frac{k^2}{h} = \frac{\text{M} k^2}{\text{M} h}$$

$$= \frac{\text{The moment of inertia relative to the rotation-axis}}{\text{The mass} \times \text{the distance of mass-centre from the axis}}. \quad (50)$$

Ex. 1. A straight heavy wire, of length $2a$, vibrates about an axis passing through its end, and perpendicular to its length : prove that the length of the simple isochronous pendulum is two-thirds of the length of the wire.

Ex. 2. A wire, in the form of the arc of a circle, vibrates about an axis passing through its middle point and perpendicular to its plane; prove that the length of the simple isochronous pendulum is that of the diameter of the circle, whatever is the length of the wire.

Let ρ and ω be the density and the area of the transverse section of the wire ; let a be the radius of the circle ; then the origin being the middle point of the wire, the equation to the wire is

$$y^2 + x^2 = 2ax;$$

$$\text{M} k^2 = \rho\omega \int (x^2+y^2)\, ds = 2a\rho\omega \int x\, ds.$$

Also

$$\text{M} h = \rho\omega \int x\, ds;$$

the limits of integration being the same in both integrals ; so that

$$l = 2a.$$

Ex. 3. Compare the times of vibration of a thin circular plate about axes passing through the circumference, and (1) touching the circle and in its plane; (2) at right angles to the plane of the circle.

The moment of inertia relative to a tangent $= \dfrac{5\pi\rho\tau a^4}{4}$;

the moment of inertia relative to a perpendicular axis $= \dfrac{3\pi\rho\tau a^4}{2}$;
and in each case $$\text{M}h = \pi\rho\tau a^3;$$
therefore, if l_1 and l_2 are the lengths of the corresponding isochronous pendulums,
$$l_1 = \frac{5a}{4}; \qquad l_2 = \frac{3a}{2};$$
and if t_1 and t_2 are the corresponding times of small vibration,
$$\frac{t_1}{t_2} = \left(\frac{l_1}{l_2}\right)^{\frac{1}{2}} = \left(\frac{5}{6}\right)^{\frac{1}{2}}.$$

Ex. 4. A right cone oscillates about an axis passing through its vertex and perpendicular to its own axis; it is required to find the length of the simple isochronous pendulum.

Let $a =$ the altitude of the cone; $b =$ the radius of the circular base; then
$$\text{M}k^2 = \frac{\pi\rho a b^2}{20}(4a^2+b^2); \qquad \text{M}h = \frac{\pi\rho a^2 b^2}{4};$$
therefore $$l = \frac{4a^2+b^2}{5a}.$$
If $a = b$, that is, if the cone is right-angled, $l = a$; and the altitude of the cone is the length of the simple isochronous pendulum: thus the centre of oscillation is in the centre of the base; so that the times of oscillation of a right-angled circular cone are equal for axes through the vertex and the centre of the base which are perpendicular to the axis of the cone.

Ex. 5. The mass of the particle at the end of a perfect pendulum of length a is M: another mass, m, which is very small in comparison with M, is placed on the rod at a distance x from the axis of suspension: determine the variation in the length of the simple isochronous pendulum, (1) when m is slightly shifted, (2) when the mass of m is slightly varied.
$$l = \frac{\text{M}a^2+mx^2}{\text{M}a+mx};$$
so that, if m is very small in comparison with M, the approximate value of l is a. From the preceding value of l, we have
$$l(\text{M}a+mx) = \text{M}a^2+mx^2;$$
(1) $$\therefore \quad \frac{dl}{dx} = \frac{m(2x-l)}{\text{M}a+mx};$$
as this $= 0$, when $x = \frac{l}{2}$, that is, approximately, $= \frac{a}{2}$, and changes sign from $-$ to $+$, it follows that the effect of m on the time of vibration is a minimum when m is placed at the

middle point of a, and that the time is increased according as it is removed from this point. It is also evident that l is a maximum when $\text{M}a+mx=0$; that is, when the centre of suspension is the centre of gravity of M and m, and that the time of oscillation is then infinite.

(2) Also, $$\frac{dl}{dm}=\frac{x(x-l)}{\text{M}a+mx};$$

hence l decreases as m increases, if m is put at any point between the place of M and the axis of suspension; if therefore m is placed at the middle point of a, where its effect on the time of vibration is the least, the addition of a small mass to m will cause a slight decrease in the length of l, and a corresponding decrease in the time of oscillation, or an increase in the rate of the clock; all these variations will be very small, and accordingly may be properly adapted to the correction of clock errors, which are generally very small.

Ex. 6. A metronome is formed of a rod of given length and mass M, having at one end a sphere of radius r and mass m, with its centre at a distance a from the rotation-axis, which is perpendicular to the rod; another sphere of radius r' and mass m', slides along the rod: to find the point at which the centre of this latter sphere must be fixed, so that the whole system may oscillate n times in a minute.

Let the metronome be represented in Fig. 24, wherein the rod, which in the position of equilibrium is vertical, is slightly inclined to the vertical.

Let the plane of the paper be the plane of vibration, and let O be the point where the rotation-axis pierces the plane. Let A be the centre of the fixed sphere, OA $=a$; let P be the centre of the sliding sphere, OP $=x$; let OB $=b$. Then, relatively to the rotation-axis,

$$\text{the moment of inertia of } m = m\left(\frac{2r^2}{5}+a^2\right)$$

$$\text{.......................... of } m' = m'\left(\frac{2r'^2}{5}+x^2\right)$$

$$\text{.......................... of M} = \text{M}\left(\frac{a^2-ab+b^2}{3}\right);$$

and the denominator of (50) in this case

$$=ma-\text{M}\frac{b-a}{2}-m'x;$$

so that

$$l = \frac{2}{15}\frac{m(6r^2+15a^2)+m'(6r'^2+15x^2)+5\text{M}(a^2-ab+b^2)}{2ma-\text{M}(b-a)-2m'x}. \qquad (51)$$

As the metronome is to oscillate n times in a minute,

$$\frac{60}{n} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}.$$

Let L = the length of the second's pendulum, so that $\pi^2\text{L} = g$; then

$$l = \frac{3600}{n^2}\text{L};$$

and if we substitute this in the left-hand member of (51), the equation contains x and known quantities; whence x may be determined; and the rod of the metronome may be graduated so that the system will oscillate in any required time.

If the rod is very thin, as is the case with the ordinary metronomes, M may be neglected; and we have

$$\frac{3600}{n^2}\text{L} = \frac{m(2r^2+5a^2)+m'(2r'^2+5x^2)}{5(ma-m'x)}.$$

Ex. 7. A pendulum consists of a rod of length a and mass m; at the end of which is a circular plate, Fig. 25, of radius r and mass M, so arranged that the plate is capable of sliding on the rod, and rests on a nut fixed at the end of the rod; the plane of the plate is always in the plane of vibration; find the length of the simple isochronous pendulum;

The moment of inertia of the plate $= \text{M}\left\{\frac{r^2}{2}+(a-r)^2\right\}$;

The moment of inertia of the rod $= m\frac{a^2}{3}$;

and $$h\Sigma.m = \frac{ma}{2}+\text{M}(a-r);$$

$$\therefore \quad l = \frac{3\text{M}(3r^2-4ar+2a^2)+2ma^2}{3\{ma+2\text{M}(a-r)\}}. \qquad (52)$$

Let us suppose the temperature to vary so that a and r are increased by da and dr respectively; M and m being unaltered; and let us suppose the pendulum to be compensating, so that l remains the same, whatever is the temperature; then, since $\text{D}l = 0$, we have from (52)

$$\{6\text{M}^2(2a^2-4ar+r^2)+m\text{M}(10a^2-8ar-9r^2)+2m^2a^2\}\,da$$
$$= \text{M}\,dr\,\{6\text{M}(3r^2-6ar+2a^2)+2ma(9r-4a)\}. \qquad (53)$$

In the most common form of compensating pendulums the straight rod is made of steel, and the weight consists of a cylinder

of mercury which is fixed at the end of the rod, the axis of the cylinder coinciding with the rod, and the base of the cylinder resting on a nut at the end of the rod. The amount of expansion of the rod and the mercury having been determined by experiment for an increase of one degree of temperature, and the length of the seconds' pendulum being also known, the quantity of mercury may be determined by a process similar to that which we have just explained*.

249.] The convertibility or the reciprocality of the centres of suspension and oscillation of a pendulum has been applied by Capt. Kater to the determination of its length; and he has hereby obtained means for determining the length of a seconds' pendulum at a given place.

Let the pendulum consist of an ordinary thin straight rod, and a heavy disc, as in Fig. 26. At the points O and C, at the distance l apart, let two knife edges be placed parallel to each other, and at right angles to the rod of the pendulum; so that the pendulum may vibrate on either of them, as in the diagram of the figure, where it rests on two horizontal and parallel plates. Let a small weight m be capable of sliding on the bar, and of being clamped to it by means of a screw. It is evident that whether O or C is the centre of suspension the length of the simple isochronous pendulum will vary according to the place of m; let the place of m be so adjusted that the times of oscillation may be the same, whether the pendulum is suspended by the knife edge at C or by that at O; so that OC $(= l)$ is the length of the simple isochronous pendulum; if then this distance OC is carefully measured, the length of a simple pendulum is accurately known: and by means of it the lengths of all other pendulums may be determined.

Thus, suppose the pendulum above described to make n oscillations in a given time, say in t; these quantities can be found by means of an astronomical or any other correct clock, by the method of coincidences: then

* For a full account of this pendulum I must refer the reader to a Memoir by Mr. Francis Bailey in the Eighth Volume of the Memoirs of the Royal Astronomical Society of London, for the year 1824: and for a description of various other kinds of compensating pendulums to "Mechanics," by Capt. Kater and Dr. Lardner; Longman and Co., London, 1830.

$$\frac{t}{n} = \pi \left(\frac{l}{g}\right)^{\frac{1}{2}}. \tag{54}$$

But if L is the length of a seconds' pendulum; $\pi^2 \mathrm{L} = g$,

$$\therefore \quad \mathrm{L} = \frac{n^2}{t^2} l; \tag{55}$$

and therefore the length of L is also known.

250.] The preceding theory of pendulum-motion also supplies the means for determining the value of g, the acceleration due to the earth's attraction. From (54), we have

$$g = \frac{n^2 \pi^2 l}{t^2}; \tag{56}$$

and thus when n and t have been determined by observation, and l by direct measurement, all the quantities in the right-hand member of this equation are known. So that from (55) and (56) the length of the seconds' pendulum, and the velocity-increment due to the earth's attraction, which is usually termed "the force of gravity," may be found at any given place. A table containing the values of L and g for a few places, with their latitudes N or S, is subjoined; the observations are reduced to the level of the sea, and to a pendulum vibrating in vacuo, at a temperature 62° of Fahrenheit*.

Name of Place.	Latitude.	Length of Pendulum in Inches.	Gravity in Feet.	Name of Observer.
Spitzbergen.........	79°49′58″N	39·2146	32·25294	Sabine.
Drontheim	63°25′54″N	39·1745	32·2198	Sabine.
London	51°31′ 8″N	39·13929	32·1910	Kater, Sabine.
Paris	48°50′14″N	39·1308	32·1838	Biot, Borda, &c.
New York	40°42′43″N	39·1016	32·1598	Sabine.
Jamaica	17°56′ 7″N	39·0351	32·1052	Sabine.
Sierra Leone	8°29′28″N	39·0199	32·0933	Sabine.
Cape of Good Hope	33°55′15″S	39·0787	32·1409	Freycinet.

* For accounts of the process by which General Sabine determined the lengths of the pendulum at those places in the following table to which his name is attached, see "An Account of Experiments to determine the Figure of the Earth by means of Pendulums vibrating seconds in different latitudes, as well as on various other subjects of Philosophical Inquiry,"

These results shew that gravity continually increases from the Equator to the Poles. And the differences between the observed results and the values calculated according to theory are found to be extremely small.

251.] By means of the preceding value for the length of a pendulum which vibrates isochronously with a body relative to a given axis, we are able to deduce experimentally the radius of gyration of a body relative to an axis; and consequently the central principal radii of gyration, and thus the constants of the central ellipsoid of gyration.

If it be possible, let the body make small oscillations about the axis relative to which the radius of gyration is to be determined: let T be the time of an oscillation, which can be observed by means of a clock; then

$$\text{T} = \pi\left(\frac{l}{g}\right)^{\frac{1}{2}}; \text{ and } \text{T}^2 = \frac{\pi^2}{g}\frac{k^2}{h}. \tag{57}$$

Let h, which is the distance of the mass-centre from the rotation-axis, be measured; then, since $\pi^2 k^2 = \text{T}^2 hg$,

$$\text{M}k^2 = \frac{\text{T}^2 h}{\pi^2}\text{W}, \tag{58}$$

if W is the weight of the body. Thus, we have the radius of gyration, and the moment of inertia of a body relative to a given axis.

If the axis passes through the mass-centre the method fails, because $h = 0$, and therefore $\text{T} = \infty$: in this case let another rotation-axis parallel to the given one be taken through the mass-centre, and at a distance h from it; then, if k' is the radius of gyration for the axis through the mass-centre,

$$\text{T}^2 = \frac{\pi^2}{g}\frac{h^2 + k'^2}{h};$$

$$\therefore \quad k'^2 = \frac{\text{T}^2 hg}{\pi^2} - h^2, \tag{59}$$

$$\text{M}k'^2 = \frac{\text{T}^2 h}{\pi^2}\text{W} - \text{M}h^2; \tag{60}$$

thus, (59) gives the radius of gyration, and (60) gives the moment of inertia about an axis passing through the mass-centre.

by Edward Sabine, F. R. S., &c., &c.; John Murray, London, 1825; at the expense of the Board of Longitude. See also three other papers by General Sabine in the Philosophical Transactions of 1827.

When these have been found for a sufficient number of axes, the central ellipsoid of a body may be constructed. Whenever therefore a body is given, however irregular be its bounding surface, whatever is the law according to which its density or the distribution of its elements varies, its central ellipsoid can always be determined by the preceding method; and consequently every curve or surface connected with it, or which may be derived from it, may always be assumed as known.

252.] Certain general properties of axes of a body with respect to vibration also require investigation. Let us refer the body to the mass-centre as origin, and to its central principal axes as coordinate axes. Let a, b, c be the three central principal radii of gyration, in the same order of magnitude and about the same axes as we have assumed in the preceding Chapter. So that if k'^2 is the radius of gyration about an axis (α, β, γ) passing through the mass-centre,

$$k'^2 = a^2(\cos\alpha)^2 + b^2(\cos\beta)^2 + c^2(\cos\gamma)^2. \tag{61}$$

Let those axes of a body, relative to which the times of vibration are equal, be called isochronal; then for an axis parallel to a line (α, β, γ), which passes through the mass-centre, and at a distance from it equal to h,

$$l = h + \frac{k'^2}{h} \tag{62}$$

$$= h + \frac{a^2(\cos\alpha)^2 + b^2(\cos\beta)^2 + c^2(\cos\gamma)^2}{h}; \tag{63}$$

and since this is true for all axes parallel to (α, β, γ), and equidistant from it, it follows that all axes lying on the surface of a right circular cylinder whose axis passes through the mass-centre are isochronal.

Let $l - h = h'$; so that $hh' = k'^2$; then as an axis at a distance h' from the mass-centre is isochronal with a parallel axis at a distance h, so all axes lying on the surface of a right circular cylinder whose radius is h', and whose axis passes through the mass-centre, are isochronal; and are isochronal with those which lie on the surface of the coaxial circular cylinder whose radius is h.

253.] From (62) it appears that $l = \infty$, when $h = 0$, and when $h = \infty$; so that there is some value of h between these limits which makes l a minimum. Let us equate to zero the h-differential of (62); then

$$\frac{dl}{dh} = 1 - \frac{k'^2}{h^2} = 0, \quad \text{if } h = k'; \tag{64}$$

$$\therefore \; l = 2k';$$

it appears then that for all parallel axes the time of oscillation is the least for those which are at a distance k' from the mass-centre of the body, where k' is the radius of gyration of the body relative to a parallel axis through the mass-centre; and that the length of the corresponding simple isochronous pendulum is $2k'$. In this case, the two coaxial cylinders of isochronal rotation-axes become identical; and from (62) we have

$$l = 2\{a^2(\cos\alpha)^2 + b^2(\cos\beta)^2 + c^2(\cos\gamma)^2\}^{\frac{1}{2}}. \tag{65}$$

Since $l = 2k'$, the time of oscillation depends on the central radii of gyration, and is least for an axis parallel to the least radius of gyration; therefore, from (65), l is the least when $\cos\beta = \cos\gamma = 0$: that is, when

$$l = 2a,$$

if a is the least central radius of gyration. And this gives the absolutely least time of oscillation of all axes about which a body can oscillate. And as of all parallel axes that which is at a distance equal to $2k'$ from the parallel central radius of gyration yields the least time of oscillation; so of all, that which is parallel to the axis of the greatest moment of inertia is the maximum minimorum, and that which is parallel to the axis of the least moment of inertia is the minimum minimorum, and the other minima are intermediate to these.

Ex. 1. Of all axes passing through and perpendicular to a thin rod of length $2a$, that at a distance $a\,3^{-\frac{1}{2}}$ from the middle point of the rod is that for which the time of oscillation of the rod is the least; and the length of the simple isochronous pendulum is

$$2a\,3^{-\frac{1}{2}}.$$

Ex. 2. Of all axes about which an elliptical plate can vibrate, the time of oscillation is the least for the axis parallel to the major axis and bisecting the semi-minor axis.

Ex. 3. For a sphere of radius a, all the radii of gyration passing through the centre are equal, and $= a\left(\frac{2}{5}\right)^{\frac{1}{2}}$; so that the axes for which the time of oscillation is the least are at a distance from the centre equal to this quantity · and

$$l = \frac{2^{\frac{3}{2}}a}{5^{\frac{1}{2}}}.$$

Ex. 4. The axis for which an ellipsoid vibrates in the shortest possible time is parallel to its greatest principal axis, and at a distance from it $= \left(\frac{b^2+c^2}{5}\right)^{\frac{1}{2}}$.

254.] Again, since all central equimomental axes lie on the surface of the right cone

$$(\text{H}-\text{A})x^2+(\text{H}-\text{B})y^2+(\text{H}-\text{C})z^2 = 0, \tag{66}$$

where H is the moment of inertia relative to any axis on the cone, this is the locus-surface of all axes of the circular cylinders of equal radius h, all lines lying on the surface of which are isochronal axes; and for which $l = h+\frac{\text{H}}{\text{M}h}$.

It is similarly the locus surface of all axes of the circular cylinders of equal radius h', all lines lying on the surfaces of which are isochronal axes; and for which

$$l' = h'+\frac{\text{H}}{\text{M}h'};$$

where $hh' = k'^2$, and the axes lying on the surfaces of all the cylinders are isochronal.

Thus, if two spheres of radii h and h' are described from the mass-centre as centre, and cones are described touching them, coaxial with and similar to the given cone, all generating lines of these two cones are isochronal.

255.] The relation between k', h, and l which is given by the equation (48), viz.,

$$k'^2 = h(l-h), \tag{67}$$

leads to the following construction for the locus of the centres of suspension, when l, the length of the pendulum, is constant; that is, for a system of isochronal axes of oscillation*.

If a, b, c are the principal central radii of gyration of a body, the equation to the central ellipsoid of gyration is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1; \tag{68}$$

and the equation to the central pedal of this ellipsoid is

$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2. \tag{69}$$

Now by Art. 187, any radius vector of this surface is the radius of gyration of the body which lies along it; and thus of all the central radii of gyration which lie in a given plane, the

* See a Paper by Böklen. Crelle Journal; Band. XCIII., p. 177.

greatest and least are those which coincide with the principal axes of the section of the pedal surface made by that plane. From G, the centre of the section, let a perpendicular to the plane be drawn, and let lengths GK_1, GK_2 be taken equal to the principal axes of the section, then K_1 and K_2 are respectively in the two sheets of the apsidal of the pedal, and the apsidal is the bounding surface of the lengths of the central radii of gyration of the body, all of which are intermediate to GK_1 and GK_2. Now the equation to the apsidal is, see (128), Art. 21,

$$\frac{x^2}{a^2-r^2}+\frac{y^2}{b^2-r^2}+\frac{z^2}{c^2-r^2}=0; \tag{70}$$

where $r^2 = x^2+y^2+z^2$, and the values of r lying along any radius are the maximum and minimum values of the radii of gyration in the plane section of the pedal surface which is perpendicular to that radius. And all other values of k' in that section are represented by lines whose ends lie between K_1 and K_2: so that the apsidal surface is the bounding surface of lines equal to the radii of gyration. If through any one of the points K lines are drawn parallel to the corresponding radii of gyration, it is evident that there is only one such line at K_1, and only one at K_2, and that these lines are perpendicular to each other; but that through any other point K lying between K_1 and K_2 there are two such lines, lying in the same plane and inclined to each other at some angle between 0° and 90°.

As the two sheets of the apsidal, see Art. 21, intersect in four cuspal points in the plane of (z, x), so at these points all the K's become coincident, and all the radii of gyration in the plane section of the pedal are equal; that is, that plane of section of the pedal is a circle. This takes place when $x^2+y^2+z^2 = b^2$, and $(b^2-a^2)x^2 = (c^2-b^2)z^2$. The number of lines which can be drawn through these points parallel to the corresponding radii of gyration in the plane section is infinite. There are evidently two such cyclic planes of the pedal.

Thus at any point on the apsidal only one line can be drawn which is parallel to the corresponding radius of gyration: at all points lying between the two sheets a pair of such lines can be drawn: and at the cuspal points the number of such lines is infinite.

256.] Now let us extend this construction, so that it should

apply to centres of suspension; let r^2 be replaced by its value $h(l-h)$, given in (67), then (68) becomes

$$\frac{x^2}{a^2-hl+h^2}+\frac{y^2}{b^2-hl+h^2}+\frac{z^2}{c^2-hl+h^2}=0; \qquad (71)$$

and let $h^2=x^2+y^2+z^2$, so that this surface is, if l is constant, the bounding surface of the centres of suspension.

If h is constant as well as l, this equation represents a quadric cone; and if h varies, a series of cones, all of which are confocal, and of which the focal lines are given by the equations

$$y=0, \quad \frac{x^2}{a^2-b^2}+\frac{z^2}{c^2-b^2}=0,$$

and these lines are the asymptotes of the focal hyperbola of the ellipsoid of gyration in the plane of (x, z), as explained in Arts. 27 and 28.

The equation to the surface may be expressed in the following form

$$\frac{x^2}{\frac{l^2}{4}-a^2-\left(h-\frac{l}{2}\right)^2}+\frac{y^2}{\frac{l^2}{4}-b^2-\left(h-\frac{l}{2}\right)^2}+\frac{z^2}{\frac{l^2}{4}-c^2-\left(h-\frac{l}{2}\right)^2}=0.$$

Let $\quad \frac{l^2}{4}-a^2=a'^2, \quad \frac{l^2}{4}-b^2=b'^2, \quad \frac{l^2}{4}-c^2=c'^2,$

$$h-\frac{l}{2}=r';$$

then $a'>b'>c'$; and the equation becomes

$$\frac{x^2}{a'^2-r'^2}+\frac{y^2}{b'^2-r'^2}+\frac{z^2}{c'^2-r'^2}=0. \qquad (72)$$

Now assuming that a'^2, b'^2, c'^2 are all positive, the form of this surface is the same as that of (70); consequently as (70) is the apsidal of the pedal derived from the ellipsoid of gyration, so is this surface the apsidal of the pedal derived from

$$\frac{x^2}{a'^2}+\frac{y^2}{b'^2}+\frac{z^2}{c'^2}=1.$$

This surface therefore consists of two sheets, having four cuspal points in which the sheets intersect, and has properties in all respects similar to those of (70).

If then we construct this latter surface, of which the principal axes are related to those of the central ellipsoid of gyration by the equations

$$a'^2+a^2=b'^2+b^2=c'^2+c^2=\frac{l^2}{4},$$

and if we find the central pedal of this surface, and determine its apsidal, the radii vectores of the surface locus of the centres of suspensions of isochronal axes are the radii vectores of this apsidal lengthened by $\frac{l}{2}$. Hence that surface-locus consists of two sheets, having four cuspal points in which the sheets intersect, and all other properties similar to those of the surface whose equation is (68). This surface-locus is the boundary of the requisite points of suspension, no point of suspension for any axis of the system lying outside of it; but all points on it and within it being points of suspension corresponding to an isochronal axis of the system.

Since h, l and k' are connected by the equation $h^2 - lh + k'^2 = 0$, there are two values of h which satisfy this equation; and these are

$$h = \frac{l}{2} \pm \left(\frac{l^2}{4} - k'^2\right)^{\frac{1}{2}},$$

and both these are real so long as $2k'$ is less than l; if $2k' = l$, there is only one value of h. If $2k'$ is less than l, there are two surfaces of the form (72) corresponding to the two different values of h; and all the properties which have been proved of one of these surfaces are equally true of the other. This also follows from the theorem that the centres of suspension and oscillation are reciprocal and convertible.

257.] Lastly, let us determine the conical surface on which lie all isochronal axes passing through the given point (x_0, y_0, z_0).

Let the equations to one of these isochronal axes referred to the mass-centre as origin be

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n} = s, \text{ (say)}; \qquad (73)$$

$$\therefore \quad h^2 = (ny_0 - mz_0)^2 + (lz_0 - nx_0)^2 + (mx_0 - ly_0)^2;$$

and

$$k'^2 = a^2 l^2 + b^2 m^2 + c^2 n^2,$$

if a, b, c are the principal central radii of gyration. In these equations, replacing l, m, n by their values from (73), we have

$$s^2 h^2 = (zy_0 - yz_0)^2 + (xz_0 - zx_0)^2 + (yx_0 - xy_0)^2;$$

$$s^2 k'^2 = a^2 (x - x_0)^2 + b^2 (y - y_0)^2 + c^2 (z - z_0)^2.$$

If therefore l is the length of the simple pendulum, isochronous with the body about each of the axes passing through (x_0, y_0, z_0),

$$lh = h^2 + k'^2;$$

$$l\{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\}^{\frac{1}{2}}$$
$$\{(zy_0-yz_0)^2+(xz_0-zx_0)^2+(yx_0-xy_0)^2\}^{\frac{1}{2}}$$
$$=(zy_0-yz_0)^2+(xz_0-zx_0)^2+(yx_0-xy_0)^2+$$
$$a^2(x-x_0)^2+b^2(y-y_0)^2+c^2(z-z_0)^2;$$

which is evidently the equation to a cone of the fourth degree.

258.] An application of these principles has been made by Capt. Robins, to the determination of the velocity with which a cannon ball leaves a gun.

A heavy board is suspended by a fixed horizontal axis; a cannon is so placed that a ball projected horizontally from the cannon strikes this board at rest at a certain point; and the board revolves through an angle, which is observed. It is required to determine the velocity of the ball. The swinging board with its axis is called a Ballistic Pendulum. A vertical section is given in Fig. 27.

We shall suppose the ball to strike the board at right angles to its plane, and to remain in the board after impact. Let

M = the sum of the masses of the pendulum and ball.

m = the mass of the ball.

$\text{M}k^2$ = the moment of inertia of the pendulum and ball.

V = the velocity of the ball at the instant of impact.

Ω = the angular velocity due to the blow of the ball.

a = the distance of the point of impact from the rotation-axis.

h = the distance of the centre of gravity of the masses of the pendulum and ball from the rotation-axis.

$$\therefore \quad \Omega = \frac{ma\text{V}}{\text{M}k^2}.$$

Then if 2α is the angle through which the body has turned, when it comes to rest, by the equation of vis viva,

$$k^2\Omega^2 = 2hg(1-\cos 2\alpha)$$
$$= 4hg(\sin\alpha)^2;$$
$$\therefore \quad \text{V} = \frac{2\text{M}k}{ma}(hg)^{\frac{1}{2}}\sin\alpha;$$

as all the quantities in the right-hand member of this equation may be observed, or are known, V is also known.

We may determine α in the following manner. At a point in the board at a distance h from the rotation-axis, let the end of a ribbon be fastened, and let the rest of the ribbon be wound tightly round a reel, so placed, that if a length c is unwound in

the ascent of the board, c is the chord of the angle 2α to the radius h, so that $c = 2h\sin\alpha$;

$$\therefore \quad \text{V} = \frac{\text{M}kc}{ma}\left(\frac{g}{h}\right)^{\frac{1}{2}};$$

k is determined by the process explained in Art. 251; and if we replace k by the value given in (58) we have

$$\text{V} = \frac{gc\text{M}}{\pi am}\text{T};$$

and if M and m are replaced by their weights, say W and w, which are proportional to them, we have

$$\text{V} = \frac{gc\text{W}}{\pi aw}\text{T}. \qquad (74)$$

If the mouth of the cannon is placed near to the pendulum, the value of V, given by this formula, must be nearly the velocity of projection. And if the distance of the pendulum from the mouth of the gun be large, so that the velocity of impact on the pendulum is less than that of projection, then, if the coefficient of resistance of the air is given, we may by the process of Art. 341, Vol. III, estimate the diminution of velocity due to the resistance of the air, and thus determine the velocity of projection.

The velocity however may be determined in the following manner, as suggested by Dr. Hutton. Let the gun itself be suspended by a horizontal axis, and thus form a pendulum; when the gun is discharged, it will oscillate by reason of the recoil; and by observing the times of these oscillations, and making the required alterations in (74), the velocity of projection will be determined: W will represent in this case the weight of the gun, and c and a must be similarly altered.

259.] The following problems are in further illustration of the principles contained in the preceding Articles.

Ex. 1. A vertical rod of length $2a$ and of mass M, which turns about a horizontal axis passing through the upper end, is struck by a blow at its centre of percussion, and ascends into its position of unstable equilibrium; determine the momentum of the blow.

Let Q = the momentum of the blow, and let Ω be the angular velocity which is thereby given to the rod about its rotation-axis. Now the centre of percussion is at a distance $\frac{4a}{3}$ from the rotation-axis; so that by (7), Art. 229, $a\text{M}\Omega = \text{Q}$; and by the prin-

ciple of vis viva, see (42), Art. 244,

$$\frac{\text{M}}{2}\frac{4a^2\Omega^2}{3} = 2a\text{M}g;$$

$$\therefore \quad \text{Q} = \text{M}(2ag)^{\frac{1}{2}}.$$

Ex. 2. A circular plate of radius a and mass M, capable of rotation about a horizontal axis which is in its plane and touches it, is struck by a blow Q at its centre of percussion, and ascends into its position of unstable equilibrium; prove that

$$\text{Q} = 4\text{M}\left(\frac{ag}{5}\right)^{\frac{1}{2}}.$$

Ex. 3. A heavy rod with one end fixed is placed horizontally, and a weight is put on it at such a point that the weight just leaves it when the rod begins to move about its fixed end: determine the position of the point.

Let x be the distance of the point from the fixed end: then for the angular acceleration of the rod we have

$$\frac{d^2\theta}{dt^2} = \frac{3g}{2a};$$

therefore $\frac{3g}{2a}x$ is the acceleration of the point where the weight is placed; but this is equal to g;

$$\therefore \quad x = \frac{2a}{3}.$$

Also at all points farther than this from the fixed end, the weight will separate from the rod, and at points nearer to the end will remain on the rod and increase its acceleration.

Ex. 4. A heavy right circular cylinder stands on a rough horizontal plane to which a jerk is given in a given direction: determine its effect on the cylinder.

Let M be the mass of the cylinder, $2a$ the height, and b the radius of the base. Let Q be the momentum of the blow which acts from the plane on the cylinder, and which being uniformly distributed over the base acts in its resultant effect at the centre. Let the line of Q be inclined to the vertical at an angle α: then the effect of Q will be (1) a vertical jerk $= \text{Q}\cos\alpha$ on the cylinder acting through the centre of the base, (2) a horizontal jerk on the base $= \text{Q}\sin\alpha$, which, as the base stands on a rough plane, gives a horizontal momentum $= \text{Q}\sin\alpha$ to the cylinder at its centre of gravity, and (3) consequentially, a rotation of the

cylinder about that tangent to its base which is perpendicular to the vertical plane containing the line of the jerk. This rotation may be determined as follows.

Let θ be the angle through which the cylinder is turned in the time t: let k be the radius of gyration about the rotation-axis, and Ω be the initial angular velocity due to the blow: then, by (6), Art. 229,

$$\text{M}k^2\Omega = \text{Q}\{a\sin\alpha - b\cos\alpha\};$$

and from the equation of vis viva, (42), Art. 244,

$$\frac{\text{M}k^2}{2}\left\{\frac{d\theta^2}{dt^2} - \Omega^2\right\} = \text{M}g\{a - a\cos\theta - b\sin\theta\};$$

whence the circumstances are known for a given value of θ.

Suppose the blow to be such that the cylinder is just upset: then $\frac{d\theta}{dt} = 0$, when $\tan\theta = \frac{b}{a} = \tan\beta$, say, where β is the angle between the axis of the cylinder and its diagonal: also let $2c$ be the diagonal, so that $a^2 + b^2 = c^2$: then

$$\text{M}k^2\Omega = \text{Q}c\sin(\alpha - \beta), \quad k^2\Omega^2 = 2g(c - a);$$

as Ω cannot be negative, the preceding value shews that α must be always greater than β; that is, the line drawn through the centre of gravity parallel to the line of the blow must fall outside the base of the cylinder if the cylinder is to be upset. Also, since $k^2 = \frac{4a^2}{3} + \frac{5b^2}{4}$, from the last two equations we have

$$\text{Q}^2 = \frac{cg\text{M}^2}{6}\frac{1 - \cos\alpha}{\{\sin(\alpha - \beta)\}^2}\{15 + (\cos\beta)^2\},$$

which determines the intensity of the jerk which is necessary for just upsetting the cylinder.

The circumstances of this problem are exemplified when a heavy body is upset by a shock of earthquake, or when a body stands on the floor of a moving railway carriage which suddenly meets with an obstacle and is brought to rest.

A similar process is applicable to a cube or a cone or a paraboloid resting on a rough plane to which a jerk is given.

260.] If a body moving about a fixed axis meets with a fixed obstacle on which it impinges, and is thereby brought to rest, the momentum of the impact may be determined as follows.

The action of the impact on the body is similar to that of a

blow, and consequently comes within the principle of the equation given in Art. 229. Hence if Q is the momentum of the impact on the obstacle, and q is the perpendicular distance from the obstacle on the fixed axis: also if m is the mass of a particle of the body which is moving with a velocity v at the instant of impact, and p is the perpendicular from the axis on the line of motion of m, then $\Sigma . mvp$ is the sum of the moments of the momenta of all the particles, which are brought to rest by the impact: consequently this must be equal to the moment of the impact, and we have

$$\text{Q}q = \Sigma . mvp;$$

but as the rotation is taking place about a fixed axis, if r is the distance of m from this axis, $v = r\frac{d\theta}{dt}$, and $p = r$,

$$\therefore \quad \text{Q}q = \Sigma . mr^2 \frac{d\theta}{dt} = \text{M}k^2 \frac{d\theta}{dt},$$

whence if q is given, Q can be determined. The following are examples in which this principle is applied:

Ex. 1. A heavy beam of mass M and length $2a$ falls from a vertical position, turning about its lower end which continues fixed, and impinges on an obstacle in the horizontal plane which passes through the lower end. Find the momentum of the impact.

The equation of vis viva is in this case

$$\frac{\text{M}}{2} \frac{4a^2}{3} \frac{d\theta^2}{dt^2} = \text{M}ga;$$

where $\frac{d\theta}{dt}$ is the angular velocity with which the beam is moving when it strikes the obstacle, therefore, if q is the distance of the obstacle from the foot of the beam,

$$\text{Q} = \frac{\text{M}(2a)^{\frac{3}{2}}}{q} \left(\frac{g}{3}\right)^{\frac{1}{2}}.$$

Ex. 2. If the obstacle is in the vertical line passing through and below the axis of rotation, then

$$\text{Q} = \frac{4\text{M}a^{\frac{3}{2}}}{q} \left(\frac{g}{3}\right)^{\frac{1}{2}}.$$

Ex. 3. A heavy right-angled cone of mass M and altitude a, rotating about a diameter of its base which is horizontal, falls from its position of unstable equilibrium to its lowest position,

and is brought to rest by its vertex impinging on a fixed obstacle. Prove that the momentum of the impact

$$= \frac{\text{M}\, a}{2q}(ag)^{\frac{1}{2}}.$$

261.] In further application of these principles let us consider the motion of the parts of some machines, in which certain rotation-axes are fixed.

Ex. 1. Two weights mg and $m'g$ are connected by a flexible and inextensible string without weight, which passes over a given pulley with a fixed axis and a rough surface; it is required to determine the circumstances of motion of each weight and of the pulley.

The pulley is supposed to be rough, so that the string does not slip over it.

Let the weights, &c. be arranged as in Fig. 17; and let the symbols be those of Ex. 1, Art. 71; and let us suppose m and m' to have the initial velocities, &c. of that example. Let M = the mass of the pulley, and a = the radius; then

$$\text{the moment of inertia of the pulley} = \text{M}\,\frac{a^2}{2}\cdot$$

Let Ω be the initial angular velocity of the pulley due to the instantaneous initial tensions of the string; then, by (6),

$$\frac{\text{M}\, a^2}{2}\,\Omega = a(\tau - \tau');$$

also $v = a\,\Omega$; so that

$$\tau = mu - ma\,\Omega, \qquad \tau' = m'u' + m'a\,\Omega;$$

$$\therefore \quad \Omega = \frac{2(mu - m'u')}{a(\text{M} + 2m + 2m')}; \tag{75}$$

whence the initial angular-velocity of the pulley, and the initial tensions of the strings are known.

Again,
$$\frac{\text{M}\, a^2}{2}\,\frac{d^2\theta}{dt^2} = a(\text{T} - \text{T}');$$

also
$$\frac{d^2x}{dt^2} = a\,\frac{d^2\theta}{dt^2}, \qquad \frac{d^2x'}{dt^2} = -a\,\frac{d^2\theta}{dt^2}; \tag{76}$$

$$\therefore \quad \text{T} = mg - ma\,\frac{d^2\theta}{dt^2}, \qquad \text{T}' = m'g + m'a\,\frac{d^2\theta}{dt^2}; \tag{77}$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{2(m - m')g}{a(\text{M} + 2m + 2m')}; \tag{78}$$

$$\therefore \quad \frac{d\theta}{dt} - \Omega = \frac{2(m-m')gt}{a(\text{M}+2m+2m')}, \tag{79}$$

$$\frac{d\theta^2}{dt^2} - \Omega^2 = \frac{4(m-m')g\theta}{a(\text{M}+2m+2m')}; \tag{80}$$

whence by a further integration θ can be determined in terms of t; and thus the space will be known through which m or m' will move in a given time. (80) is the equation of vis viva.

If we replace $\frac{d^2\theta}{dt^2}$ in (77) by its value, given in (78), we shall find the tensions of the strings at any time t.

If the weights of the strings are taken into account, the equation of angular motion assumes the following form: Let $\rho =$ the density, $\omega =$ the area of a transverse section of the string; $c =$ the whole length; then, if $\text{M}k^2$ is the moment of inertia of the pulley,

$$(\text{M}k^2 + ma^2 + m'a^2 + \rho\omega c a^2)\frac{d^2\theta}{dt^2} = a(m-m')g + a\rho\omega g x - a\rho\omega g x'$$
$$= a\{m - m' + \rho\omega(x-x')\}g.$$

Ex. 2. To investigate the circumstances of motion of a wheel and axle, the weights of the strings being neglected, and $\text{M}k^2$ being the moment of inertia of the machine relative to its axis.

Let us use the same symbols as in Ex. 2, Art. 71, and those of the last example; and let Fig. 18 represent the plan of the wheel and axle when projected on the vertical plane of the paper.

$$\tau = mu - mc\Omega, \qquad \tau' = m'u' + m'c'\Omega; \tag{81}$$

$$\therefore \quad \Omega = \frac{cmu - c'm'u'}{\text{M}k^2 + mc^2 + m'c'^2}; \tag{82}$$

whereby the initial angular-velocity of the machine, and the initial tensions of the strings are known.

Again, $$\text{T} = mg - mc\frac{d^2\theta}{dt^2}, \quad \text{T}' = m'g + m'c'\frac{d^2\theta}{dt^2}; \tag{83}$$

$$\text{M}k^2\frac{d^2\theta}{dt^2} = c\text{T} - c'\text{T}';$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{(cm - c'm')g}{\text{M}k^2 + mc^2 + m'c'^2}; \tag{84}$$

whence all the circumstances of motion may be determined.

If $\text{P} =$ the pressure on the axis at the time t, it is equal to the weight of the wheel and axle together with the tensions of the strings; therefore

$$\text{P} = \text{M}g + \text{T} + \text{T}'$$

$$= (\text{M} + m + m')g - \frac{(mc - m'c')^2}{\text{M}k^2 + mc^2 + m'c'^2} g\,; \qquad (85)$$

$$= \text{M}g + \frac{mm'(c + c')^2 + (m + m')\,\text{M}k^2}{\text{M}k^2 + mc^2 + m'c'^2}\, g\,; \qquad (86)$$

that is, the pressure on the axis is less than it would be if the machine were at rest; but it can never vanish.

Ex. 3. It is required to determine the motion of a system of wheels and pinions, such as a crane, or the like, the power attached to the first wheel being P, and the weight attached to the last pinion or axle being W.

Whatever is the form of the system, it may always be arranged as in Fig. 28; where we have taken four wheels and pinions: C_1, C_2, C_3, C_4 are the centres of the successive wheels and pinions, C_1 being that of the axle to which the weight W is attached. Let the pressures between the successive wheels and pinions, whether due to the action of teeth or to friction, be T_1, T_2, T_3; let a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4 be the radii of the several pinions and wheels in order; and let H_1, H_2, H_3, H_4 be their moments of inertia; let T be the tension of the string to which the weight is attached, and t the tension of that by which P acts; let M be the mass of the weight W, and let $\text{P} = mg$; let us suppose W to descend in the time dt through a space dx, and P to ascend through a space dx'; and let $d\theta_1, d\theta_2, d\theta_3, d\theta_4$ be the angles through which the wheels rotate in that time; then

$$dx = a_1 d\theta_1;\ b_1 d\theta_1 = a_2 d\theta_2;\ b_2 d\theta_2 = a_3 d\theta_3;\ b_3 d\theta_3 = a_4 d\theta_4;\ b_4 d\theta_4 = -dx'. \quad (87)$$

For the translation of W and P, we have

$$\text{T} = \text{W} - \text{M}\frac{d^2 x}{dt^2} = \text{W} - a_1 \text{M}\frac{d^2\theta_1}{dt^2}\,; \qquad (88)$$

$$t = \text{P} - m\frac{d^2 x'}{dt^2} = \text{P} + b_4 m\frac{d^2\theta_4}{dt^2}\,. \qquad (89)$$

And for the rotation of the pulleys, we have

$$\left.\begin{aligned}
\text{H}_1\frac{d^2\theta_1}{dt^2} &= a_1\text{T} - b_1\text{T}_1, \\
\text{H}_2\frac{d^2\theta_2}{dt^2} &= a_2\text{T}_1 - b_2\text{T}_2, \\
\text{H}_3\frac{d^2\theta_3}{dt^2} &= a_3\text{T}_2 - b_3\text{T}_3, \\
\text{H}_4\frac{d^2\theta_4}{dt^2} &= a_4\text{T}_3 - b_4 t\,;
\end{aligned}\right\} \qquad (90)$$

whence, by a simple elimination, we have

$$\frac{d^2x}{dt^2}\{\text{H}_1(a_2a_3a_4)^2+\text{H}_2(a_3a_4b_1)^2+\text{H}_3(a_4b_1b_2)^2+\text{H}_4(b_1b_2b_3)^2$$
$$+\text{M}(a_1a_2a_3a_4)^2+m(b_1b_2b_3b_4)^2\}$$
$$=a_1a_2a_3a_4\{a_1a_2a_3a_4\text{W}-b_1b_2b_3b_4\text{P}\};\quad(91)$$

and, by integration, the space described by W in the time t may be found.

Also, from (87),
$$\frac{d^2x'}{dt^2}=-\frac{b_1b_2b_3b_4}{a_1a_2a_3a_4}\frac{d^2x}{dt^2};\quad(92)$$

and thus the motion of P may be determined.

A similar process may of course be applied, whatever is the number of the wheels and pinions.

If in the preceding example the wheels are all equal, and all the pinions are equal,

$$\frac{d^2x}{dt^2}\{\text{H}(a^6+a^4b^2+a^2b^4+b^6)+\text{M}a^8+mb^8\}=a^4\{\text{W}a^4-\text{P}b^4\}.\quad(93)$$

Ex. 4. A heavy flexible and inextensible string of given length a, is wound round a solid cylinder of mass M and radius c, which is capable of rotation about its axis, which is horizontal; a piece of the string of length b hangs down, so that the cylinder begins to rotate; it is required to determine the motion of the string and of the cylinder.

Let the circumstances at the time t be represented in Fig. 29; and let $\text{M}k^2$ be the moment of inertia of the cylinder. In the time t let a chain of length $= x = c\theta$ be unwound from the cylinder; let $\omega =$ the area of a transverse section, $\rho =$ the density of the string; then the weight of the string which hangs vertically at the time $t = \rho\omega g(b+c\theta)$. Let T = the tension of the string at the point P, CP $= b+x$.

$$\therefore\ \text{T}=\rho\omega(b+c\theta)\left\{g-c\frac{d^2\theta}{dt^2}\right\};$$

and the moment of inertia of the cylinder, and the chain wound round it at the time t, $= \text{M}k^2+\rho\omega(a-b-c\theta)c^2$; so that the equation of rotation of the cylinder is

$$\{\text{M}k^2+\rho\omega(a-b-c\theta)c^2\}\frac{d^2\theta}{dt^2}=c\text{T}$$
$$=\rho\omega c(b+c\theta)\left\{g-c\frac{d^2\theta}{dt^2}\right\};$$

$$\therefore \quad \frac{d^2\theta}{dt^2} = \frac{\rho\omega c(b+c\theta)g}{\mathrm{M}k^2+\rho\omega ac^2};$$

$$\therefore \quad \frac{d\theta^2}{dt^2} = \frac{\rho\omega cg}{\mathrm{M}k^2+\rho\omega ac^2}(2b\theta+c\theta^2), \tag{94}$$

since we have assumed $\frac{d\theta}{dt} = 0$, when $\theta = 0$; therefore when $c\theta = a-b$, that is, when the whole chain is unwound,

$$\frac{d\theta^2}{dt^2} = \frac{\rho\omega g(a^2-b^2)}{\mathrm{M}k^2+\rho\omega ac^2}.$$

Again, from (94), for the whole time spent in unwinding the string, we have

$$\left\{\frac{\rho\omega cg}{\mathrm{M}k^2+\rho\omega ac^2}\right\}^{\frac{1}{2}} t = \int_0^{\frac{a-b}{c}} \frac{d\theta}{(2b\theta+c\theta^2)^{\frac{1}{2}}}.$$

$$= \frac{1}{c^{\frac{1}{2}}}\log\frac{a+(a^2-b^2)^{\frac{1}{2}}}{b}, \tag{95}$$

which gives the time.

By a similar process we may determine the length of string which a cylinder, rotating with a given angular velocity, would wind up before it is brought to rest.

Ex. 5.] A balance has equal weights in the scales, and oscillates through small angles, the beam and scales moving in a plane which is perpendicular to the axis of vibration; it is required to determine the circumstances of motion.

Let the balance, &c. be represented in Fig. 31, in which the plane of the paper is the plane of motion of the beam and scales, and the axis of vibration is perpendicular to the plane of the paper. Let O be the point where this axis pierces the paper; let G be the centre of gravity of the balance without the weights; let

M = the mass of the balance.

m = the mass of each weight in the scales.

$\mathrm{M}k^2$ = the moment of inertia of the balance relative to the rotation-axis.

a = the length of each arm = AB = BA′.

OB = b. OG = h.

Let θ = the angle between OB and the vertical line; which angle, as well as its t-differential, we shall assume to be infinitesimal, so that the squares and higher powers may be neglected.

Let $\frac{dx}{dt}, \frac{dx'}{dt}$ be the vertical velocities of the weights in the scales at P and P′ respectively; let T and T′ be the tensions of the strings at A and A′ respectively. We shall neglect the oscillations of the scales about the points A and A′.

The perpendicular distances from O on AP and A′P′ respectively are $a\cos\theta - b\sin\theta$, and $a\cos\theta + b\sin\theta$; which quantities, as θ is infinitesimal, are $a - b\theta$ and $a + b\theta$. So that the equation of rotation is

$$\mathrm{M}k^2 \frac{d^2\theta}{dt^2} = \mathrm{T}(a - b\theta) - \mathrm{T}'(a + b\theta) - \mathrm{M}gh\theta. \tag{96}$$

Now $\quad \mathrm{T} = m\left(g - \frac{d^2x}{dt^2}\right), \qquad \mathrm{T}' = m\left(g - \frac{d^2x'}{dt^2}\right);$

but $\quad dx = d(a\sin\theta + b\cos\theta), \qquad dx' = d(-a\sin\theta + b\cos\theta);$

$$\frac{dx}{dt} = (a\cos\theta - b\sin\theta)\frac{d\theta}{dt}, \qquad \frac{dx'}{dt} = (-a\cos\theta - b\sin\theta)\frac{d\theta}{dt};$$

$$\frac{d^2x}{dt^2} = (a\cos\theta - b\sin\theta)\frac{d^2\theta}{dt^2}, \quad \frac{d^2x'}{dt^2} = (-a\cos\theta - b\sin\theta)\frac{d^2\theta}{dt^2};$$

so that (96) becomes

$$\frac{d^2\theta}{dt^2} = -\frac{2mb + \mathrm{M}h}{2ma^2 + \mathrm{M}k^2} g\theta. \tag{97}$$

Let α be the value of θ when $\frac{d\theta}{dt} = 0$; then

$$\frac{d\theta^2}{dt^2} = \frac{(2mb + \mathrm{M}h)g}{2ma^2 + \mathrm{M}k^2}(\alpha^2 - \theta^2); \tag{98}$$

$$\therefore \quad t = \left\{\frac{2ma^2 + \mathrm{M}k^2}{(2mb + \mathrm{M}h)g}\right\}^{\frac{1}{2}} \cos^{-1}\frac{\theta}{\alpha};$$

and therefore the time of an oscillation

$$= \pi\left\{\frac{2ma^2 + \mathrm{M}k^2}{(2mb + \mathrm{M}h)g}\right\}^{\frac{1}{2}}; \tag{99}$$

and therefore if $l =$ the length of the simple isochronous pendulum,

$$l = \frac{2ma^2 + \mathrm{M}k^2}{2mb + \mathrm{M}h}. \tag{100}$$

262.] We must now return to equations (35) and (36), which determine the pressures borne by the fixed points of the axis during motion of this kind.

As the z-axis is fixed, the particles of the system have no

motion in a direction parallel to that axis, so that for all particles $\frac{d^2 z}{dt^2} = 0$; and therefore from the last of (35),

$$\Sigma . \text{P} \cos \gamma = \Sigma . m\text{Z}; \tag{101}$$

and as neither $\text{P}_1 \cos \gamma_1$, nor $\text{P}_2 \cos \gamma_2$, enters into the other equations, this shews that the sum of the components of the pressures along the z-axis is equal to the sum of the similar axial components of the impressed momentum-increments on all the particles; but as the sum only is given, each pressure is indeterminate. This case is similar to that of Art. 231, and admits of a similar explanation.

I may observe in passing, that if the axis is capable of sliding in the direction of its length, then the motion of all the particles of the body along that line will be derived from the equation

$$\Sigma . m \left(\text{Z} - \frac{d^2 z}{dt^2}\right) = 0;$$

and as $\frac{d^2 z}{dt^2}$ will be the same for all the particles, if $\text{M} =$ the mass of the body,

$$\frac{d^2 z}{dt^2} = \frac{\Sigma . m\text{Z}}{\text{M}}; \tag{102}$$

whereby the longitudinal displacement of the axis may be determined.

The components of the pressures at the fixed points, which are perpendicular to the axis, enter into the first two equations of both (35) and (36). In these let us replace $\frac{d^2 x}{dt^2}$ and $\frac{d^2 y}{dt^2}$ by their equivalents, given in Art. 241, in terms of ω; then these four equations become

$$\left.\begin{aligned} \text{P}_1 \cos \alpha_1 + \text{P}_2 \cos \alpha_2 &= \omega^2 \Sigma . mx + \frac{d\omega}{dt} \Sigma . my + \Sigma . m\text{X}, \\ \text{P}_1 \cos \beta_1 + \text{P}_2 \cos \beta_2 &= \omega^2 \Sigma . my - \frac{d\omega}{dt} \Sigma . mx + \Sigma . m\text{Y}; \end{aligned}\right\} \tag{103}$$

$$\left.\begin{aligned} \text{P}_1 z_1 \cos \beta_1 + \text{P}_2 z_2 \cos \beta_2 &= \omega^2 \Sigma . myz - \frac{d\omega}{dt} \Sigma . mxz - \text{L}, \\ \text{P}_1 z_1 \cos \alpha_1 + \text{P}_2 z_2 \cos \alpha_2 &= \omega^2 \Sigma . mxz + \frac{d\omega}{dt} \Sigma . myz + \text{M}; \end{aligned}\right\} \tag{104}$$

from which equations the components of the pressures perpendicular to the rotation-axis may be determined. It is worth

while to consider the forms which the preceding equations take relatively to certain axes of the body.

(1) Let us suppose the rotation-axis to be a principal axis of the body; and let us moreover take the origin at its principal point; then $\Sigma . myz = \Sigma . mzx = 0$; and (104) become

$$\left.\begin{aligned} P_1 z_1 \cos\beta_1 + P_2 z_2 \cos\beta_2 &= -L, \\ P_1 z_1 \cos\alpha_1 + P_2 z_2 \cos\alpha_2 &= M; \end{aligned}\right\} \tag{105}$$

from which, with (103), the pressures may be determined.

(2) Let us suppose the rotation-axis to be a central principal axis, and the mass-centre to be the origin; then

$$\Sigma . myz = \Sigma . mzx = 0; \quad \Sigma . mx = \Sigma . my = 0;$$

then (104) and (103) become respectively (105), and

$$\left.\begin{aligned} P_1 \cos\alpha_1 + P_2 \cos\alpha_2 &= \Sigma . mX, \\ P_1 \cos\beta_1 + P_2 \cos\beta_2 &= \Sigma . mY; \end{aligned}\right\} \tag{106}$$

whence we have

$$P_1 \cos\alpha_1 = \frac{z_2 \Sigma . mX - M}{z_2 - z_1}, \qquad P_1 \cos\beta_1 = \frac{z_2 \Sigma . mY + L}{z_2 - z_1}; \tag{107}$$

$$P_2 \cos\alpha_2 = \frac{-z_1 \Sigma . mX + M}{z_2 - z_1}, \qquad P_2 \cos\beta_2 = \frac{-z_1 \Sigma . mY - L}{z_2 - z_1}. \tag{108}$$

If the points of support of the axis are equally distant from the centre of gravity, so that $z_2 = -z_1$, then

$$P_1 \cos\alpha_1 = \frac{z_1 \Sigma . mX + M}{2 z_1}, \qquad P_1 \cos\beta_1 = \frac{z_1 \Sigma . mY - L}{2 z_1}; \tag{109}$$

$$P_2 \cos\alpha_2 = \frac{z_1 \Sigma . mX - M}{2 z_1}, \qquad P_2 \cos\beta_2 = \frac{z_1 \Sigma . mY + L}{2 z_1}. \tag{110}$$

(3) If no forces act on the system, so that $X = Y = Z = 0$ for all particles; and $L = M = N = 0$; then the body rotates about the fixed axis with the constant initial velocity Ω.

And if moreover the rotation-axis passes through the mass-centre, so that $\Sigma . mx = \Sigma . my = 0$; then, from (103) and from (101), we have

$$\left.\begin{aligned} P_1 \cos\alpha_1 + P_2 \cos\alpha_2 &= 0, \\ P_1 \cos\beta_1 + P_2 \cos\beta_2 &= 0, \\ P_1 \cos\gamma_1 + P_2 \cos\gamma_2 &= 0; \end{aligned}\right\} \tag{111}$$

whence we have

$$\left.\begin{aligned} &P_1 = -P_2; \\ \alpha_1 = \alpha_2, \quad &\beta_1 = \beta_2, \quad \gamma_1 = \gamma_2; \end{aligned}\right\} \tag{112}$$

so that the pressures at the fixed points are equal and opposite, and act along parallel straight lines; they therefore form a

couple, the effect of which would be to alter the rotation-axis of the body, were two points on the axis not fixed.

(4) Moreover, if the rotation-axis is a central principal axis and no forces act on the system, in addition to (111) we have, from (104),

$$\left.\begin{aligned} P_1 z_1 \cos\beta_1 + P_2 z_2 \cos\beta_2 = 0, \\ P_1 z_1 \cos a_1 + P_2 z_2 \cos a_2 = 0; \end{aligned}\right\} \tag{113}$$

and therefore $P_1 = -P_2 = 0$; and no pressure exists at the fixed points in the rotation-axis. This result agrees with that of Art. 175, wherein it is proved that the couple of the centrifugal forces vanishes for all points on a central principal axis. Hence it is that such axes are called permanent axes, and are axes of no pressure; they are therefore those axes about which a body will rotate freely, and without fixed points in them, when no forces act.

(5) In the case of a heavy body rotating about a fixed horizontal axis, being the problem which has been considered in Art. 243, if the pressures can be reduced to a single resultant acting at the point where the vertical plane through the centre of gravity intersects the axis, the pressure may be conveniently determined in the following manner. Let S and L be the pressures or stresses respectively perpendicular to and along the line passing through the centre of gravity, where θ is the inclination of that line to the vertical; then, as these pressures are the excesses in these directions of the impressed over the expressed momentum-increments,

$$S = Mg\sin\theta + Mh\,\frac{d^2\theta}{dt^2} = \frac{Mg\,k^2}{h^2+k^2}\sin\theta; \tag{114}$$

$$L = Mg\cos\theta + Mh\,\frac{d\theta^2}{dt^2}, \tag{115}$$

$$= Mg\left\{\cos\theta + \frac{2h^2}{h^2+k^2}(\cos\theta - \cos a)\right\}; \tag{116}$$

$\frac{d^2\theta}{dt}$ and $\frac{d\theta^2}{dt^2}$ having the values which are given in (39) and (40) Art. 243. From these quantities the component of pressure in any direction, and also the line of action of the whole pressure, can be determined. The whole pressure is of course $(S^2+L^2)^{\frac{1}{2}}$.

263.] The following are applications of the preceding results.

Ex. 1. A heavy wire in the form of a semicircle has its two ends attached to a vertical axis about which it revolves with an

angular velocity ω: determine the horizontal tensions at the points of attachment.

Let P_1 and P_2 be the pressures at the upper and lower points respectively; and let a be the radius of the semicircle, and h the distance of the centre of gravity from the axis; then $h = \frac{2a}{\pi}$; and taking moments about the lower point,

$$2aP_1 = Mgh + M\omega^2 ha,$$

also

$$P_1 + P_2 = M\omega^2 h;$$

$$\therefore \quad P_1 = \frac{M}{\pi}(a\omega^2 + g); \qquad P_2 = \frac{M}{\pi}(a\omega^2 - g).$$

Ex. 2. A heavy sphere revolves uniformly about a vertical chord, which is fixed at the two points where it meets the sphere. Determine the pressures at the points.

Let a and M be the radius and mass of the sphere respectively, $2c$ = the length of the chord; so that the distance of the chord from the centre $= (a^2 - c^2)^{\frac{1}{2}}$. Let ω = the angular velocity of the sphere. Let P_1 and P_2 be the horizontal pressures, and R_1 and R_2 the vertical pressures at the upper and lower points of attachment respectively; then, as in the preceding example,

$$P_1 = \frac{M}{2c}(c\omega^2 + g)(a^2 - c^2)^{\frac{1}{2}},$$

$$P_2 = \frac{M}{2c}(c\omega^2 - g)(a^2 - c^2)^{\frac{1}{2}},$$

$$R_1 + R_2 = Mg.$$

Ex. 3. A heavy thin bar of mass M and length $2a$, having one end fixed, about which it moves in a vertical plane, falls from a horizontal position; determine the pressure at the fixed point for any position of the bar.

In this case let the pressure be resolved along and perpendicular to the bar; then applying (114) and (115), we have

$$\frac{d^2\theta}{dt^2} = -\frac{3g\sin\theta}{4a}, \qquad \frac{d\theta^2}{dt^2} = \frac{3g\cos\theta}{2a},$$

$$S = \frac{Mg\sin\theta}{4}, \qquad L = \frac{5Mg\cos\theta}{2};$$

so that when the bar has in its motion a vertical position, the pressure at the fixed point is wholly vertical, and is $\frac{5Mg}{2}$; viz. two and a half times the weight of the bar; and when the bar is horizontal $L = 0$, and $S = \frac{Mg}{4}$.

Hence if ϕ is the angle at which the resultant pressure is inclined to the bar

$$\tan\phi = \frac{\text{S}}{\text{L}} = 10\tan\theta.$$

These results give the answer to the following problem.

Ex. 4. A heavy bar of length $2a$ and mass M is fixed at its two ends in a horizontal position; one support is removed and the bar turns about the other end; find the pressure at this end when the motion begins.

The preceding example shews that the pressure is wholly vertical, and is equal to one-fourth of the weight of the bar; so that while the rod is at rest on its two ends, each support bears one-half of the weight; but if one support is removed and the bar begins to fall, the pressure on the other end is immediately diminished to one-fourth of the weight.

Ex. 5. A heavy cube, whose side is $2a$ and mass is M, just makes a complete revolution about an edge which is horizontal and is fixed to points at the two corners of the cube; find the pressures on these points in the various positions of the cube.

The axis is evidently a principal axis, and its principal plane bisects the side of the cube and contains the centre of gravity of the cube. Hence the two pressures are similar in all respects, and are equivalent to a single resultant, passing through the middle point of the side of the cube;

$$h = a\,2^{\frac{1}{2}}; \qquad k^2 = \frac{8a^2}{3};$$

$$\frac{d\theta^2}{dt^2} = \frac{3g}{2a2^{\frac{1}{2}}}(1+\cos\theta), \qquad \frac{d^2\theta}{dt^2} = -\frac{3g}{4a\,2^{\frac{1}{2}}}\sin\theta;$$

$\therefore$ from (114) and (115),

$$\text{S} = \frac{\text{M}g\sin\theta}{4}, \qquad \text{L} = \frac{\text{M}g(3+5\cos\theta)}{2};$$

and the corresponding pressures at the fixed points are each one-half of these. When the cube is in the highest and lowest positions, $\text{S} = 0$, and $\text{L} = -\text{M}g$, and $4\text{M}g$ respectively, so that when the cube is in in its lowest position, the pressure on each fixed point is wholly vertical and is twice the weight of the cube.

Ex. 6. An elliptic plate is supported with its plane vertical and major axis horizontal by two pins at its two foci; one pin is withdrawn, and the plate begins to rotate about the other pin,

but no change of pressure takes place; find the eccentricity of the plate.

Taking the value of s given in (114), and making $\sin\theta = 1$, we have

$$\frac{\text{M}g}{2} = \text{s} = \text{M}g - \frac{4\text{M}g\,a^2 e^2}{4a^2 e^2 + a^2 + b^2};$$

$$\therefore \quad 5e^2 = 2.$$

Ex. 7. A heavy rectangular plate rests in a horizontal position on four legs at its four corners, two are suddenly removed; prove that the pressure on each of the other two immediately becomes one-eighth of the weight.

SECTION 3.—*On internal stresses in a wire or thin bar, caused by the motion of the bar.*

264.] In the demonstration of D'Alembert's principle given in Section 1, Chapter III, we have pointed out how in the motion of a single material particle the expressed momentum-increment of the particle is exactly equal to that impressed on it, but how when the particle is a member of a system or of a body that equality cannot be asserted; and how in this latter case the difference between the impressed and the expressed momentum-increments causes an action between the particle and the surrounding particles of the character of a pressure or of a tension, which is called a stress. I also pointed out in Article 70 certain varieties of stresses, viz. normal stresses, tangential stresses or shears, and bending stresses, the former two being actions of translation and the last an action of rotation. Now the general investigation into the action of these and similar stresses belongs to another part of the Mechanics, viz. the theory of elastic bodies, and the action of strains and stresses; one portion, however, which is the most simple, arises so immediately out of the questions considered in the present Chapter, that it is desirable to enquire into it, and this is the action of stresses which occur in thin bars or wires in motion, and which are due to their motion. Now a principal and a fundamental property of a stress is that it is referred to an area, and it is measured by reference to an unit-area; thus, suppose a body to be under the action of force and to be strained; at any point within the body there is a consequent stress; but what is the nature of this, and how is it

measured? Through the point let there be drawn a plane, which divides the body into two parts; let it be imagined that one part is taken away, and that the other remains in the same condition as it was before the removal of the former part, generally certain forces must act upon the surface of the former part to keep it in the state that it was in; these forces are the internal stresses, and as they are distributed through the dividing area, they may vary from point to point in it both in intensity and direction.

For the complete consideration of all these we must divide the cutting plane into small elements, and consider only those which act on a small area at the point through which the dividing plane has been drawn; it is evident that the stress will be distributed over the area, and that the amount will vary directly as the area, and that the intensity will be measured by the amount which acts on an unit-area, supposing the stress to be uniform within that area; it is evident also that generally the intensity and direction of the stress will vary according to the direction in which the dividing plane passing through the point is drawn. The general enquiry into all these questions belongs to the theory of elastic bodies. I propose here to consider their application to thin bars only, when the stress is caused by the motion of the bars, and when that stress is the difference between the impressed and the expressed momentum-increments of the particle at the point.

We suppose the bar to be thin, and without determining the exact form of the transverse section we suppose it to be small, and the same at all points of the bar, so that there will be no variation of stress due to the variation of section. Let us take the transverse section at a point P; then the stresses acting on this transverse section will be (1) a stress normal to the section, being a pressure or a pull along a line which is approximately tangential to the curve of the bar at the given point; (2) a stress whose action-line is in the section and tends to make it slide or slip on its adjacent transverse section; this is a shear, or a shearing stress, and a twist, when one section rotates on its adjacent stratum; (3) a stress-couple, or a bending-couple, which tends to make the transverse section turn about an axis in its own plane. Each of these in its own way tends to produce a disruption of the bar; for the bar may give way either by being parted or

torn asunder, or by being sheared, or by being snapped into two parts by over-bending and so breaking. As these stresses arise from the differences between the impressed and the expressed momentum-increments, the latter have to be calculated, and the problem is an application of the principles which have been developed in the preceding pages. The following examples will elucidate the process more completely than any further general remarks. It is however to be observed that the stresses on a section may be estimated by considering the lost momenta on either one side or the other of the section, as it may be most convenient, for at the section the action and reaction are of course equal.

265.] Illustrative examples.

Ex. 1. A heavy horizontal bar of length $2a$ and mass M falls through a vertical distance h, and impinges at its middle point on a fixed obstacle; find the shearing stress and the stress-couple at any point of the bar.

Let ρ be the density and κ the area of the transverse section of the bar, so that $\text{M} = 2a\rho\kappa$. Let x be the distance from the middle point, at which the stresses are to be estimated, and let us take account of the lost momenta between the point and the end of the bar. Let S and G be respectively the shearing stress, and the moment of the stress-couple; and let $\text{V}^2 = 2gh$; then

$$\text{S} = \int_0^{a-x} \rho\kappa \text{V} d\xi = \rho\kappa\text{V}(a-x);$$

$$\text{G} = \int_0^{a-x} \rho\kappa\text{V}\xi d\xi = \frac{\rho\kappa\text{V}}{2}(a-x)^2.$$

Hence, at the middle point, the stress-couple is the greatest, and is $\frac{\text{MV}a}{4}$; so that the bar will break unless its resisting strength against breaking is greater than this quantity.

Ex. 2. If the bar meets with fixed obstacles at its ends, instead of one obstacle in the middle, then employing the same notation as in the preceding example, and observing that the upward pressure of each obstacle is $\frac{\text{MV}}{2}$, we have

$$\text{S} = \int_0^{a-x} \rho\kappa\text{V} d\xi - \frac{\text{MV}}{2} = -\frac{\text{MV}}{2a}x,$$

that is, varies as the distance from the middle point, and at the middle point vanishes.

$$\text{G} = \int_0^{a-x} \rho\kappa\text{V}\xi d\xi - \frac{\text{MV}}{2}(a-x)$$

$$= -\frac{\text{MV}}{4a}(a^2-x^2).$$

Hence the moment of the stress-couple varies as the product of the segments into which the bar is divided at the point, and is a maximum at the middle point, where it is $-\frac{\text{MV}a}{4}$.

Ex. 3. A thin wire in the form of a semicircle revolves with an angular velocity ω about an axis which passes through its middle point and is perpendicular to its plane. Find the moment of the bending stress (the tendency to break) at any point of the wire.

Let a be the radius of the circle, so that $\text{M} = \pi\rho a\kappa$; let the point for which the bending stress is to be estimated be at the distance $a\phi$ from the middle point along the arc, and let the moment be estimated from the end of the wire up to this point; then

$$\text{G} = 8a^3\omega^2\rho\kappa \sin\frac{\phi}{2}\int_{\frac{\pi}{4}}^{\frac{\pi-\phi}{2}} \cos\theta\cos\left(\theta+\frac{\phi}{2}\right)d\theta$$

$$= \frac{\text{M}a^2\omega^2}{\pi}\left\{1-\cos\phi-\sin\phi+\frac{\pi-2\phi}{2}\sin\phi\right\}.$$

Hence $\text{G}=0$, when $\phi=0$, and when $\phi=\frac{\pi}{2}$; and is a maximum when $\phi=\frac{\pi-2}{2}$, that is when the arc $a\phi$ is half the excess of the semicircle over the diameter; at this point

$$\text{G} = \frac{2\text{M}a^2\omega^2}{\pi}\left(\sin\frac{\phi}{2}\right)^2.$$

Ex. 4. A heavy rod of mass M and length $2a$ hangs vertically by its upper end from an elastic string attached to a fixed point above it, whose natural length is c; the rod is pulled downwards and then released; y is the distance of the centre of gravity of the rod below the fixed point at the time t; determine the longitudinal stress at any point of the rod at that time.

If T is the tension of the string at the time t, $\text{T} = \text{E}\frac{y-a-c}{c}$; and

$$\text{M}\frac{d^2y}{dt^2} = \text{M}g - \text{T}.$$

Therefore on a section at the distance x from the top of the rod

$$\text{The longitudinal stress} = \int_x^{2a} \left(g - \frac{d^2y}{dt^2}\right) \rho\kappa d\xi$$

$$= \int_x^{2a} \frac{\text{T}}{2a} d\xi$$

$$= \text{E}\frac{y-a-c}{c}\frac{2a-x}{2a}.$$

Hence the longitudinal stress on any section varies as the distance of the section from the bottom of the rod; and at the top of the rod is equal to the tension of the string.

Ex. 5. A heavy rod of mass M and length $2a$ rotates in a vertical plane about its upper end. Find the longitudinal and shearing stresses, and the stress-couple at any point of the rod.

Let L and S be the longitudinal and shearing stresses, and let G be the moment of the stress-couple, on a section at a distance x from the upper end of the rod; let θ be the angle between the rod and the vertical line; then the equations of angular acceleration and of vis viva are respectively

$$\frac{d^2\theta}{dt^2} = -\frac{3g\sin\theta}{4a}; \qquad \frac{d\theta^2}{dt^2} = \frac{3g}{2a}(\cos\theta - \cos\alpha).$$

Then taking account of that part of the rod which lies beyond the point, we have

$$\text{L} = \int_0^{2a-x} \rho\kappa \left\{g\cos\theta + (x+\xi)\frac{d\theta^2}{dt^2}\right\} d\xi$$

$$= \frac{\text{M}}{2a}\left\{g\cos\theta(2a-x) + \frac{4a^2-x^2}{2}\frac{3g}{2a}(\cos\theta - \cos\alpha)\right\}$$

$$= \frac{\text{M}g}{8a^2}(2a-x)\{4a\cos\theta + 3(2a+x)(\cos\theta - \cos\alpha)\}.$$

$$\text{S} = \int_0^{2a-x} \rho\kappa \left\{g\sin\theta + (x+\xi)\frac{d^2\theta}{dt^2}\right\} d\xi$$

$$= \frac{\text{M}}{2a}\left\{g\sin\theta(2a-x) - \frac{4a^2-x^2}{2}\frac{3g}{4a}\sin\theta\right\}$$

$$= \frac{\text{M}g\sin\theta}{16a^2}(2a-x)(2a-3x).$$

$$\text{G} = \int_0^{2a-x} \rho\kappa \left\{g\sin\theta - (x+\xi)\frac{3g}{4a}\sin\theta\right\} \xi d\xi$$

$$= \frac{\text{M}}{2a}\left\{\frac{g\sin\theta}{2}(2a-x)^2 - \frac{3g\sin\theta}{4a}\left(\frac{x}{2}(2a-x)^2 + \frac{(2a-x)^3}{3}\right)\right\}$$

$$= -\frac{\text{M}g\sin\theta}{16a^2}x(2a-x)^2.$$

From these values it appears that the shear and the stress-couple vary as the angle between the rod and the vertical varies, so that both vanish when the rod in its motion becomes vertical; that the shear vanishes when $x = \frac{2a}{3}$, and is a maximum when $x = \frac{4a}{3}$; that the moment of the bending couple is a maximum when $x = \frac{2a}{3}$, that is, at the point where the shear vanishes. Also that the shear and the bending couple are independent of the initial position of the rod.

If $x = 0$, that is, at the point of suspension

$$\text{L} = \frac{\text{M}g}{2}(5 \cos\theta - 3 \cos\alpha), \quad \text{S} = \frac{\text{M}g \sin\theta}{4}, \quad \text{G} = 0;$$

the first two of which give the pressures on the point of suspension, along and perpendicular to the bar. See Ex. 3, Art. 263.

SECTION 4.—*On changes in motion due to sudden changes in constraint.*

266.] In the previous sections of this chapter we have considered the motion of bodies about fixed axes, when the initial circumstances are given in immediate reference to these axes. The problem which we have now to consider is that wherein when a body has been moving about a fixed axis, the axis becomes suddenly set free, and another line becomes fixed immediately about which the body rotates; and the question is the determination of the consequent circumstances of motion.

The principle of solution is the same in all cases, however greatly the special circumstances may vary according as the new axis is parallel to the former, the new axis intersects the former, or the two axes do not intersect at all; and the principle is that which is contained in the equation (6), Art. 229. The impressed momenta are due to the effective momenta of the several particles at the instant when one axis is set free, and the other becomes fixed, and are the impulses which would reduce each of these particles to rest. Thus if m is a type-particle, v the component of its velocity in a plane perpendicular to the new axis, and p the perpendicular from the axis on the line of this component, the moment of the impressed momenta is $\Sigma.mvp$; and if ω is the

resulting angular velocity and $\text{M} k^2$ the moment of inertia of the body about the new axis, then

$$\text{M} k^2 \omega = \Sigma . m v p. \tag{117}$$

In the case of parallel axes, the theorems given in Articles 95 and 97 relative to the moments of the momenta of the particles of a system and to their transformation are directly applicable. The following are the forms which these theorems assume:

(1) Let the axis about which the angular velocity is Ω pass through the mass-centre, and let the other axis about which the angular velocity is ω be at the distance h from it, then

$$\Omega k'^2 = \omega (h^2 + k'^2), \tag{118}$$

where k' is the radius of gyration about the axis passing through the mass-centre.

(2) If ω is the angular velocity about any axis, and Ω is the angular velocity about a parallel axis through the mass-centre in reference to which k' is the radius of gyration, and h is the distance between these axes, then, if v is the consequent velocity of the mass-centre in a plane at right-angles to these axes,

$$\omega (h^2 + k'^2) = v h + \Omega k'^2. \tag{119}$$

I would however observe that as the circumstances vary in most cases, it is generally best to have recourse to first principles; the application of these is shewn in the following examples.

267.] Illustrative examples.

Ex. 1. A thin bar of length a is revolving in a plane about one end with an angular velocity Ω, when suddenly the end is set free and the other end is fixed. Find the new angular velocity, ω.

By (117)
$$\text{M}\frac{a^2}{3}\omega = \int_0^a \rho\kappa (a-x)\Omega x\, dx = \text{M}\frac{a^2}{6}\Omega;$$
$$\therefore \quad 2\omega = \Omega.$$

Ex. 2. A rectangular plate whose sides are $2a$ and $2b$ is revolving with angular velocity Ω about an axis through its centre and parallel to the side $2b$, when one of the parallel sides to the axis becomes suddenly fixed, and the former axis is set free; find ω the angular velocity about the new axis.

The moment of inertia about the axis through the mass-centre is $\text{M}\frac{a^2}{3}$: hence and from (117),

$$\text{M}\frac{a^2}{3}\Omega = \text{M}\left(\frac{a^2}{3} + a^2\right)\omega;$$
$$\therefore \quad \Omega = 4\omega.$$

Ex. 3. If the plate is a square of side a, and revolves with an angular velocity Ω about a diagonal, and strikes on an obstacle with one of its angular points, so that the plate revolves with an angular velocity ω, about an axis passing through the angular point, then

$$M\frac{a^2}{12}\Omega = M\left(\frac{a^2}{12}+\frac{a^2}{2}\right)\omega;$$

$$\therefore \quad \Omega = 7\omega.$$

Ex. 4. An elliptical plate is revolving with an angular velocity Ω about a latus rectum, when the other latus rectum suddenly becomes fixed, the former being set free; determine the angular velocity ω about the new axis.

By (117) we have

$$M\left(\frac{a^2}{2}+a^2e^2\right)\omega = \iint(ae+x)(ae-x)\rho\tau\,dy\,dx\,\Omega,$$

where x is the distance of the element from the minor-axis, and the limits of integration comprise the whole area.

$$\therefore \quad (4e^2+1)\omega = (4e^2-1)\Omega;$$

which determines the ratio of the new angular velocity to the former; and shews that the new angular velocity has the same or opposite sign to the former angular velocity according as $2e$ is greater or less than unity. If $2e=1$, $\omega=0$, the new axis passing through the focus which is the centre of percussion, see Ex. 2, Art. 239, and the plate having been brought to rest.

Ex. 5. A circular disc is revolving with an angular velocity Ω about an axis through its centre and at right angles to its plane, when a point in its circumference suddenly becomes fixed, determine the angular velocity ω about this point.

Applying (117) we have

$$M\left(\frac{a^2}{2}+a^2\right)\omega = \int_0^a\int_0^{2\pi}\rho\tau r\,dr\,d\theta\, r\Omega(r+a\cos\theta) = M\Omega\frac{a^2}{2};$$

$$\therefore \quad 3\omega = \Omega.$$

Ex. 6. A rectangular plate whose sides are a and b is revolving about the side a with an angular velocity Ω when the conterminous side b becomes fixed, and the side a is set free; determine the subsequent angular velocity ω about b.

$$M\frac{a^2}{3}\omega = \int_0^a\int_0^b\rho\tau\Omega xy\,dy\,dx = M\Omega\frac{ab}{4};$$

$$\therefore \quad 4a\omega = 3b\Omega.$$

Ex. 7. An isosceles triangular plate is revolving with an angular

velocity Ω about the bisector of the vertical angle, when one of the equal sides becomes suddenly fixed; find the angular velocity ω with which the plate revolves about this side.

If a is the semi-vertical angle

$$4\omega\cos a = \Omega.$$

Ex. 8. The octant of an ellipsoid is revolving with an angular velocity Ω about the axis a, when the axis b suddenly becomes fixed, the axis a being set free. With what angular velocity ω does the body revolve about the axis b?

Let M be the mass of the octant; then the moment of inertia about the b-axis is $M\frac{c^2+a^2}{5}$; and let h_2, see Art. 94, be the sum of the moments of the momenta of all the particles about the same axis; then

$$h_2 = \Sigma . m\left(z\frac{dx}{dt} - x\frac{dz}{dt}\right);$$

but from Art. 53,
$$\frac{dx}{dt} = z\Omega_y - y\Omega_z = 0,$$

$$\frac{dz}{dt} = y\Omega_x - x\Omega_y = y\Omega;$$

$$\therefore \quad h_2 = -\Sigma . mxy\Omega;$$

and
$$M\frac{c^2+a^2}{5}\omega = h_2 = -\Omega\Sigma . mxy = -\frac{2Mab}{5\pi}\Omega;$$

$$\therefore \quad \omega = -\frac{2ab\Omega}{\pi(c^2+a^2)};$$

the negative sign shewing that if Ω is positive according to the convention of signs given in Art. 44, ω is negative according to the same convention.

Ex. 9. A heavy bar AB of length a falls about its lower end B from a vertical to a horizontal position, when the end A is suddenly fixed and B is set free, so that the bar falls into a vertical position AB as at first; then A is set free, and B is fixed, so that the bar again falls about B into a horizontal position, when the end A is suddenly fixed, and B is set free, and so on; find the angular velocity ω of the bar about the upper end, when it takes a vertical position for the nth time.

$$\omega^2 = \frac{4g}{a}\left\{1 + \left(\frac{1}{4}\right)^{4n-2}\right\}.$$

CHAPTER VI.

THE ROTATION OF A RIGID BODY, OR OF AN INVARIABLE SYSTEM, ABOUT A FIXED POINT.

SECTION 1.—*The rotation of a rigid body about a fixed point under the action of instantaneous forces.*

268.] WHEN a rigid body, or any system of particles of invariable form, moves with one point of it fixed, it is evident that it admits only of rotation about an axis passing through that fixed point; generally, the position of this axis will continuously vary, and will describe one cone fixed in the moving body, and another cone fixed in space, which two cones touch each other, and the line of contact of which is the instantaneous axis: it is also evident that any given particle of the system will move on the surface of a sphere whose centre is the fixed point.

We shall suppose the form, matter, and density of every part of the moving system to be given; and therefore the position of the principal axes, and the principal moments of inertia relative to the fixed point, will also be assumed to be known: these latter we shall take to be A, B, C, as in Chap. IV; and we shall assume the order of magnitude to be the same as that of Art. 184; viz.,

$$A < B < C: \tag{1}$$

we shall also assume the position of the principal axes, as well as the values of the principal moments of inertia, to be given at every point of the system.

Let the fixed point be the origin; and at it let two systems of coordinate axes originate; one of which we assume to be fixed absolutely in space, and the other to be fixed in the body and to move with it: this latter system we will take to be the system of principal axes which originates at the point, because our expressions will be much simplified thereby. The motion of the body will be in the first place referred to this latter system in terms of the angular velocities about the principal axes; and the incidents of its motion in space will be thence inferred by means of the connecting equations (120)... (125) of Art. 64, or some equivalents of them.

The general investigation will consist of two parts, according as the system is under the action of instantaneous forces or of finite accelerating forces. We shall consider the effects of instantaneous forces in the present section, and in the succeeding section those of finite accelerating forces; and in each case I shall inquire into the resulting angular velocity, the position of the rotation-axis, the pressure on the fixed point, and the other incidents of motion.

269.] For the sake of simplicity, we will suppose the body to be at rest at the time when the instantaneous forces act on it. Let us first refer all the elements to the system of axes fixed in space; and let the axial components of the impressed momenta be $\Sigma.\text{X}$, $\Sigma.\text{Y}$, $\Sigma.\text{Z}$, and let the expressed momenta be $\Sigma.m\text{V}_x$, $\Sigma.m\text{V}_y$, $\Sigma.m\text{V}_z$; and if the force is a single blow which impresses a momentum Q, let Q_x, Q_y, Q_z be its components. Let (x, y, z) be the initial place of m; let P be the pressure at the origin due to the forces, and let λ, μ, ν be the direction-angles of its line of action; then the equations of motion are

$$\left.\begin{aligned}\Sigma.(\text{X}-m\text{V}_x)-\text{P}\cos\lambda &= 0,\\ \Sigma.(\text{Y}-m\text{V}_y)-\text{P}\cos\mu &= 0,\\ \Sigma.(\text{Z}-m\text{V}_z)-\text{P}\cos\nu &= 0;\end{aligned}\right\}\quad (2)$$

$$\left.\begin{aligned}\Sigma\{y(\text{Z}-m\text{V}_z)-z(\text{Y}-m\text{V}_y)\} &= 0,\\ \Sigma\{z(\text{X}-m\text{V}_x)-x(\text{Z}-m\text{V}_z)\} &= 0,\\ \Sigma\{x(\text{Y}-m\text{V}_y)-y(\text{X}-m\text{V}_x)\} &= 0.\end{aligned}\right\}\quad (3)$$

Let these equations be transformed into their equivalents in terms of angular velocities, as in Art. 147. Let Ω be the angular velocity which results from the instantaneous forces, and let Ω_x, Ω_y, Ω_z be its axial components; then (2) become

$$\left.\begin{aligned}\Sigma.\text{X}-\Omega_y\Sigma.mz+\Omega_z\Sigma.my-\text{P}\cos\lambda &= 0,\\ \Sigma.\text{Y}-\Omega_z\Sigma.mx+\Omega_x\Sigma.mz-\text{P}\cos\mu &= 0,\\ \Sigma.\text{Z}-\Omega_x\Sigma.my+\Omega_y\Sigma.mx-\text{P}\cos\nu &= 0;\end{aligned}\right\}\quad (4)$$

from which, or from (2), the pressure at the fixed point, and the direction-cosines of its line of action, may be determined.

270.] Let us now consider the motion in reference to the principal axes of the body at the fixed point, and replace (3) by their equivalents in terms of angular velocities about these axes; let G be the moment of the couple of the impressed momenta, or of the blow, if this motion is due to a blow; and let L_1, M_1, N_1 be the axial components of the moment of this couple relative

to the three principal axes; and let Ω_1, Ω_2, Ω_3 be the axial components of the instantaneous angular velocity. Then, by the reduction which has already been made in Art. 174, the equations (3) become

$$\mathrm{A}\,\Omega_1 = \mathrm{L}_1, \qquad \mathrm{B}\,\Omega_2 = \mathrm{M}_1, \qquad \mathrm{C}\,\Omega_3 = \mathrm{N}_1: \tag{5}$$

these equations determine the axial components of the angular velocity relative to the principal axes fixed in the body and moving with it. From these the angular velocities relative to the three axes fixed in space may be found, by means of the equations given in (87), Art. 58; and thus the position of the initial rotation-axis is absolutely determined.

The preceding equations admit of dissection, and of deduction from first principles, in a manner similar to that which has been employed in Arts. 158 and 159. It is consequently unnecessary to repeat it.

271.] From (5) we have

$$\Omega_1 = \frac{\mathrm{L}_1}{\mathrm{A}}, \qquad \Omega_2 = \frac{\mathrm{M}_1}{\mathrm{B}}, \qquad \Omega_3 = \frac{\mathrm{N}_1}{\mathrm{C}}; \tag{6}$$

$$\therefore \quad \Omega^2 = \Omega_1^{\,2} + \Omega_2^{\,2} + \Omega_3^{\,2}$$

$$= \frac{\mathrm{L}_1^{\,2}}{\mathrm{A}^2} + \frac{\mathrm{M}_1^{\,2}}{\mathrm{B}^2} + \frac{\mathrm{N}_1^{\,2}}{\mathrm{C}^2}; \tag{7}$$

and if α, β, γ are the direction-angles of the instantaneous rotation-axis,

$$\cos\alpha = \frac{\Omega_1}{\Omega}, \qquad \cos\beta = \frac{\Omega_2}{\Omega}, \qquad \cos\gamma = \frac{\Omega_3}{\Omega}, \tag{8}$$

$$= \frac{\mathrm{L}_1}{\mathrm{A}\,\Omega}; \qquad = \frac{\mathrm{M}_1}{\mathrm{B}\,\Omega}; \qquad = \frac{\mathrm{N}_1}{\mathrm{C}\,\Omega}; \tag{9}$$

whence Ω and the direction-angles of its rotation-axis are known.

Hence the equations to the initial rotation-axis are

$$\frac{\mathrm{A}x}{\mathrm{L}_1} = \frac{\mathrm{B}y}{\mathrm{M}_1} = \frac{\mathrm{C}z}{\mathrm{N}_1}. \tag{10}$$

And as L_1, M_1, N_1 are the axial components of the moment of the impressed couple, or, as Poinsot calls it, the couple of impulsion, the equations to its axis are

$$\frac{x}{\mathrm{L}_1} = \frac{y}{\mathrm{M}_1} = \frac{z}{\mathrm{N}_1}; \tag{11}$$

and the equation to its plane is

$$\mathrm{L}_1 x + \mathrm{M}_1 y + \mathrm{N}_1 z = 0. \tag{12}$$

272.] Now these expressions admit of a geometrical interpretation by means of the momental ellipsoid at the fixed point similar to that given in Art. 153.

In reference to the principal axes fixed in the body, the equation to the momental ellipsoid is

$$\mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 = \mu; \tag{13}$$

relatively to which the equations to the axis conjugate to the plane (12) are, see Art. 22,

$$\frac{\mathrm{A}x}{\mathrm{L}_1} = \frac{\mathrm{B}y}{\mathrm{M}_1} = \frac{\mathrm{C}z}{\mathrm{N}_1}; \tag{14}$$

but these are the equations to the instantaneous initial axis. Hence we have the following theorem:

The instantaneous axis of rotation, due to a given impressed couple, is the axis of the momental ellipsoid at the fixed point which is conjugate to the plane of the couple. Hence, if the momental ellipsoid of the moving system is constructed, and a plane is drawn touching it, and parallel to the plane of the couple of impulsion, the central radius vector of the ellipsoid drawn to the point of contact is the instantaneous axis. See also Art. 153.

The point where the instantaneous axis meets the ellipsoid is called the instantaneous pole.

The initial angular velocity varies as the square of the central radius vector of the momental ellipsoid, which coincides with the initial rotation-axis.

For if θ is the angle between the initial rotation-axis and the axis of the couple of impulsion, by (16), Art. 147,

$$\Omega = \frac{\mathrm{G}\cos\theta}{\Sigma . m r^2} = \frac{\mathrm{R}^2\,\mathrm{G}\cos\theta}{\mu}, \tag{15}$$

if R is the central radius of the ellipsoid which coincides with the initial rotation-axis; that is, the initial angular velocity varies as the square of the central radius vector of the ellipsoid which coincides with the initial rotation-axis, and as the component relative to that axis of the moment of the couple of impulsion.

Hence, if a body rotates about an axis passing through a fixed point, the plane of the momental ellipsoid conjugate to that axis is the plane of the couple which instantaneously impressed in an opposite direction will bring the body to rest; and

if the axial components of the angular velocity of the body at that instant are ω_1, ω_2, ω_3, and G' is the moment of the couple which reduces the body to rest,

$$G'^2 = A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2; \tag{16}$$

and the equation to the plane in which the couple must be impressed is

$$A\omega_1 x + B\omega_2 y + C\omega_3 z = 0. \tag{17}$$

If the plane of the couple of impulsion is a principal plane of the momental ellipsoid, the instantaneous rotation-axis lies along the axis of the couple; but for no other plane of impulsion will the axis of the couple lie along the instantaneous rotation-axis. Hence, if a body is rotating about a principal axis, it may be brought to rest by a couple whose axis is that rotation-axis; but in no other case will the axis of the couple which brings the body to rest coincide with the rotation-axis.

Hence also, if a body is rotating about a given axis, and a blow is given to the body whence a couple of impulsion arises, if the plane of this couple is conjugate to the original axis of rotation, no change of rotation-axis is caused; but if the plane of the couple of the blow is not conjugate to the previous rotation-axis, a change of axis takes place. And therefore if a body is rotating about a principal axis, and a blow is given to the body which produces a couple whose axis is that principal axis, the position of the rotation-axis will be unaltered, and there will only be a change of angular velocity.

I may observe, that if the axes fixed in the body are not principal axes, equations (32), Art. 151, will take the place of (5), and the equation of the momental ellipsoid would be (114), Art. 181; it is therefore unnecessary to repeat them here. From these equations however the same geometrical interpretation as that which we have just arrived at may be deduced.

It is to be observed, that $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are independent of G, the moment of the momentum of the couple of impulsion; so that if the body is put into motion by a blow, the position of the instantaneous rotation-axis is the same, whatever is the intensity of the blow, provided that its line of action is the same. Not so however the initial angular velocity.

273.] The following examples are in illustration of the preceding.

Ex. 1. A right-angled triangular plate, fixed at its mass-

centre, is struck at its right angle by a blow perpendicularly to its plane: it is required to find the position of its initial instantaneous axis.

Let $3a$, $3b$, see Fig. 32, be respectively the sides CA, CB of the triangle, $\rho =$ the density, $\tau =$ the thickness of the plate. Let the origin be taken at the mass-centre; and let the coordinate axes be parallel to the sides. As these axes are not principal, we must recur to (32), Art. 151, for Ω_x, Ω_y, Ω_z. In reference to the origin and axes which we have chosen, the equations to the sides BC, CA, AB respectively are $x = -a$, $y = -b$, $\frac{x}{a} + \frac{y}{b} = 1$; and consequently

$$\text{A} = \Sigma . m (y^2 + z^2) = \frac{9\rho\tau a b^3}{4}, \qquad \text{D} = \Sigma . myz = 0,$$

$$\text{B} = \Sigma . m (z^2 + x^2) = \frac{9\rho\tau a^3 b}{4}, \qquad \text{E} = \Sigma . mzx = 0,$$

$$\text{C} = \Sigma . m (x^2 + y^2) = \frac{9\rho\tau ab(a^2 + b^2)}{4}, \quad \text{F} = \Sigma . mxy = -\frac{9\rho\tau a^2 b^2}{8}.$$

Let Q be the momentum impressed by the blow at C, which is $(-a, -b)$, in a line perpendicular to the plane of the plate, and parallel to the axis of z in a positive direction; so that we have

$$\text{L}_1 = -b\text{Q}, \qquad \text{M}_1 = a\text{Q}, \qquad \text{N}_1 = 0;$$

$$\therefore \quad \Omega_x = -\frac{8\text{Q}}{27\rho\tau ab^2}, \qquad \Omega_y = \frac{8\text{Q}}{27\rho\tau a^2 b}, \qquad \Omega_z = 0;$$

and therefore the equation to the instantaneous initial rotation-axis is

$$\frac{x}{a} + \frac{y}{b} = 0;$$

consequently the initial rotation-axis, which of course passes through the fixed point, is parallel to the hypothenuse of the triangle.

Ex. 2. Let us consider the general case of a plate of infinitesimal thickness, which has one point fixed, and which is struck at a given point (x_0, y_0) by a blow Q, in a line perpendicular to the plane of the plate.

The plane of the couple of impulsion is evidently that passing through the point of the blow and the fixed point, and perpendicular to the plane of the plate. Thus its equation is

$$\frac{x}{x_0} - \frac{y}{y_0} = 0.$$

Now the axis of the momental ellipsoid (13), which is conjugate to this plane, is
$$\text{A}x_0 x + \text{B}y_0 y = 0; \tag{18}$$
and this therefore is the initial rotation-axis.

Also, if $\text{Q} =$ the momentum impressed by the blow at the point (x_0, y_0) in a line perpendicular to the plane of the plate, and in a direction parallel to the positive direction of the axis of z,
$$\text{L}_1 = y_0 \text{Q}, \qquad \text{M}_1 = -x_0 \text{Q}, \qquad \text{N}_1 = 0;$$
$$\therefore \quad \Omega_1 = \frac{y_0 \text{Q}}{\text{A}}, \qquad \Omega_2 = \frac{-x_0 \text{Q}}{\text{B}}, \qquad \Omega_3 = 0;$$
$$\therefore \quad \Omega = \left(\frac{y_0^2}{\text{A}^2} + \frac{x_0^2}{\text{B}^2}\right)^{\frac{1}{2}} \text{Q}.$$
Thus, if the plate is elliptical and fixed at its centre,
$$\text{A} = \frac{\pi \rho \tau a b^3}{4}, \qquad \text{B} = \frac{\pi \rho \tau a^3 b}{4};$$
and consequently the equation of the initial rotation-axis is
$$\frac{x x_0}{a^2} + \frac{y y_0}{b^2} = 0;$$
that is, the initial axis is conjugate to the axis passing through the place of impact of the blow.

If the plate is parabolic, and fixed at its vertex, and if a and b are severally the length and the extreme ordinate of the plate,
$$\text{A} = \frac{4 \rho \tau a b^3}{15}, \qquad \text{B} = \frac{4 \rho \tau a^3 b}{7};$$
and the equation to the initial instantaneous axis is
$$\frac{7 x x_0}{3 a^2} + \frac{5 y y_0}{b^2} = 0.$$

Ex. 3. A cube fixed at its mass-centre is struck by a blow, whose momentum is Q, along an edge: it is required to determine the initial instantaneous axis.

Let $2a =$ the edge of the cube; let the origin be taken at the mass-centre, and let the coordinate axes be parallel to the edges; let the line of the blow Q be parallel to the axis of z; and let its point of application be $(a, a, 0)$; so that
$$\text{L}_1 = a \text{Q}, \qquad \text{M}_1 = -a \text{Q}, \qquad \text{N}_1 = 0;$$
$$\text{A} = \text{B} = \text{C} = \frac{16 \rho \tau a^5}{3};$$
$$\therefore \quad \Omega_1 = \frac{3 \text{Q}}{16 \rho \tau a^4}, \qquad \Omega_2 = -\frac{3 \text{Q}}{16 \rho \tau a^4}, \qquad \Omega_3 = 0;$$

and therefore the equation to the initial rotation-axis is

$$x+y=0:$$

this result is evident from the theorem, that the instantaneous initial axis is conjugate to the plane of the impulsive couple; for the momental ellipsoid is in this case a sphere, and therefore the instantaneous rotation-axis coincides with the axis of the couple of impulsion.

274.] Let us now consider the equations (2), or their equivalents (4), by means of which the pressure at the fixed point, which is due to the impulsive forces, is to be determined.

Let M be the mass of the body; and let its mass-centre be $(\bar{x}, \bar{y}, \bar{z})$; then

$$\Sigma . mx = \text{M}\bar{x}, \qquad \Sigma . my = \text{M}\bar{y}, \qquad \Sigma . mz = \text{M}\bar{z}; \tag{19}$$

and (4) become

$$\left.\begin{aligned} \text{P}\cos\lambda &= \Sigma.\text{X}-\text{M}(\Omega_y\bar{z}-\Omega_z\bar{y}), \\ \text{P}\cos\mu &= \Sigma.\text{Y}-\text{M}(\Omega_z\bar{x}-\Omega_x\bar{z}), \\ \text{P}\cos\nu &= \Sigma.\text{Z}-\text{M}(\Omega_x\bar{y}-\Omega_y\bar{x}). \end{aligned}\right\} \tag{20}$$

Now the last terms of these three equations are evidently the axial components of the momentum of the whole moving mass condensed into its mass-centre: so that the pressure which acts at the fixed point is the excess of the impressed momentum over the momentum of the whole mass condensed at its mass-centre, which is due to the initial angular velocity.

To apply these formulæ, let us take Ex. 1 of the preceding Article: in that case

$$\bar{x}=\bar{y}=\bar{z}=0; \quad \Sigma.\text{X}=\Sigma.\text{Y}=0, \quad \Sigma.\text{Z}=\text{Q};$$

$$\therefore \quad \text{P}\cos\lambda=\text{P}\cos\mu=0; \qquad \text{P}\cos\nu=\text{Q};$$

$$\therefore \quad \text{P}=\text{Q};$$

and the line of pressure is perpendicular to the plate.

If there is no pressure at the fixed point, then

$$\left.\begin{aligned} \Sigma.\text{X}-\text{M}(\Omega_y\bar{z}-\Omega_z\bar{y}) &= 0, \\ \Sigma.\text{Y}-\text{M}(\Omega_z\bar{x}-\Omega_x\bar{z}) &= 0, \\ \Sigma.\text{Z}-\text{M}(\Omega_x\bar{y}-\Omega_y\bar{x}) &= 0; \end{aligned}\right\} \tag{21}$$

whence we have

$$\bar{x}\,\Sigma.\text{X}+\bar{y}\,\Sigma.\text{Y}+\bar{z}\,\Sigma.\text{Z}=0; \tag{22}$$

$$\Omega_x\Sigma.\text{X}+\Omega_y\Sigma.\text{Y}+\Omega_z\Sigma.\text{Z}=0; \tag{23}$$

and it appears that the line of action of the resultant of the impressed momenta, or of the blow, if the motion is due to a single blow, is perpendicular to the plane containing the fixed

point, the mass-centre, and the rotation-axis of the initial angular velocity.

If the fixed point is the mass-centre of the body, then $\bar{x} = \bar{y} = \bar{z} = 0$; and

$$\text{P} \cos \lambda = \Sigma.\text{X}, \quad \text{P} \cos \mu = \Sigma.\text{Y}, \quad \text{P} \cos \nu = \Sigma.\text{Z};$$

and the pressure is in intensity, direction, and line of action, equal to the resultant of the impressed momenta.

SECTION 2.—*The rotation of a rigid body about a fixed point under the action of finite accelerating forces.*

275.] Let us, as in the preceding section, refer the motion of the body, or material system, to two sets of coordinate axes, originating at the fixed point: one of which is fixed in space, and the other is fixed in the body, and moves with it: let this latter system be the principal system relative to the fixed point.

Let P be the pressure at the fixed point at the time t, and let λ, μ, ν be the direction-angles of its line of action relatively to the axes fixed in space: relatively to the same axes let (x, y, z) be the place of m, and let X, Y, Z be the axial components of the impressed velocity-increment; then the equations of motion are

$$\left.\begin{aligned} \Sigma.m\left(\text{X} - \frac{d^2x}{dt^2}\right) - \text{P}\cos\lambda = 0, \\ \Sigma.m\left(\text{Y} - \frac{d^2y}{dt^2}\right) - \text{P}\cos\mu = 0, \\ \Sigma.m\left(\text{Z} - \frac{d^2z}{dt^2}\right) - \text{P}\cos\nu = 0; \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} \Sigma.m\left\{y\left(\text{Z} - \frac{d^2z}{dt^2}\right) - z\left(\text{Y} - \frac{d^2y}{dt^2}\right)\right\} = 0, \\ \Sigma.m\left\{z\left(\text{X} - \frac{d^2x}{dt^2}\right) - x\left(\text{Z} - \frac{d^2z}{dt^2}\right)\right\} = 0, \\ \Sigma.m\left\{x\left(\text{Y} - \frac{d^2y}{dt^2}\right) - y\left(\text{X} - \frac{d^2x}{dt^2}\right)\right\} = 0. \end{aligned}\right\} \quad (25)$$

Let equations (24) be transformed into their equivalents in terms of angular velocities, as in Art. 154; let ω be the angular velocity about the instantaneous axis at the time t, of which let $\omega_x, \omega_y, \omega_z$ be the axial components; then (24) become

$$\left.\begin{aligned}
\ldots.m\text{X}-\frac{d\omega_y}{dt}\Sigma.mz+\frac{d\omega_z}{dt}\Sigma.my-\omega_x\Sigma.m\{\omega_x x+\omega_y y+\omega_z z\}+\omega^2\Sigma.mx&=\text{P}\cos\lambda,\\
\ldots.m\text{Y}-\frac{d\omega_z}{dt}\Sigma.mx+\frac{d\omega_x}{dt}\Sigma.mz-\omega_y\Sigma.m\{\omega_x x+\omega_y y+\omega_z z\}+\omega^2\Sigma.my&=\text{P}\cos\mu,\\
\ldots.m\text{Z}-\frac{d\omega_x}{dt}\Sigma.my+\frac{d\omega_y}{dt}\Sigma.mx-\omega_z\Sigma.m\{\omega_x x+\omega_y y+\omega_z z\}+\omega^2\Sigma.mz&=\text{P}\cos\nu;
\end{aligned}\right\}\quad(26)$$

from which, or from (24), the pressure at the fixed point and the direction-cosines of its line of action are to be determined.

Let us replace (25) by what they become in terms of angular velocities about the principal axes fixed in the body and moving with it: this reduction has been made in Art. 174; and for the equivalents of (25) we have

$$\left.\begin{aligned}
\text{A}\frac{d\omega_1}{dt}+(\text{C}-\text{B})\omega_2\omega_3&=\text{L},\\
\text{B}\frac{d\omega_2}{dt}+(\text{A}-\text{C})\omega_3\omega_1&=\text{M},\\
\text{C}\frac{d\omega_3}{dt}+(\text{B}-\text{A})\omega_1\omega_2&=\text{N};
\end{aligned}\right\}\qquad(27)$$

where L, M, N are the axial components of the moment of the couple of the impressed momentum-increments relative to the principal axes.

As these equations have been already analysed and deduced from first principles in Art. 157—161, it is unnecessary to give further explanation of them. I may observe that the instantaneous angular velocity, and its axial components relative to the moving principal axes, may be (theoretically at least) derived from (27); and thence the axial components relative to the axes fixed in space by means of the equations given in Art. 58; or, as we shall find more convenient, we may determine θ, ϕ, and ψ by means of Art. 64, and thus determine the position of the body in space as well as the incidents of its motion at the time t.

As the pressure at the fixed point, which is to be determined by means of (26), depends on the angular velocity and its axial components at the instant, these must be first determined; and consequently I proceed to consider (27) and (25).

The general solution is beyond the present powers of Mathematical Analysis; and we can investigate only those more simple cases which can be solved either partially or wholly.

276.] Let us first take the simple case in which $\text{L}=\text{M}=\text{N}=0$;

that is, when the conditions explained in Art. 88 are satisfied; in these cases either no forces act, or the action-lines of the forces pass through the fixed point, and thus produce no effect in causing a change of rotation or a change of position of the instantaneous axis. Hence this case includes that of a heavy body, the centre of gravity of which is at the fixed point.

In this case, as no work is done by the force or by the pressure acting at the fixed point, and as $L = M = N = 0$, the theorems of the invariability of the moment of the momentum of the system, of the fixed position of the invariable axis and the invariable plane, and of the constancy of the vis viva or the kinetic energy of the system, as developed in Chap. III, Section 3, hold good; and taking the value of the kinetic energy of the system, which is due only to the rotation of it about the instantaneous axis, to be that explained in Art. 219, and denoting its constant value by k^2, we have

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = k^2. \tag{28}$$

Also taking the value of the moment of momentum of the system about the instantaneous axis to be that which is determined in the same Article, and denoting its invariable value by h^2, we have

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = h^2; \tag{29}$$

and if α, β, γ are the direction-angles of the invariable axis with respect to the principal axes at the time t, then, as shewn in Art. 219,

$$\cos\alpha = \frac{A\omega_1}{h}, \qquad \cos\beta = \frac{B\omega_2}{h}, \qquad \cos\gamma = \frac{C\omega_3}{h}. \tag{30}$$

Hence with respect to the principal axes of the body at the fixed point, the equations to the instantaneous axis, to the invariable axis, and to the invariable plane, are respectively

$$\left.\begin{array}{l} \dfrac{x}{\omega_1} = \dfrac{y}{\omega_2} = \dfrac{z}{\omega_3}; \\ \dfrac{x}{A\omega_1} = \dfrac{y}{B\omega_2} = \dfrac{z}{C\omega_3}; \\ A\omega_1 x + B\omega_2 y + C\omega_3 z = 0. \end{array}\right\} \tag{31}$$

If the system is put into motion by certain initial impulses, or is otherwise in such a condition that at a given instant, say when $t = 0$, $\omega_1 = \Omega_1$, $\omega_2 = \Omega_2$, $\omega_2 = \Omega_3$, then

$$\left.\begin{array}{l} A\,\Omega_1^2 + B\,\Omega_2^2 + C\,\Omega_3^2 = k^2, \\ A^2\Omega_1^2 + B^2\Omega_2^2 + C^2\Omega_3^2 = h^2; \end{array}\right\} \tag{32}$$

which assign values to k and h: and the equations to the invariable axis are at that instant

$$\frac{x}{\text{A}\Omega_1} = \frac{y}{\text{B}\Omega_2} = \frac{z}{\text{C}\Omega_3}; \tag{33}$$

and the equation to the invariable plane is

$$\text{A}\Omega_1 x + \text{B}\Omega_2 y + \text{C}\Omega_3 z = 0. \tag{34}$$

Thus the invariable elements are given in terms of the circumstances at a given epoch.

277.] These results may also be derived directly from Euler's equations of motion given in (27), when $\text{L} = \text{M} = \text{N} = 0$.

For multiplying them severally, (1) by $\omega_1, \omega_2, \omega_3$; and (2) by $\text{A}\,\omega_1, \text{B}\omega_2, \text{C}\omega_3$, and adding, we have

$$\left.\begin{aligned} \text{A}\,\omega_1 \frac{d\,\omega_1}{dt} + \text{B}\,\omega_2 \frac{d\,\omega_2}{dt} + \text{C}\,\omega_3 \frac{d\,\omega_3}{dt} &= 0, \\ \text{A}^2\omega_1 \frac{d\,\omega_1}{dt} + \text{B}^2\omega_2 \frac{d\,\omega_2}{dt} + \text{C}^2\omega_3 \frac{d\,\omega_3}{dt} &= 0; \end{aligned}\right\} \tag{35}$$

whence by integration, and taking the limits at the times $t = t$ and $t = 0$, we have

$$\begin{aligned} \text{A}\omega_1{}^2 + \text{B}\omega_1{}^2 + \text{C}\,\omega_3{}^2 &= \text{A}\Omega_1{}^2 + \text{B}\Omega_2{}^2 + \text{C}\Omega_3{}^2 \\ &= k^2. \end{aligned}$$

Similarly integrating the second equation of (35), and taking the same limits of integration, we have

$$\begin{aligned} \text{A}^2\omega_1{}^2 + \text{B}^2\omega_2{}^2 + \text{C}^2\omega_3{}^2 &= \text{A}^2\Omega_1{}^2 + \text{B}^2\Omega_2{}^2 + \text{C}^2\Omega_3{}^2 \\ &= h^2; \end{aligned}$$

k^2 and h^2 having the same meaning as in the preceding Article.

If h_1, h_2, h_3 are the moments of momentum of the system at the time t about the principal axes, then $h_1 = \text{A}\omega_1$, $h_2 = \text{B}\,\omega_2$, $h_3 = \text{C}\,\omega_3$; and

$$\cos\alpha = \frac{h_1}{h} = \frac{\text{A}\,\omega_1}{h}, \qquad \cos\beta = \frac{h_2}{h} = \frac{\text{B}\,\omega_2}{h}, \qquad \cos\gamma = \frac{h_3}{h} = \frac{\text{C}\,\omega_3}{h};$$

and the equation to the invariable plane is

$$\text{A}\omega_1 x + \text{B}\,\omega_2 y + \text{C}\,\omega_3 z = 0;$$

which are the same equations as those determined in the preceding Article.

278.] Also, conversely, when a body is in motion with one point fixed, and is not acted on by any forces save those which pass through the fixed point, so that the equations of conservation of moment of momentum hold good, Euler's equations of motion

may be deduced from these equations, as pointed out by Professor Sylvester in the Philosophical Transactions, Vol. CLVI, p. 767.

Let the fixed point be the origin: and let the principal axes at it be the coordinate axes. Let a, β, γ be the direction-angles of the invariable axis in reference to them at the time t; so that if $\omega_1, \omega_2, \omega_3$ are the principal angular velocities at that time, and h is the invariable moment of the momentum of the system,

$$\cos a = \frac{\text{A}\,\omega_1}{h}, \quad \cos\beta = \frac{\text{B}\,\omega_2}{h}, \quad \cos\gamma = \frac{\text{C}\,\omega_3}{h}. \tag{36}$$

It is to be observed that an angular motion about a principal axis produces by its motion no change of angular velocity about either of the other principal axes; this is evident from the values of L″, M″, N″ given in Art. 174, and also from the fact that such an effect could only be due to quantities of the forms $\Sigma . myz$, $\Sigma . mzx$, $\Sigma . mxy$, and that these quantities vanish when the body is referred to a system of principal axes.

About the fixed point O, as centre, describe a spherical surface of radius unity; and let this surface intersect the principal axes and the invariable axis in the points A, B, C, I; let these points be joined, two and two, by arcs of great circles; so that IA $= a$, IB $= \beta$, IC $= \gamma$. In the time dt, let the body rotate about the principal axes through angles respectively equal to $\omega_1 dt$, $\omega_2 dt$, $\omega_3 dt$; and let us consider the results of these separately. When the rotation takes place about OA, a is constant, but β and γ vary; their variations may be found as follows: let the angle IAC $= \theta$, so that

$$\cos\beta = \sin a \sin\theta, \quad \cos\gamma = \sin a \cos\theta, \quad \text{and } d\theta = \omega_1 dt;$$

$$\therefore \left.\begin{aligned} d.\cos\beta &= \sin a\cos\theta\, d\theta = \cos\gamma\,\omega_1\, dt, \\ d.\cos\gamma &= -\sin a \sin\theta\, d\theta = -\cos\beta\,\omega_1\, dt; \end{aligned}\right\} \tag{37}$$

similarly for rotations through angles $\omega_2 dt$, $\omega_3 dt$ about OB and OC respectively,

$$\left.\begin{aligned} d.\cos\gamma &= \cos a\,\omega_2 dt, \\ d.\cos a &= -\cos\gamma\,\omega_2 dt; \end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned} d.\cos a &= \cos\beta\,\omega_3 dt, \\ d.\cos\beta &= -\cos a\,\omega_3 dt; \end{aligned}\right\} \tag{39}$$

whence, taking the total variations of each direction-cosine, we have

$$\left.\begin{aligned} d.\cos a &= (\omega_3\cos\beta - \omega_2\cos\gamma)\, dt, \\ d.\cos\beta &= (\omega_1\cos\gamma - \omega_3\cos a)\, dt, \\ d.\cos\gamma &= (\omega_2\cos a - \omega_1\cos\beta)\, dt; \end{aligned}\right\} \tag{40}$$

and replacing $\cos\alpha$, $\cos\beta$, $\cos\gamma$ by their values given in (36), we have

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\omega_3 &= 0,\\ B\frac{d\omega_2}{dt}+(A-C)\,\omega_3\omega_1 &= 0,\\ C\frac{d\omega_3}{dt}+(B-A)\,\omega_1\omega_2 &= 0;\end{aligned}\right\} \qquad (41)$$

which are Euler's equations, when there are no couples of impressed momenta, and the conservation of moments of momenta holds good.

279.] The instantaneous angular velocity being ω, the direction-cosines of the instantaneous axis relative to the principal axes are $\frac{\omega_1}{\omega}$, $\frac{\omega_2}{\omega}$, $\frac{\omega_3}{\omega}$; and as the direction-cosines of the invariable axis in reference to the same axes are those given in (30), if θ is the angle between these axes,

$$\cos\theta = \frac{A\omega_1^2+B\omega_2^2+C\omega_3^2}{\omega h} = \frac{k^2}{\omega h};$$

$$\therefore\ \omega\cos\theta = \frac{k^2}{h}; \qquad (42)$$

that is, the component of the angular velocity at any instant about the invariable axes is constant. This theorem is called the conservation of the angular velocity about the invariable axis.

280.] Although the properties of the couple of the momenta arising from the centrifugal forces have been generally investigated in Articles 174 and 215, yet there are some peculiarities in the form of them in the present problem which require attention.

In accordance with the notation of Art. 174,

$$L''=(B-C)\,\omega_2\omega_3,\quad M''=(C-A)\,\omega_3\omega_1,\quad N''=(A-B)\,\omega_1\omega_2; \qquad (43)$$

$$\left.\begin{aligned} \therefore\quad L''\omega_1+M''\omega_2+N''\omega_3 &= 0,\\ \text{and}\quad L''A\omega_1+M''B\omega_2+N''C\omega_3 &= 0;\end{aligned}\right\} \qquad (44)$$

so that the axis of the centrifugal couple is perpendicular to the instantaneous axis and to the invariable axis; and consequently to the plane which contains these two lines; this plane therefore is the plane of the couple.

Now the centrifugal couple is at each instant of the nature of an external impulsive couple which, acting about a given axis, produces a rotation about another axis related to the former

in the manner described in Article 272; viz. in reference to the momental ellipsoid, the axis conjugate to the plane of the couple of impulsion is the corresponding rotation-axis. Thus if $A\omega_1$, $B\omega_2$, $C\omega_3$ are proportional to the direction-cosines of the axis of the couple of impulsion, $\omega_1, \omega_2, \omega_3$ are proportional to the direction-cosines of the corresponding rotation-axis. Hence the direction-cosines of the rotation-axis of the centrifugal couple are proportional to

$$\frac{B-C}{A}\omega_2\omega_3, \quad \frac{C-A}{B}\omega_3\omega_1, \quad \frac{A-B}{C}\omega_1\omega_2; \tag{45}$$

and as the sum of these quantities severally multiplied by $A\omega_1$, $B\omega_2$, $C\omega_3$ vanishes, it follows that this axis is perpendicular to the invariable axis, and consequently lies in the invariable plane.

281.] Also from (43), if G'' is the moment of the centrifugal couple,

$$\begin{aligned} G''^2 &= (B-C)^2\omega_2^2\omega_3^2 + (C-A)^2\omega_3^2\omega_1^2 + (A-B)^2\omega_1^2\omega_2^2 \\ &= \omega^2(A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2) - (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)^2 \\ &= \omega^2 h^2 - k^4 \\ &= \omega^2 h^2 - \omega^2 h^2(\cos\theta)^2 = \omega^2 h^2 (\sin\theta)^2 \qquad (46) \\ &= k^4(\tan\theta)^2, \text{ by } (42); \qquad (47) \end{aligned}$$

that is, the moment of the couple of the centrifugal forces varies as the tangent of the angle between the instantaneous and the invariable axes; and (46) shews that it is represented in magnitude by the area of the parallelogram whose sides are the line-representatives of the instantaneous angular velocity, and of the moment of the couple of impulsion. (See Art. 217.)

As there is no external couple producing acceleration, that is, as $L = M = N = 0$, the only couple of acceleration is that due to the centrifugal forces, and consequently the direction-cosines of the axis of this couple are proportional to $A\frac{d\omega_1}{dt}$, $B\frac{d\omega_2}{dt}$, $C\frac{d\omega_3}{dt}$.

282.] The complete solution of the problem requires that the position of the body should be determined at the time t; and for this purpose it is necessary to refer it to axes fixed in space. As our choice of these is free, we will take the system which is suggested in the preceding Articles; and taking the fixed point as the origin, we will take the invariable axis to be the axis of z, and the invariable plane to be the plane of (x, y), the axis of x being a line chosen arbitrarily in that plane; and we shall sup-

pose $\omega_1, \omega_2, \omega_3$ to have been determined in terms of t. Now the position of the body with its three principal axes fixed in it will be most conveniently determined by means of the three angles θ, ϕ, ψ of Articles 3 and 64; and since α, β, γ, as given in Art. 278, are the direction-angles of the invariable axis, which is the axis of z,

$$\cos\alpha = \sin\phi\sin\theta, \quad \cos\beta = \cos\phi\sin\theta, \quad \cos\gamma = \cos\theta; \tag{48}$$

$$\therefore \quad \text{A}\omega_1 = h\sin\phi\sin\theta, \quad \text{B}\omega_2 = h\cos\phi\sin\theta, \quad \text{C}\omega_3 = h\cos\theta. \tag{49}$$

$$\therefore \quad \cos\theta = \frac{\text{C}\omega_3}{h}, \qquad \tan\phi = \frac{\text{A}\omega_1}{\text{B}\omega_2}, \tag{50}$$

which determine θ and ϕ in terms of t.

ψ does not enter into these equations, because the position of the axis of x in the invariable plane from which ψ is measured is arbitrary. The t-variation of ψ, that is the angular velocity of ON, see Fig. 1, about the invariable axis may thus be found. From (124), Art. 64, we have

$$\frac{d\psi}{dt} = \frac{\omega_1\sin\phi + \omega_2\cos\phi}{\sin\theta}; \tag{51}$$

$$= \frac{\text{A}\omega_1^2 + \text{B}\omega_2^2}{\text{A}^2\omega_1^2 + \text{B}^2\omega_2^2}h; \tag{52}$$

and if ω_1 and ω_2 are replaced by their values in terms of t, this is a function of t only; and if it is integrated with the proper limits of the time, the precessional angle due to that time will be determined.

283.] From the equations given in Articles 276 and 277, which express the conservation of the kinetic energy of the system and the invariability of the moment of the momentum and of the position of its axis, M. Poinsot has, by means of the momental ellipsoid at the fixed point, in his admirable "Théorie nouvelle de la Rotation des corps*," derived a complete image of the motion of the system.

At the fixed point let the momental ellipsoid be described, the axes of coordinates being the principal axes of the body at that point. As the interpretation of the equations will be geometrical, it is desirable to preserve homogeneity as far as convenient. Let therefore the equation to the momental ellipsoid be, see Art. 184,

$$\text{A}x^2 + \text{B}y^2 + \text{C}z^2 = \text{M}g^4, \tag{53}$$

* Liouville's Journal des Mathématiques, XVI; and Paris, Bachelier, 1851. See also Briot, Liouville, tome VII.

where M is the mass of the body, and g is an arbitrary constant of one dimension in space; so that if a, b, c are the principal radii of gyration, and $\text{A} = \text{M}a^2$, $\text{B} = \text{M}b^2$, $\text{C} = \text{M}c^2$, equation (53) becomes

$$a^2x^2 + b^2y^2 + c^2z^2 = g^4. \tag{54}$$

We shall use either (53) or (54) as we find convenient.

Let the radius-vector of the momental ellipsoid which lies along the instantaneous axis of rotation at the time t be r; let (x, y, z) be its extremity, and let this point be called the instantaneous pole; then

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3} = \frac{r}{\omega} = \frac{\{\text{A}x^2 + \text{B}y^2 + \text{C}z^2\}^{\frac{1}{2}}}{\{\text{A}\omega_1{}^2 + \text{B}\omega_2{}^2 + \text{C}\omega_3{}^2\}^{\frac{1}{2}}} \tag{55}$$

$$= \frac{g^2}{k}\text{M}^{\frac{1}{2}}; \tag{56}$$

so that the coordinates to the instantaneous pole are as follows: viz.

$$x = \frac{g^2}{k}\text{M}^{\frac{1}{2}}\omega_1, \quad y = \frac{g^2}{k}\text{M}^{\frac{1}{2}}\omega_2, \quad z = \frac{g^2}{k}\text{M}^{\frac{1}{2}}\omega_3; \tag{57}$$

also

$$\omega = \frac{kr}{g^2\text{M}^{\frac{1}{2}}}. \tag{58}$$

Thus the instantaneous angular velocity is proportional to the radius vector of the momental ellipsoid which lies along the instantaneous axis. Hence critical values of the angular velocity coincide with critical values of the radius vector of the ellipsoid, and of all possible values of the angular velocity the greatest is that when its axis of rotation lies along the axis of least moment of inertia, and the least is that which lies along the axis of greatest moment of inertia; this theorem is evident from general considerations founded on the conservation of kinetic energy.

The equation to the tangent plane at the instantaneous pole is

$$\text{A}\omega_1\xi + \text{B}\omega_2\eta + \text{C}\omega_3\zeta = kg^2\text{M}^{\frac{1}{2}}, \tag{59}$$

which is evidently parallel to the invariable plane given in (31); and if p_0 is the perpendicular distance from the origin on this plane,

$$p_0{}^2 = \frac{k^2\text{M}g^4}{h^2}, \tag{60}$$

and is constant; consequently the tangent plane to the momental ellipsoid at the instantaneous pole is parallel to the invariable plane and at a constant distance p_0 from it, both planes being perpendicular to the invariable axis; it is consequently fixed in space.

Hence the area of the section of the ellipsoid made by the invariable plane (31) is constant, see (122), Art. 18, and is equal to

$$\frac{\pi h g^4}{abck\text{M}^{\frac{1}{2}}}.$$

Also for all positions of the instantaneous pole on the surface of the ellipsoid the product of the principal radii of curvature is constant. See Vol. I, Art. 405.

Hence also
$$\omega = \frac{k^2}{h\cos\theta} = \frac{k^2 r}{h p_0}. \qquad (61)$$

284.] These facts supply us with the following image of the body's motion.

Imagine the momental ellipsoid at the fixed point to be constructed; as the preceding equations contain only the principal moments of inertia at the fixed point we may suppose the body to be replaced by its momental ellipsoid; then the motion of the latter will correctly and completely represent the motion of the former.

At the fixed point also let the invariable axis and the invariable plane be drawn; and let a plane be drawn perpendicular to the invariable axis, cutting it at a distance p_0 from the origin on either side of it; this plane is that whose equation is (59); and if the momental ellipsoid be imagined to be placed so as to touch the plane, the point of contact is the instantaneous pole at the corresponding instant, and is for the instant at rest. If then we imagine the momental ellipsoid whose centre is fixed to roll without sliding on this plane which it always touches, we shall have a true image of the body's motion.

Moreover, as the point of contact is the instantaneous pole and the radius vector to it is the instantaneous axis, the angular velocity varies as the length of the radius vector by reason of (58); and the component of the angular velocity about the invariable axis is constant by reason of (42).

These circumstances are delineated in Fig. 33; O is the fixed point and is the centre of the momental ellipsoid ABC; OG is the invariable axis, which we have in the figure taken to be vertical. Along OG, vertically downwards from O, OG is taken equal to p_0, and through G a plane is drawn perpendicular to OG, and consequently parallel to the invariable plane; this plane is that whose equation is (59), and is that on which the momental ellip-

soid rolls. I is the point of contact at the time t, and is the instantaneous pole; OI is the instantaneous rotation-axis, and is that to which the angular velocity ω is proportional by reason of (58); GOI is the angle θ of (42), so that if ω is resolved into two components, one of which is in the invariable plane and the other about the invariable axis, the latter is $\omega \cos\theta$, which is constant, see (42), throughout the motion, and the former is $\omega \sin\theta$ which is, see (46), proportional to the moment of the centrifugal couple. The plane GOI which contains the instantaneous and the invariable axes is the plane of the centrifugal couple, see Art. 280; and the moment of the centrifugal couple varies as the area of the triangle GOI.

The construction mentioned on Art. 272 is the initial case, or the case at a given epoch, of that motion of which this construction gives an image at all times.

As the ellipsoid rolls on the plane GEF, successive points of it come continuously into contact with the plane; and these points of contact lie in two curves, one being EIF in the plane GEF, and the other being the curve IQP on the surface of the ellipsoid: the former is evidently generally a plane curve of an undulating character, and the latter is a closed curve on the ellipsoid. These curves have been named by M. Poinsot the Herpolhode and the Polhode respectively.

285.] As the polhode is the locus of those points on the ellipsoid, at which the tangent planes are all at the same distance p_0 from the centre, it will generally be a curve of double curvature, being the line of intersection of the two quadrics,

$$\left.\begin{aligned} \mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 &= \mathrm{M}g^4, \\ \mathrm{A}^2x^2 + \mathrm{B}^2y^2 + \mathrm{C}^2z^2 &= \frac{h^2}{k^2}\mathrm{M}g^4; \end{aligned}\right\} \tag{62}$$

which are the equations to two concentric and coaxial ellipsoids.

The projections of the curve of intersection of these two ellipsoids on the coordinate planes, which are the principal planes of each ellipsoid, are given by the following equations, viz.

$$\mathrm{B}(\mathrm{B}-\mathrm{A})y^2 + \mathrm{C}(\mathrm{C}-\mathrm{A})z^2 = \left(\frac{h^2}{k^2} - \mathrm{A}\right)\mathrm{M}g^4; \tag{63}$$

$$\mathrm{C}(\mathrm{C}-\mathrm{B})z^2 - \mathrm{A}(\mathrm{B}-\mathrm{A})x^2 = \left(\frac{h^2}{k^2} - \mathrm{B}\right)\mathrm{M}g^4; \tag{64}$$

$$\mathrm{A}(\mathrm{C}-\mathrm{A})x^2 + \mathrm{B}(\mathrm{C}-\mathrm{B})y^2 = \left(\mathrm{C} - \frac{h^2}{k^2}\right)\mathrm{M}g^4; \tag{65}$$

the coefficients being put in each case in positive forms, it being presumed, as previously, that C is the greatest and A is the least principal moment of inertia at the fixed point.

Now since in the general case of an ellipsoid of three unequal axes the perpendicular from the centre on the tangent plane can never be greater than the greatest and never less than the least semi-axis, and may have all intermediate values, it follows that h^2 is never less than $\mathrm{A}k^2$ and never greater than $\mathrm{C}k^2$, and may have any value intermediate to these; hence (63) and (65) always represent ellipses in the planes of (y, z) and of (x, y) respectively, and (64) is the equation to a hyperbola in the plane of (x, z), of which the real axis lies along the axis of x or the axis of z, according as h^2 is less than or greater than $\mathrm{B}k^2$. If $h^2 = \mathrm{A}k^2$, (63) represents a point in the plane of (y, z), viz. the origin, the tangent plane in this case touching the ellipsoid at the end of the A-axis; if $h^2 = \mathrm{B}k^2$, the equation represents two planes passing through the B-axis, and making with the plane of (x, y) angles $\tan^{-1} \pm \left\{\frac{\mathrm{A}(\mathrm{B}-\mathrm{A})}{\mathrm{C}(\mathrm{C}-\mathrm{B})}\right\}^{\frac{1}{2}}$, so that the polhode in each case is an ellipse, being the sections of the momental ellipsoid by these planes; and if $h^2 = \mathrm{C}k^2$, (65) represents a point in the plane of (x, y), viz. the origin, the tangent plane in this case touching the ellipsoid at the end of the C-axis.

We thus obtain three distinct cases:—

(1) That in which p_0 is greater than the mean principal semi-axis and not greater than the greatest principal semi-axis, that is, where h^2 is less than $\mathrm{B}k^2$ and greater than $\mathrm{A}k^2$; in this case each polhode corresponding to a value of p_0 within the assigned limits is a closed symmetrical curve around the vertex A as a centre, and situated symmetrically in four quarters relatively to the principal planes of the ellipsoid which intersect at right angles at A. The projection of this curve on the plane of (y, z) is a complete ellipse; on the plane of (z, x) is an arc of a hyperbola whose real axis lies along the axis of x; and on the plane of (x, y) is the arc of an ellipse. According as p_0 varies within its limits there is a family of similar and similarly situated closed curves, of which the limiting forms are the vertex A, and the semi-ellipses mentioned above, which is the form of the polhode when $h^2 = \mathrm{B}k^2$. Fig. 33 indicates the mode of generation of these curves.

(2) If p_0 is equal to the mean principal semi-axis, that is if $h^2 = \text{B}k^2$, the polhode becomes a plane curve, since in that case (64) becomes

$$\text{C}(\text{C}-\text{B})z^2-\text{A}(\text{B}-\text{A})x^2 = 0, \tag{66}$$

which represents two planes equally inclined to the plane of (x, y). These planes cut the ellipsoid in two equal ellipses, of which the semi-axis major and the semi-axis minor are respectively

$$\left\{\frac{\text{A}+\text{C}-\text{B}}{\text{AC}}\text{M}g^4\right\}^{\frac{1}{2}} \quad \text{and} \quad \left\{\frac{\text{M}g^4}{\text{B}}\right\}^{\frac{1}{2}}, \tag{67}$$

the common minor axis of the two ellipses coinciding with the mean axis of the ellipsoid. These ellipses we shall call the dividing ellipses of the ellipsoid, because they divide the surface of the ellipsoid into four parts, each of which has its own separate family of polhodes. Thus the instantaneous pole in this case moves in a plane curve. The planes of the dividing ellipses are the cyclic planes of the ellipsoid of gyration, which is the reciprocal surface to the momental-ellipsoid.

(3) If p_0 is less than the mean principal semi-axis, and not less than the least principal semi-axis, that is if h^2 is greater than $\text{B}k^2$ and not greater than $\text{C}k^2$, each polhode corresponding to a value of p_0 within the assigned limits is a closed symmetrical curve around the vertex C as a centre, and situated symmetrically in four quarters relatively to the principal planes of the ellipsoid which intersect at right angles at C. The projection of each of these polhodes is an arc of an ellipse on the plane of (y, z), is an arc of a hyperbola on the plane of (z, x) whose real axis lies along the axis of z, and is a complete ellipse on the plane of (x, y). According as p_0 varies within the assigned limits, there is a family of similar and similarly situated closed curves of which the limiting forms are the vertex C and the semi-ellipses mentioned above.

The surface of the ellipsoid is thus divided into four districts by the dividing ellipses, each pair of opposite districts having similar polhodes, of which the centres are respectively the A- and the C-vertices of the ellipsoid.

As the projections of the polhode on the three principal planes of the momental ellipsoid are arcs of conics, see (63), (64), (65), it follows that at no point in the polhode is the radius of curvature zero or infinite; or, in other words, no point of the polhode is a stationary point.

286.] Of the family of polhodes described about the vertex A, if r_1 and r_2 are respectively the maximum and minimum central radii vectores,

$$r_1^2 = \frac{\text{M}g^4}{\text{CA}}\left\{\text{C}+\text{A}-\frac{h^2}{k^2}\right\}, \qquad r_2^2 = \frac{\text{M}g^4}{\text{BA}}\left\{\text{B}+\text{A}-\frac{h^2}{k^2}\right\},$$

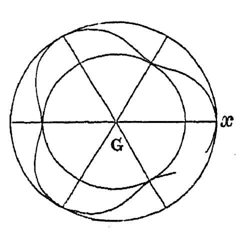
since $r_1^2 - r_2^2$ is positive.

And for the family of polhodes about the vertex C, if r_1 and r_2 are respectively the maximum and minimum central radii vectores,

$$r_1^2 = \frac{\text{M}g^4}{\text{CA}}\left\{\text{C}+\text{A}-\frac{h^2}{k^2}\right\}, \qquad r_2^2 = \frac{\text{M}g^4}{\text{CB}}\left\{\text{C}+\text{B}-\frac{h^2}{k^2}\right\},$$

since $r_1^2 - r_2^2$ is positive.

287.] As the herpolhode is the locus on the fixed plane of the point of contact of the ellipsoid as it rolls on the plane, it is evidently generally a plane curve of an undulating form, such as is delineated in the annexed figure, consisting of equal and regularly recurring portions of which the maximum and minimum radii vectores are equal for each portion. Thus the curve, as shewn in the figures, Art. 292, regularly winds between two concentric circles, the common centre of which is the point of intersection of the fixed plane with the invariable axis, meeting alternately one and the other. The maximum and minimum radii vectores of the herpolhode are the radii of these circles. These singular values recur regularly at equal angles on the fixed plane, and correspond to the maximum and minimum central radii vectores of the polhode which take place when the polhode intersects a principal plane of the ellipsoid. Thus if ρ_1 and ρ_2 are respectively the maximum and minimum radii vectores of the herpolhode corresponding to a polhode about the vertex A,



$$\rho_1^2 = \frac{\text{M}g^4}{\text{CA}}\left\{\text{C}+\text{A}-\frac{h^2}{k^2}\right\} - \frac{\text{M}g^4 k^2}{h^2},$$

$$\rho_2^2 = \frac{\text{M}g^4}{\text{BA}}\left\{\text{B}+\text{A}-\frac{h^2}{k^2}\right\} - \frac{\text{M}g^4 k^2}{h^2};$$

ρ_1 and ρ_2 are the radii of the external and internal bounding circles. Similarly may the radii of the bounding circles be found for the herpolhodes corresponding to polhodes about the vertex C. If the angle between two consecutive maximum radii

of the herpolhode is a multiple or an aliquot part of a right angle, the herpolhode re-enters, and forms a closed curve; but if this angle is not commensurable with a right angle, the curve never re-enters; and although the instantaneous rotation-axis returns periodically to the same plane in the body yet it never returns to a position which it has previously held in space.

288.] If $h^2 = \text{A}k^2$, or if $h^2 = \text{C}k^2$, the herpolhode is in each case reduced to a point, viz. G, at which the momental ellipsoid touches the fixed plane; in these cases the rotation-axis of the body has the same position throughout the motion, both in the body and in space; it is one of the principal axes of the body and is thus a permanent axis.

289.] But if $h^2 = \text{B}k^2$, that is if the perpendicular distance from the fixed point on the fixed plane is equal to the mean principal semi-axis of the momental ellipsoid, the herpolhode takes the following form; the polhodes in this case being the dividing ellipses.

Let x, y, z, r, s refer to the polhode relatively to the principal axes of the momental ellipsoid, and let ρ, ϕ, σ refer to the herpolhode relatively to the fixed plane on which the ellipsoid rolls, G being the pole; then as the ellipsoid rolls, $ds = d\sigma$; and also

$$r^2 = \rho^2 + p_0{}^2 = \rho^2 + \frac{\text{M}g^4}{\text{B}}. \tag{68}$$

Thus the equations to the polhode take the following forms,

$$\left.\begin{aligned} \text{A}x^2 + \text{B}y^2 + \text{C}z^2 &= \text{M}g^4, \\ \text{A}^2x^2 + \text{B}^2y^2 + \text{C}^2z^2 &= \text{BM}g^4, \\ x^2 + y^2 + z^2 = r^2 &= \rho^2 + \frac{\text{M}g^4}{\text{B}}; \end{aligned}\right\} \tag{69}$$

whence

$$\left.\begin{aligned} x^2 &= \frac{\text{BC}\rho^2}{(\text{B}-\text{A})(\text{C}-\text{A})}, \\ y^2 &= \frac{(\text{C}-\text{B})(\text{B}-\text{A})\text{M}g^4 - \text{ABC}\rho^2}{\text{B}(\text{C}-\text{B})(\text{B}-\text{A})}, \\ z^2 &= \frac{\text{AB}\rho^2}{(\text{C}-\text{A})(\text{C}-\text{B})}. \end{aligned}\right\} \tag{70}$$

And since $ds^2 = d\sigma^2 = d\rho^2 + \rho^2 d\phi^2$;

$$\begin{aligned} \therefore\quad \rho^2 d\phi^2 &= ds^2 - d\rho^2 \\ &= dx^2 + dy^2 + dz^2 - d\rho^2 \\ &= \frac{\text{CAM}g^4\, d\rho^2}{(\text{C}-\text{B})(\text{B}-\text{A})\text{M}g^4 - \text{ABC}\rho^2}. \end{aligned}$$

Let $\frac{\text{ABC}}{(\text{C}-\text{B})(\text{B}-\text{A})\,\text{M}g^4} = \frac{1}{a^2}, \qquad \frac{(\text{C}-\text{B})(\text{B}-\text{A})}{\text{CA}} = \mu^2;$

then
$$-\mu\, d\phi = \frac{a\, d\rho}{\rho(a^2-\rho^2)^{\frac{1}{2}}},$$
$$\rho = \frac{2a}{e^{\mu\phi}+e^{-\mu\phi}}, \tag{71}$$
the limits of integration being such that $\rho = a$ when $\phi = 0$. This is the equation to the herpolhode in the fixed plane on which the ellipsoid rolls. The maximum radius vector is taken as the prime radius, and this of course corresponds to the maximum radius vector of the polhode, taken from the centre of the ellipsoid; for when $\phi = 0$,
$$\rho^2 = \frac{(\text{C}-\text{B})(\text{B}-\text{A})}{\text{ABC}}\text{M}g^4$$
$$= \left\{\frac{\text{C}+\text{A}-\text{B}}{\text{AC}} - \frac{1}{\text{B}}\right\}\text{M}g^4, \tag{72}$$
so that the greatest value of ρ is the distance between the focus and the centre of the dividing ellipse. The curve lies symmetrically on the two sides of this maximum radius vector; as ϕ increases, ρ decreases, and ultimately $\rho = 0$, when $\phi = \infty$. The curve is consequently a spiral, such as is drawn in Fig. 34, of which GE is the maximum radius, relatively to which the curve is symmetrical. The branches, both in the positive and negative directions fall into the pole G after an infinite number of convolutions, so that the pole is an asymptotic point; and this occurs when the instantaneous rotation-axis having passed along the dividing ellipse falls into the mean axis of the ellipsoid, and coincides with the invariable axis.

If then the instantaneous pole is at any time on this curve it will move along it; and although the length of the herpolhode is finite, being equal to the length of the dividing ellipse, yet, as will be shewn hereafter, the instantaneous pole will never fall into the pole of the curve, that is, into the point G. If however the instantaneous axis ever coincides with the invariable axis, it will always do so, unless some other external couple acts and produces a change of position in the rotation-axis.

290.] The momental ellipsoid also gives a perfect geometrical image of the kinematics of the problem in the following form:

Let the fixed point O be, as heretofore, the centre of the momental

ellipsoid; let P be the instantaneous pole and OP the instantaneous axis; let the tangent plane be drawn at P, and let OG be the perpendicular from O on it, so that OG is the invariable axis, and $OG = p_0$. Through O let a plane be drawn parallel to the tangent plane; that plane is the invariable plane, the tangent plane being also a fixed plane in which is the herpolhode; G is the pole and GP is the radius vector of that curve. The plane OPG is the plane of the centrifugal couple, the moment of the couple being proportional to the area of the triangle OPG. Let OH be the radius vector of the ellipsoid which lies along the intersection of the plane OPG with the invariable plane, so that OH is parallel to GP, the two planes being perpendicular to each other. Through O draw OL perpendicular to the plane OPG, so that the axis of the centrifugal couple lies along this line, and let a plane be drawn at right angles to this line and touch the ellipsoid in K; join OK; then OK is the axis conjugate to the plane OPG, and as OH and OP are conjugate axes of the elliptic section made by that plane, OP, OH, and OK are a system of conjugate axes; hence the rotation-axis due to the centrifugal couple lies along OK, and OK is in the invariable plane OLH, since that plane is parallel to the tangent plane at P. Also, as OL is parallel to the tangent plane at P, OL and OP are conjugate axes of the elliptic section made by the plane OLP, and as they are at right angles to each other, they are the principal semi-axes of that elliptic section. See Figure in Art. 291.

291.] As the right-hand member in the equation to the momental ellipsoid, which is given in (117), Art. 182, is indeterminate, we will assume it to be unity, so that the expressions in the following investigations may be as simple as possible. Thus the equations to the polhode take the form

$$\left.\begin{aligned} \text{A}x^2 + \text{B}y^2 + \text{C}z^2 &= 1; \\ \text{A}^2x^2 + \text{B}^2y^2 + \text{C}^2z^2 &= \frac{1}{{p_0}^2} = \frac{h^2}{k^2}. \end{aligned}\right\} \qquad (73)$$

Let θ be as heretofore the angle POG. Then the equations to the several straight lines mentioned above are as follows;

To OP, $$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3} = \frac{r}{\omega}.$$

To OG, $$\frac{x}{\text{A}\omega_1} = \frac{y}{\text{B}\omega_2} = \frac{z}{\text{C}\omega_3} = \frac{r}{h}.$$

To OH, which lies in the plane OPG and is perpendicular to OG,

$$\frac{x}{\omega_1(h^2-\text{A}k^2)} = \frac{y}{\omega_2(h^2-\text{B}k^2)} = \frac{z}{\omega_3(h^2-\text{C}k^2)} = \frac{r}{hk^2\tan\theta} = \frac{r}{h\text{G}''},$$

since
$$\begin{aligned}\omega_1^2(h^2-\text{A}k^2)^2+\omega_2^2(h^2-\text{B}k^2)^2+\omega_3^2(h^2-\text{C}k^2)^2 &= h^4\omega^2-k^4h^2 \\ &= h^2k^4(\tan\theta)^2 \\ &= h^2\text{G}''^2.\end{aligned}$$

To OL, which is the axis of the centrifugal couple,

$$\frac{x}{(\text{B}-\text{C})\omega_2\omega_3} = \frac{y}{(\text{C}-\text{A})\omega_3\omega_1} = \frac{z}{(\text{A}-\text{B})\omega_1\omega_2} = \frac{r}{k^2\tan\theta} = \frac{r}{\text{G}''}.$$

To OK, which is the axis of the rotation arising from the centrifugal couple,

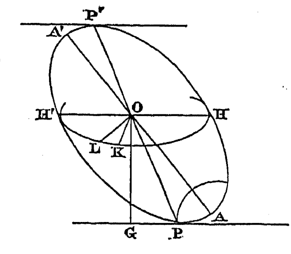
$$\frac{\text{A}\omega_1 x}{\text{B}-\text{C}} = \frac{\text{B}\omega_2 y}{\text{C}-\text{A}} = \frac{\text{C}\omega_3 z}{\text{A}-\text{B}} = \frac{r}{\omega''},$$

where ω'' is the angular velocity due to the centrifugal couple.

The equations to OP, OH and OK, which three lines constitute a system of conjugate axes, satisfy the conditions of such axes which are given in Art. 23, viz.

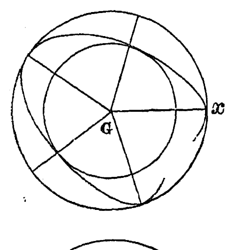
$$\text{A}\lambda\lambda'+\text{B}\mu\mu'+\text{C}\nu\nu' = 0.$$

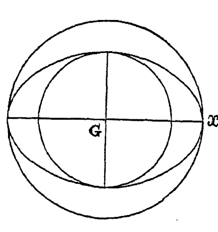
Also the equations shew that OK is perpendicular to OG; and that OL and OP are conjugate axes, and being at right angles to each other are the principal semi-axes of the section. The annexed figure exhibits the position, &c., of these various lines.



292.] These equations will enable us to investigate further properties of the herpolhode curve, and other characteristics of the motion. The herpolhode is, as already stated, a curve which winds between two concentric circles, coming into contact with, and touching alternately one and the other. In some cases each portion which lies between the two circles will be concave towards the poles in the parts nearer to the outer circle, and convex towards the pole in the parts nearer to the inner circle, so that there will be a point of inflexion, and the radius of curvature will change sign by passing through infinity; in other cases the curve will be always concave towards the pole, and there will be no point of inflexion; these two classes of curve are shewn in the annexed figures. In the latter case the curve is of the same form as that

of the moon's path projected on the plane of the ecliptic, that path being always concave towards the sun. If the angle between two successive maximum radii vectores is two right angles, the path of the herpolhode is in the form of an ellipse, of which the radii of the circles are the principal semi-axes, as in the annexed figures.





Let, as in preceding Articles, x, y, z, r, s refer to the instantaneous pole at the time t, as a point on the polhode; and let ρ, ϕ, σ refer to it at the same time as a point on the herpolhode in the fixed plane, ϕ being the angle which the plane OGP makes with some fixed plane which contains the invariable axis; then $\rho = \text{GP}$, G being the pole of the curve; let all these be considered as functions of t, and so varying with t. Let t-differential coefficients be denoted by accents, as in Article 126.

Let $\quad \text{B}-\text{C} = \alpha, \quad \text{C}-\text{A} = \beta, \quad \text{A}-\text{B} = \gamma;$

so that $\quad \alpha+\beta+\gamma = 0; \quad \text{A}\alpha+\text{B}\beta+\text{C}\gamma = 0.$

Now since

$$\frac{\omega_1}{x} = \frac{\omega_2}{y} = \frac{\omega_3}{z} = \frac{\omega}{r} = (\text{A}\omega_1^2+\text{B}\omega_2^2+\text{C}\omega_3^2)^{\frac{1}{2}} = k;$$

$$\therefore \quad \omega_1 = kx, \quad \omega_2 = ky, \quad \omega_3 = kz, \quad \omega = kr.$$

Therefore, by the equations (41), Art. 278,

$$\text{A}x' = k\alpha yz, \quad \text{B}y' = k\beta zx, \quad \text{C}z' = k\gamma xy. \tag{74}$$

Hence in respect of the herpolhode,

$$\begin{aligned} \rho^2 &= r^2 - p_0^2 \\ &= x^2+y^2+z^2-(\text{A}^2x^2+\text{B}^2y^2+\text{C}^2z^2)^{-1} \\ &= (\alpha^2y^2z^2+\beta^2z^2x^2+\gamma^2x^2y^2)\,p_0^2 \qquad (75) \\ &= \text{U}^2p_0^2, \end{aligned}$$

if $$\alpha^2y^2z^2+\beta^2z^2x^2+\gamma^2x^2y^2 = \text{U}^2; \tag{76}$$

$$\therefore \quad \rho = \text{U}p_0 = \frac{k\text{U}}{h};$$

but $$\rho = p_0 \tan\theta;$$

$$\therefore \quad \text{U} = \tan\theta, \quad \text{and} \quad \rho = \frac{k\tan\theta}{h};$$

where θ is the angle between the invariable and the instantaneous axes.

293.] Again, projecting on the fixed plane the sectorial area described by the instantaneous axis in the time dt about the fixed point, firstly directly, and secondly by the intervention of the principal axes of the ellipsoid, in reference to which the direction-cosines of the invariable axis are $\frac{A\omega_1}{h}, \frac{B\omega_2}{h}, \frac{C\omega_3}{h}$, and equating these values, we have

$$\rho^2 d\phi = \frac{A\omega_1}{h}(y\,dz - z\,dy) + \frac{B\omega_2}{h}(z\,dx - x\,dz) + \frac{C\omega_3}{h}(x\,dy - y\,dx);$$

$$\therefore \rho^2\phi' = \frac{k^2}{hABC}(\alpha^2 BC y^2 z^2 + \beta^2 CA z^2 x^2 + \gamma^2 AB x^2 y^2) = \frac{k^2 V^2}{hABC}, \quad (77)$$

if $$\alpha^2 BC y^2 z^2 + \beta^2 CA z^2 x^2 + \gamma^2 AB x^2 y^2 = V^2; \quad (78)$$

$$\therefore \quad \phi' = \frac{h V^2 (\cot\theta)^2}{ABC}. \quad (79)$$

This equation is equivalent to the condition $d\sigma = ds$, and may be derived from it.

Equations (75) and (77) are those to the herpolhode in terms of x, y, z, which are in respect of that curve subsidiary quantities: for these may, by means of (73), together with

$$x^2 + y^2 + z^2 = r^2 = \rho^2 + p_0^2,$$

be expressed in terms of ρ and known quantities.

The value of V^2 may be expressed in terms of θ and of OH, the radius vector of the momental ellipsoid which is parallel to ρ; let this radius vector be R; then, substituting in the equation to the ellipsoid the direction-cosines of R as given in Art. 291, we have

$$A\omega_1^2(h^2 - Ak^2)^2 + B\omega_2^2(h^2 - Bk^2)^2 + C\omega_3^2(h^2 - Ck^2)^2 = \frac{h^2k^4(\tan\theta)^2}{R^2};$$

therefore

$$(A\omega_1^2 + B\omega_2^2 + C\omega_3^2)(A^3\omega_1^2 + B^3\omega_2^2 + C^3\omega_3^2) - (A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2)^2 = \frac{h^2k^4(\tan\theta)^2}{R^2};$$

$$\alpha^2 BC\omega_2^2\omega_3^2 + \beta^2 CA\omega_3^2\omega_1^2 + \gamma^2 AB\omega_1^2\omega_2^2 = \frac{h^2k^2(\tan\theta)^2}{R^2},$$

$$\alpha^2 BC y^2 z^2 + \beta^2 CA z^2 x^2 + \gamma^2 AB x^2 y^2 = \frac{h^2(\tan\theta)^2}{k^2 R^2};$$

$$\therefore \quad \text{V} = \frac{h \tan\theta}{k\text{R}} = \frac{h\text{G}''}{k^3\text{R}},$$

and
$$\phi' = \frac{h^3}{\text{ABC}k^2\text{R}^2}; \tag{80}$$

as all the factors which occur on the right-hand member of this equation are positive, ϕ' is positive, and therefore ϕ increases as the time increases.

Also, since $\text{R}^2\phi' = \frac{h^3}{\text{ABC}k^2}$ and is constant, it follows that equal sectorial areas are described by OH in equal times.

294.] The existence of ϕ' is the effect of the centrifugal forces, the couple due to which is G'', as given in (47), Art. 281, and of which L'', M'', N'' are the axial components: for if there were no centrifugal couple the instantaneous axis would be a permanent axis fixed in the body, the angle between the instantaneous and the invariable axes would be constant, the two limiting circles of the herpolhode would coalesce, and the coalescent circle would be the herpolhode; and under particular values of h and k, the herpolhode would become a point.

As G'' is the centrifugal couple, which causes the angular velocity ω'' about the axis OK, if r'' is the radius vector of the momental ellipsoid lying along OK, we have from (15), Art. 272,

$$\begin{aligned}\omega'' &= \text{G}''r''^2 \cos\text{KOL} \\ &= \text{G}''r''^2 \frac{\alpha^2\text{BC}\omega_2^2\omega_3^2 + \beta^2\text{CA}\omega_3^2\omega_1^2 + \gamma^2\text{AB}\omega_1^2\omega_2^2}{\text{ABC}\omega''\text{G}''};\end{aligned}$$

$$\therefore \quad \omega''^2 = \frac{k^4\text{V}^2r''^2}{\text{ABC}} = \frac{h^2k^4(\tan\theta)^2r''^2}{\text{ABC}k^2\text{R}^2} \tag{81}$$

$$= \frac{h^2\text{G}''^2r''^2}{\text{ABC}k^2\text{R}^2}. \tag{82}$$

Hence it appears that $\omega'' = 0$, if $\text{G}'' = 0$.

From (81) we have

$$\frac{\omega''}{\omega} = \frac{h^2\sin\theta}{k(\text{ABC})^{\frac{1}{2}}}\frac{\text{OK}}{\text{OH}},$$

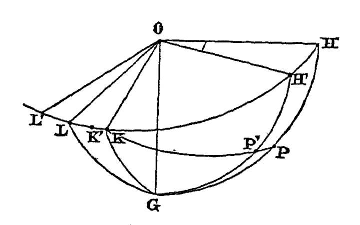
where OK and OH are the radii vectores of the ellipsoid which lie along OK, the rotation-axis of the centrifugal couple, and OH respectively.

Let ω' be the angular velocity at the time $t+dt$; then ω' is the resultant of ω and ω'', the axes of all three being in one plane. From O as a centre describe a sphere with radius unity,

and let the axes of ω, ω'', ω' intersect the surface in the points P, K, P′, and let these points be joined by an arc of a great circle: then

$$\frac{\omega}{\sin \text{KP}'} = \frac{\omega''}{\sin \text{P}'\text{P}} = \frac{\omega'}{\sin \text{PK}}.$$

Let the invariable axis, the projection of the instantaneous axis on the invariable plane, the axis of the centrifugal couple, and the rotation-axis of that couple, intersect the spherical surface in G, H, L, K, all these being in these positions at the time t; and let the corresponding positions at the time $t+dt$ intersect the surface in H′, L′, K′; join by great circles GPH, GP′H′, GK, GL, HH′KL, the last being in the invariable plane. Then GP $= \theta$, GP′ $= \theta'$, HH′ $= d\phi$, $d\phi$ being the angle through which OH, and consequently the radius vector of the herpolhode, revolves in the time dt. All these points and lines are drawn in the annexed figure. Since OL is perpendicular to OH, and OL′ is perpendicular to OH′, all these lines being in the invariable plane, the angle between OL and OL′ is equal to that between OH and OH′, so that LOL′ $=$ HOH′ $= d\phi = \phi' dt$.



Since ω' is the resultant of ω and ω'', its axial components are equal to the sum of the corresponding components of the other quantities, so that

$$\omega_1' = \omega_1 + \omega_1'', \quad \omega_2' = \omega_2 + \omega_2'', \quad \omega_3' = \omega_3 + \omega''^3;$$

therefore

$$\omega_1' = \omega_1 + \frac{a\,\omega_2\omega_3}{\text{A}}, \qquad \omega_2' = \omega_2 + \frac{\beta\,\omega_3\omega_1}{\text{B}}, \qquad \omega_3' = \omega_3 + \frac{\gamma\,\omega_1\omega_2}{\text{C}};$$

$$\therefore \quad \text{A}\omega_1\omega_1' + \text{B}\omega_2\omega_2' + \text{C}\omega_3\omega_3' = \text{A}\omega_1^2 + \text{B}\omega_2^2 + \text{C}\omega_3^2;$$

that is,
$$\omega' \cos\theta' = \omega\cos\theta = k^2;$$

the same theorem as that already proved in Art. 279.

295.] The preceding values of ρ and ϕ' lead directly to the conclusion that in the special form of the herpolhode corresponding to the momental ellipsoid, the curve is throughout concave towards the pole, so that there is no point of inflexion, and the curve is of the general form of the second figure in Art. 292. The value of $d\tau$ which is given in Art. 298, Vol. I, is

$$\frac{d\tau}{dt} = \frac{\rho^2\phi'^3 + \rho\rho'\phi'' + 2\rho'^2\phi' - \rho\rho''\phi'}{\sigma'^2}. \tag{83}$$

Now from (75) we have

$$\rho^2 = \frac{k^2}{h^2}(\alpha^2 y^2 z^2 + \beta^2 z^2 x^2 + \gamma^2 x^2 y^2);$$

$$\therefore \quad \rho\rho' = -\frac{k\alpha\beta\gamma}{\text{ABC}} xyz,$$

$$\rho\rho'' + \rho'^2 = -\frac{k^2\alpha\beta\gamma}{\text{A}^2\text{B}^2\text{C}^2}(\alpha\text{BC}y^2z^2 + \beta\text{CA}z^2x^2 + \gamma\text{AB}x^2y^2).$$

Also from (77) we have

$$\rho^2\phi' = \frac{k^2}{h\text{ABC}}(\alpha^2\text{BC}y^2z^2 + \beta^2\text{CA}z^2x^2 + \gamma^2\text{AB}x^2y^2);$$

$$\therefore \quad 2\rho\rho'\phi' + \rho^2\phi'' = -\frac{2k^3\alpha\beta\gamma xyz}{h\text{ABC}}.$$

Now substituting these values in (83), and replacing h^2 by its value given in (73), we have after all reductions:

$$\frac{d\tau}{dt} = \frac{hk^2}{\text{A}^2\text{B}^2\text{C}^2\text{U}^4\sigma'^2}\{\text{V}^2\text{K} + 2x^2y^2z^2(\text{V}^2\text{W}^2 + \alpha^2\beta^2\gamma^2\text{U}^2)\}, \qquad (84)$$

where $\text{K} = \alpha^4(\text{B}+\text{C}-\text{A})y^4z^4 + \beta^4(\text{C}+\text{A}-\text{B})z^4x^4 + \gamma^4(\text{A}+\text{B}-\text{C})x^4y^4$,

$$\text{W}^2 = \beta^2\gamma^2\text{A}x^2 + \gamma^2\alpha^2\text{B}y^2 + \alpha^2\beta^2\text{C}z^2.$$

Now every term in (84) is positive by reason of its form, except K; but this is also positive, because the sum of any two of the principal moments of inertia is always greater than the third moment, see Art. 150; hence $d\tau$ is always positive, and the herpolhode is always concave towards the pole of the curve, and there is no point of inflexion.

Other properties of the herpolhode might be investigated by means of the preceding equations, but it would be beyond the scope of the present work to do so, as they do not represent any important kinetical theorems.

296.] From the properties of the polhode and the herpolhode the following inferences can be drawn as to the change of position of the instantaneous pole, and consequently of the instantaneous axis, when a disturbing external couple is brought to act on the system.

We will suppose the body to be rotating about its mean principal axis, and to be acted on by a couple of given moment, whereby the instantaneous pole is moved from B (say), along one of the dividing ellipses of the ellipsoid; then the instantaneous pole will travel along this ellipse until it falls into the opposite vertex B′, and the body is completely overturned;

in this motion the instantaneous pole will have completely described the herpolhode (71), on the plane of rolling. This is the greatest derangement which a body having one point fixed can undergo; but this complete bouleversement, as will be shewn hereafter, requires an infinite time.

If however the instantaneous pole is moved from the extremity of the mean axis of the ellipsoid to a point which is not in either of the dividing ellipses, then the following consequences will occur. Let the ellipsoid be divided into four districts by means of the planes of the dividing ellipses; two of these districts will contain the vertices A and A′ of the ellipsoid, and two will contain the vertices C and C′; which are respectively the vertices of the axes of least and of greatest moment. Now if the instantaneous pole is shifted into either of the districts which contain C or C′, the polhode will be a closed curve towards and around C or C′, and the instantaneous pole will perform a complete circuit of this curve, and will periodically return to its first position: the more nearly too to C the instantaneous pole is shifted from B, the less will be the subsequent motion of the instantaneous axis; and should the newly impressed couple be such as just to move the pole from B to C, the instantaneous axis will then be the principal axis of greatest moment, and will become permanent.

But if the instantaneous pole is shifted from B to a position within the districts which include A or A′, the polhode then becomes a closed curve towards and around these points, a complete circuit of which the instantaneous pole describes. And if the instantaneous rotation-axis is shifted, so as to coincide with the principal axis of least moment, its position becomes permanent.

Now these several results depend on the perpendicular distance between the invariable plane and the fixed plane parallel to it, on which the momental ellipsoid rolls. This distance is given in (60), and the least and the greatest values of it are respectively $g^2\left(\frac{\text{M}}{\text{C}}\right)^{\frac{1}{2}}$ and $g^2\left(\frac{\text{M}}{\text{A}}\right)^{\frac{1}{2}}$: to adapt these mathematical expressions to the image of the present Article, I shall suppose initially $h^2 = \text{B}\,k^2$, and the body to be rotating permanently about its axis of mean moment. Let us suppose the position of the rotation-axis to be shifted by the action of a new couple, whereby h becomes h' and k becomes k'; hereby both the direction-cosines of the invariable axis, and the distance of the plane on which the

ellipsoid rolls from the fixed point, will be changed; no change however will be made in A, B, or C, or in the magnitude of the momental ellipsoid, or in the position of the principal axes of the body relatively to the body. If then after the momentum has been impressed by the new couple,

$$\frac{h'^2}{k'^2} = \frac{h^2}{k^2} = \text{B},$$

the instantaneous axis will move along the plane of one of the dividing ellipses, and ultimately a complete bouleversement of the body will take place. If however

$$\frac{k'^2}{h'^2} \text{ is greater than } \frac{k^2}{h^2},$$

the plane on which the ellipsoid rolls is moved to a greater distance from the fixed point; whereby the instantaneous pole is shifted into one or other of the two districts of the surface of the momental ellipsoid in which is A or A′, and the instantaneous pole moves in a closed curve about A or A′. And if $\frac{k'^2}{h'^2}$ takes the greatest value which it admits of, viz. $\frac{1}{\text{A}}$, the rotation-axis becomes the principal axis of least moment of the ellipsoid, and is permanent.

Again, if
$$\frac{k'^2}{h'^2} \text{ is less than } \frac{k^2}{h^2},$$

the instantaneous pole is shifted into one or other of the districts which contain C or C′, and moves in a closed curve about C or C′. And if the impressed couple is such that $\frac{k'^2}{h'^2}$ takes the least possible value, viz. $\frac{1}{\text{C}}$, the rotation-axis becomes the principal axis of greatest moment, and is permanent.

The angles $\tan^{-1} \pm \left\{\frac{\text{A}(\text{B}-\text{A})}{\text{C}(\text{C}-\text{B})}\right\}^{\frac{1}{2}}$ which determine the position of the dividing ellipses may be taken as the measures of the stability of rotation of the body relatively to the axes of greatest and least moment. Thus the larger these angles are, the larger is the district surrounding the axis of least moment within which, if the instantaneous pole is, the centre of the polhode will be A or A′; and consequently the smaller will be the districts within which the polhodes will have C or C′ for their centres.

And thus we say that the larger these angles are, the more stable is the body relatively to the axis of least moment; and the smaller the angles are, the greater is the stability of rotation relatively to the axis of greatest moment.

Hence, the axes of greatest and of least moments are stable axes; and the axis of mean moment is an unstable axis.

As however a given couple will produce a less deviation with respect to OC as a permanent axis than with respect to OA, the stability of OC is greater than that of OA.

297.] Hence it is evident, that if a body is in motion with one point fixed, and is not under the action of any external force, the principal axes at the point are the only stable or permanent axes. If therefore the body rotates permanently about an axis which is not a principal axis, that permanence can be maintained only by the action of an external force, the couple due to which is equal to and neutralises that which arises from the centrifugal force and disturbs the position of the instantaneous axis. Hence, if L″, M″, N″ are the moments of the components of this couple about the coordinate axes,

$$L'' = (C-B)\omega_2\omega_3, \qquad M'' = (A-C)\omega_3\omega_1, \qquad N'' = (B-A)\omega_1\omega_2;$$

so that if G″ is the moment of the couple,

$$G''^2 = (C-B)^2\omega_2^2\omega_3^2 + (A-C)^2\omega_3^2\omega_1^2 + (B-A)^2\omega_1^2\omega_2^2. \qquad (85)$$

This result also follows from the circumstance that as the position of the instantaneous axis does not vary with the time,

$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0;$$

and the preceding values follow from equations (27), Art. 275.

The axes about which G″ is a minimum have been called by Mr. Walton* axes of greatest or least reluctance, and are indeed those about which the moment of the centrifugal couple is a maximum or a minimum. The positions of these axes, as determined from (85), are given by the following values:

$$(1)\ \omega_1 = 0, \quad \omega_2^2 = \omega_3^2 = \frac{\omega^2}{2}, \quad G'' = \pm(C-B)\frac{\omega^2}{2};$$

$$(2)\ \omega_2 = 0, \quad \omega_3^2 = \omega_1^2 = \frac{\omega^2}{2}, \quad G'' = \pm(A-C)\frac{\omega^2}{2};$$

$$(3)\ \omega_3 = 0, \quad \omega_1^2 = \omega_2^2 = \frac{\omega^2}{2}, \quad G'' = \pm(B-A)\frac{\omega^2}{2};$$

* Quarterly Journal of Pure and Applied Mathematics, Vol. VII. p. 376.

so that there are six such axes, two being in each principal plane of the momental ellipsoid, and bisecting the angles between the principal axes in these planes.

298.] As the preceding theorems have been derived from properties of the momental ellipsoid, other theorems corresponding to them may be derived from the reciprocal ellipsoid, which is the ellipsoid of gyration, by the process of Reciprocation. The following are due to Professor Maccullagh *. We also hereby obtain another image of the motion of the body.

Let a, b, c be the principal axes of gyration, as in Art. 184, and let M be the mass of the body, so that $A = Ma^2$, $B = Mb^2$, $C = Mc^2$; then the equations of motion (41) are

$$\left.\begin{aligned} a^2\frac{d\omega_1}{dt}+(c^2-b^2)\,\omega_2\omega_3 &= 0, \\ b^2\frac{d\omega_2}{dt}+(a^2-c^2)\,\omega_3\omega_1 &= 0, \\ c^2\frac{d\omega_3}{dt}+(b^2-a^2)\,\omega_1\omega_2 &= 0; \end{aligned}\right\} \tag{86}$$

whence if k^2 and h are, as heretofore, respectively the vis viva and the moment of momentum of the whole body, which are constant,

$$a^2\omega_1^{\,2}+b^2\omega_2^{\,2}+c^2\omega_3^{\,2} = \frac{k^2}{M}, \tag{87}$$

$$a^4\omega_1^{\,2}+b^4\omega_2^{\,2}+c^4\omega_3^{\,2} = \frac{h^2}{M^2}; \tag{88}$$

and the equation to the ellipsoid of gyration is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1. \tag{89}$$

Let this ellipsoid be constructed at the given point, and let a line be drawn from the centre lying along the instantaneous axis at the time t; let the plane be drawn perpendicular to this instantaneous axis and touching the ellipsoid; let (x, y, z) be its point of contact, r be the distance of that point from the centre, and p be the length of the perpendicular from the centre on this tangent plane; then

$$\frac{x}{a^2\omega_1} = \frac{y}{b^2\omega_2} = \frac{z}{c^2\omega_3} = \frac{Mr}{h} = \frac{1}{p\omega} = \frac{M^{\frac{1}{2}}}{k}; \tag{90}$$

$$\therefore\quad \omega = \frac{k}{pM^{\frac{1}{2}}}, \qquad r = \frac{h}{kM^{\frac{1}{2}}}. \tag{91}$$

* See Collected Works of James Maccullagh, Dublin, 1880, p. 329.

Hence

(1) If a tangent plane is drawn to the ellipsoid of gyration perpendicular to the instantaneous axis the angular velocity varies inversely as the perpendicular from the fixed point on this tangent plane, and

(2) The radius vector from the centre to the point of contact of the plane is constant, and $= \frac{h}{k\text{M}^{\frac{1}{2}}}$; we will call this quantity r_0.

Also from the first three of the equalities (90), as r lies along the invariable axis, which is fixed in space, it follows that the point of contact is a fixed point on the invariable axis. Let this point be called the invariable point, r_0 being its distance from the fixed point. Hence, taking the value of p_0 which is given in (60) and lies along the same line as r_0, $r_0 p_0 = g^2$; and hence

(3) The ellipsoid of gyration passes always through a fixed point on the invariable axis, the instantaneous axis being perpendicular to the plane which touches the ellipsoid at that point, and the angular velocity being inversely proportional to the perpendicular from the centre on the tangent plane.

Hence the distance of the fixed point from the centre cannot be less than a or greater than c; that is h^2 is not less than $\text{M}a^2 k^2$, and not greater than $\text{M}c^2 k^2$; and it may have these values or any value intermediate to them.

These results are identical with those previously deduced from the properties of the momental ellipsoid.

299.] As the body moves, the locus of the points on its surface which pass through the fixed point is the curve of intersection of the ellipsoid and the sphere whose equations are respectively

$$\left.\begin{aligned} \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} &= 1, \\ x^2+y^2+z^2 &= \frac{h^2}{\text{M}k^2}; \end{aligned}\right\} \tag{92}$$

$\frac{h^2}{\text{M}k^2}$ lying within the limits given above; and the curve is consequently a sphero-conic.

If $\frac{h^2}{\text{M}k^2} = b^2$, this sphero-conic becomes a plane curve, and we have the two circles contained in the cyclic planes; which correspond to the dividing ellipses in the preceding illustration;

and according as $\frac{h^2}{\text{M}k^2}$ is greater or less than b^2 and within its possible limits, we have closed curves about the vertices A and C respectively, and these closed curves degenerate into the vertices A and C when $h^2 = \text{M}a^2k^2$ and $h^2 = \text{M}c^2k^2$ respectively.

Also from (91), if θ is the angle between r and p,

$$\omega \cos \theta = \omega \frac{p}{r} = \frac{k^2}{h},$$

that is, as shewn in Article 279, the component of the angular velocity about the invariable axis is constant throughout the motion.

The criteria as to stability and the measure of stability may be inferred from these expressions by reasoning exactly analogous to that of the preceding articles, and it is not necessary to repeat it.

300.] The following images of the motion of the body also deserve attention.

Since one point of the body is fixed every other point in it moves on the surface of a sphere whose radius is the distance of the point from the fixed point, and the position of any point may be determined by reference to this spherical surface; and if the places of two points not lying in the same central radius vector were given, the position of the body would be determined. These two points might be taken at equal distances from the fixed point, and would thus be on the same spherical surface; and if they were joined by the arc of a great circle, the position of this arc will determine the position of the body. This mode of determination is convenient for astronomical problems.

Also since one point is fixed, a line drawn from that point to any point in the body describes a cone as the body moves, the vertex of the cone being at the given point. Now these cones are of three different characters; (1) a line may be fixed in the body, and by reason of the motion of the body may move in space; (2) a line may be fixed in space, and may move in the body in consequence of the motion of the body; (3) a line may move both in the body and in space; the principal axes of the body are of the first kind; the invariable axis is of the second; the instantaneous axis of rotation, the moment-axis of the centrifugal couple and its rotation-axis are of the third kind. All these lines generate cones as the body moves, some of which are

fixed in the body and others are fixed in space. We propose to investigate these, and especially those which present images of the motion of the body by means of rolling cones, as pointed out in Article 51.

301.] Let us first consider those cones which are generated by the instantaneous axis, both in the body and in space.

As to the former the equations to the instantaneous axis are

$$\frac{x}{\omega_1} = \frac{y}{\omega_2} = \frac{z}{\omega_3}. \tag{93}$$

But from (28) and (29), eliminating the right-hand members, we have

$$\text{A}\,(h^2 - \text{A}\,k^2)\,\omega_1{}^2 + \text{B}\,(h^2 - \text{B}\,k^2)\,\omega_2{}^2 + \text{C}\,(h^2 - \text{C}\,k^2)\,\omega_3{}^2 = 0, \tag{94}$$

whence, by means of (93),

$$\text{A}\,(h^2 - \text{A}\,k^2)\,x^2 + \text{B}\,(h^2 - \text{B}\,k^2)\,y^2 + \text{C}\,(h^2 - \text{C}\,k^2)\,z^2 = 0, \tag{95}$$

which is the equation to a quadric cone, coaxial with the momental ellipsoid, and having its internal axis coincident with the A- or the C-axis according as h^2 is less than or greater than $\text{B}\,k^2$. If $h^2 = \text{B}\,k^2$, the cone degenerates into two planes, which are those of the dividing ellipses. This cone is evidently that whose vertex is at the fixed point, and whose director-curve is the polhode; hence it is called the polhode cone. If two principal moments are equal the cone becomes circular.

The cone which is generated by the instantaneous axis, and is fixed in space, is that which has the fixed point for its vertex, and the herpolhode for its director-curve. The form of it may be a fluted circular cone, having ridges and furrows on its surface which correspond to the undulations of the herpolhode and converge to the vertex; or, as in the case of the momental ellipsoid, the surface may be always concave towards its axis. The equation to this cone is transcendental or finite according to the character of that of the herpolhode. This cone is called the herpolhode cone. If $h^2 = \text{B}\,k^2$, the cone has a convolution ultimately terminating in the line of the invariable axis.

As the instantaneous axis lies on the surface of each of these cones along the line of their tangency, and is at rest for the instant of contact, the motion of the body may be represented by the rolling, without sliding or slipping, of these cones, one on the other; and as they have a common fixed vertex, there is no sliding along the common generating line or line of contact; ω

is the angular velocity with which one rolls on the other; and ϕ' is the angular velocity at the time t of the projection of the instantaneous axis on the invariable plane. The rolling of these cones one on the other gives a perfect representation of the motion of the body.

302.] The following is an instance in which motion of this kind may be exhibited by means of a rolling cone.

Suppose the centre of the earth to be at rest, and not to be moving in its elliptical course in the plane of the ecliptic about the sun; but let the earth have its daily rotation about its own rotation-axis; then this last motion is not affected by the former supposition. Now this axis is with slight variations inclined to the normal of the ecliptic at an angle 23° 28′, which is called the obliquity of the ecliptic; and it is found by observation that the earth's rotation-axis, while retaining the same principal inclination to the normal of the ecliptic, moves backward, that is, in a direction contrary to the earth's rotation, annually through an angle of 50·1″; that is, the earth's axis describes a circular cone fixed in space, whose axis is the normal of the ecliptic and semi-vertical angle is 23° 28′. This cone is called the precessional cone. Now as, by the preceding Article, there can be no change of position of an axis in space without a change of position of it in the body, it follows that the rotation-axis of the earth daily changes its position in the earth, and daily describes a cone about the earth's geometrical axis as its axis. Hence arises a daily variation in terrestrial latitude, and the determination of this variation is a question of interest and importance. It takes the form of two cones rolling on each other, one being fixed in space, and the other fixed in the earth; and the latter rolling outside the former, as the precessional motion is in a direction contrary to the rotation of the earth; and the problem is, having given one cone and the rolling motion of the other on it, to determine the vertical angle of the other cone.

From the centre of the earth, as a centre, describe a sphere whose radius is the earth's polar radius, viz. 3949·6 miles, intersecting the cones in circles. Let R and r be the radii of these circles on their chordal planes. Then, as the annual precession is 50·1″,

$$\frac{\text{arc described in 1 day}}{2\pi \text{R}} = \frac{50{\cdot}1}{365 \times 360 \times 60 \times 60};$$

but the arc described in 1 day $= 2\pi r$, and

$$\text{R} = 3949{\cdot}6 \times \sin 23^\circ 28' \text{ miles};$$

$$\therefore \quad r = \frac{50{\cdot}1 \times 3949{\cdot}6 \times \sin 23^\circ 28' \times 1760 \times 3}{365 \times 360 \times 60 \times 60} \text{ feet}$$

$$= {\cdot}879518 \text{ feet} = 10{\cdot}5542 \text{ inches};$$

so that the instantaneous rotation-pole describes daily on the earth's surface a circle whose radius is 10·554 inches, and whose centre is the geometrical pole. The circumference of this circle is 5·52618 feet, and is the distance through which the instantaneous pole travels in a day, on the hypothesis of an uniform precession of the equinoxes, and of an uniform diurnal rotation of the earth as a rigid body. See Poinsot, Théorie de Rotation, première partie, 33.

303.] The equation to the cone generated in the body by the invariable axis, as the body moves, may thus be found. If (x, y, z) is a point in the invariable axis, the equations to the invariable axis are

$$\frac{x}{\text{A}\,\omega_1} = \frac{y}{\text{B}\,\omega_2} = \frac{z}{\text{C}\,\omega_3}; \tag{96}$$

whence, by substitution in (94), we have

$$\frac{h^2 - \text{A}\,k^2}{\text{A}} x^2 + \frac{h^2 - \text{B}\,k^2}{\text{B}} y^2 + \frac{h^2 - \text{C}\,k^2}{\text{C}} z^2 = 0, \tag{97}$$

which is the equation to a quadric cone in the body, coaxial with the momental ellipsoid, on which the invariable axis in its motion lies. If $h^2 = \text{B}\,k^2$, the cone degenerates into two planes passing through the B-axis of the ellipsoid; and if two principal axes are equal, the cone becomes circular.

As (97) may be put into the form

$$\frac{h^2}{\text{M}} \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right\} - k^2 \left(x^2 + y^2 + z^2 - \frac{h^2}{\text{M}\,k^2} \right) = 0, \tag{98}$$

on comparing this with (92) it appears that the curve of intersection of the ellipsoid of gyration with the concentric sphere, which passes through the invariable point in Maccullagh's interpretation, lies on the surface of this cone; or, in other words, this curve is the direction-curve of the cone.

304.] Hence also we derive the following image of the body's motion. Let the instantaneous angular velocity ω at the time t be resolved into two components, the axis of one of which is the

invariable axis, and the axis of the other is in the invariable plane. Now if θ is the angle between the instantaneous and the invariable axes, $\omega \cos \theta$ is the former component, and is, by reason of (42), Art. 279, constant: the latter component is $\omega \sin \theta$; about the axis of which the body rotates in the time dt through an angle $\omega \sin \theta \, dt$; and thus the invariable axis moves over a surface element of the cone (97); and as this resolution may be continued, so will the invariable axis describe that conical surface. Also the axis of the latter component will continuously change its position in the body, although it is always in the invariable plane; and as it is always perpendicular to a generating line of the cone (97), so will it in its successive positions generate a cone in the moving body which is the reciprocal of (97); and of which consequently the equation is

$$\frac{A x^2}{h^2 - A k^2} + \frac{B y^2}{h^2 - B k^2} + \frac{C z^2}{h^2 - C k^2} = 0; \qquad (99)$$

and this is evidently a cone coaxial with (97), having the same internal axis, and whose major and minor external axes are respectively the minor and major external axes of (97). If $h^2 = B k^2$, $y = 0$; in which case (99) represents two straight lines.

Now in the motion of the body the surface of the cone (97) always contains the invariable axis; and the surface of the second cone is always in contact with the invariable plane. Thus the motion of the body may be represented by the rolling of the cone (99) on the invariable plane, with its own proper angular velocity, while the invariable plane turns about the invariable axis with the constant angular velocity $\omega \cos \theta$. Thus the cone rolls on a plane which at the same time has a constant revolving motion of its own and slides under the cone; and the cone slides as well as rolls, and consequently is called the rolling and sliding cone. While the line of contact describes in the body a cone whose equation is (99), it describes in space the invariable plane.

As the line of contact of this cone with the invariable plane is the line OH of Art. 291, whose equations are therein given, so the amount of the sliding of the cone is measured by the excess of the absolute angular velocity of the body about the invariable axis over that of the component of the angular velocity ω about the same axis: the former is ϕ', as given in (80), Art. 293, and the latter is $\omega \cos \theta$, as given in (42), Art. 279; thus the sliding

on the invariable plane of a point in the cone at a distance r from the fixed point varies as

$$\pm\left(\frac{h^3}{\text{ABC}k^2\text{R}^2}-\frac{k^2}{h}\right)r;$$

where R is the radius vector of the momental ellipsoid which lies along OH.

305.] This mode of representing the motion, by the rolling and sliding on the invariable plane of a cone fixed in the body may also be considered from the following point of view. Let us investigate the equation to the cone which is the envelope of the invariable plane; then we have the three equations,

$$\left.\begin{aligned}\text{A}\,\omega_1 x+\text{B}\,\omega_2 y+\text{C}\,\omega_3 z &= 0,\\ \text{A}\,\omega_1^2+\text{B}\,\omega_2^2+\text{C}\,\omega_3^2 &= k^2,\\ \text{A}^2\omega_1^2+\text{B}^2\omega_2^2+\text{C}^2\omega_3^2 &= h^2;\end{aligned}\right\}\tag{100}$$

whence differentiating, and eliminating $\omega_1, \omega_2, \omega_3$ by the usual process of indeterminate multipliers, the equation to the envelope is found to be

$$\frac{\text{A}\,x^2}{h^2-\text{A}\,k^2}+\frac{\text{B}\,y^2}{h^2-\text{B}\,k^2}+\frac{\text{C}\,z^2}{h^2-\text{C}\,k^2}=0,\tag{101}$$

which is the equation of a quadric cone, and is the surface-envelope of the invariable plane. This cone is of course the same as (99). The equation to the cone which is reciprocal to this, and is generated by the invariable axis, is of course that given in (97). Thus the motion may be represented by the rolling of the cone (101) on the invariable plane; but as the invariable plane has also an uniform rotation about the invariable axis, which lies in the surface of (97), the cone has an uniform sliding motion, as also a variable rolling motion on the invariable plane.

306.] The rotation-axis due to the centrifugal forces also describes a cone in the body, and in space describes the invariable plane. As to the cone described in the body, the equations to the axis are those to OK, see Art. 291,

$$\frac{\text{A}\,\omega_1}{\text{B}-\text{C}}\,x=\frac{\text{B}\,\omega_2}{\text{C}-\text{A}}\,y=\frac{\text{C}\,\omega_3}{\text{A}-\text{B}}\,z:\tag{102}$$

whence eliminating $\omega_1, \omega_2, \omega_3$ by means of (94), we have

$$(h^2-\text{A}k^2)\frac{(\text{B}-\text{C})^2}{\text{A}x^2}+(h^2-\text{B}k^2)\frac{(\text{C}-\text{A})^2}{\text{B}y^2}+(h^2-\text{C}k^2)\frac{(\text{A}-\text{B})^2}{\text{C}z^2}=0,\tag{103}$$

which is the equation to a cone of the fourth order; and becomes the two planes of the dividing ellipse, when $h^2 = \text{B}k^2$.

As the rotation-axis of the centrifugal forces, see Art. 291, is always perpendicular to the invariable axis, it lies in the invariable plane, and the invariable plane is consequently the surface generated by it and fixed in space.

307.] The following are certain properties of the principal axes of the momental ellipsoid in reference to the invariable axis and the invariable plane.

Let $\text{R}_1, \text{R}_2, \text{R}_3$ be the principal semi-axes of the ellipsoid, and let $\text{R}_a, \text{R}_b, \text{R}_c$ be the radii vectores to the fixed plane of rolling which lie along the principal axes; then if a, b, c are the principal radii of gyration, we have from Art. 184

$$a^2\text{R}_1{}^2 = b^2\text{R}_2{}^2 = c^2\text{R}_3{}^2 = g^4; \tag{104}$$

and from (59) we have

$$a^2\omega_1\text{R}_a = b^2\omega_2\text{R}_b = c^2\omega_3\text{R}_c = \frac{kg^2}{\text{M}^{\frac{1}{2}}}, \tag{105}$$

and if λ, μ, ν are the direction-angles of the invariable axis with reference to the principal axes,

$$\frac{\cos\lambda}{a^2\omega_1} = \frac{\cos\mu}{b^2\omega_2} = \frac{\cos\nu}{c^2\omega_3} = \frac{\text{M}}{h};$$

hence

$$(\text{R}_1\cos\lambda)^2 + (\text{R}_2\cos\mu)^2 + (\text{R}_3\cos\nu)^2 = \frac{\text{M}k^2g^4}{h^2}$$

$$= p_0{}^2 \text{ by } (60);$$

that is, the sum of the squares of the projections of the principal axes of the momental ellipsoid on the invariable axis is equal to the square of the perpendicular from the fixed point on the fixed plane of rolling.

Also from (105)

$$\frac{1}{\text{R}_a{}^2} + \frac{1}{\text{R}_b{}^2} + \frac{1}{\text{R}_c{}^2} = \frac{h^2}{\text{M}k^2g^4};$$

also

$$\frac{1}{a^2\text{R}_a{}^2} + \frac{1}{b^2\text{R}_b{}^2} + \frac{1}{c^2\text{R}_c{}^2} = \frac{1}{g^4};$$

$$\frac{1}{a^4\text{R}_a{}^2} + \frac{1}{b^4\text{R}_b{}^2} + \frac{1}{c^4\text{R}_c{}^2} = \frac{r^2}{g^8},$$

where r is the central radius vector of the instantaneous pole.

308.] We proceed now to the investigation of the problem which has been stated in Art. 282, viz., to the determination of $\omega_1, \omega_2, \omega_3$ as functions of the time, and consequently to the values

of θ, ϕ and ψ in terms of t, as given in (50) and (52) of that Article.

Let equations (41) be severally multiplied by $\frac{\omega_1}{\text{A}}$, $\frac{\omega_2}{\text{B}}$, $\frac{\omega_3}{\text{C}}$, and added; then

$$\omega_1 \frac{d\omega_1}{dt} + \omega_2 \frac{d\omega_2}{dt} + \omega_3 \frac{d\omega_3}{dt} = -\left\{ \frac{\text{C}-\text{B}}{\text{A}} + \frac{\text{A}-\text{C}}{\text{B}} + \frac{\text{B}-\text{A}}{\text{C}} \right\} \omega_1 \omega_2 \omega_3$$
$$= \frac{(\text{C}-\text{B})(\text{A}-\text{C})(\text{B}-\text{A})}{\text{ABC}} \omega_1 \omega_2 \omega_3. \qquad (106)$$

But since $\omega^2 = \omega_1^2 + \omega_2^2 + \omega_3^2$, the left-hand member $= \omega \frac{d\omega}{dt}$;

therefore
$$\omega \frac{d\omega}{dt} = \frac{(\text{C}-\text{B})(\text{A}-\text{C})(\text{B}-\text{A})}{\text{ABC}} \omega_1 \omega_2 \omega_3. \qquad (107)$$

Now $\omega_1, \omega_2, \omega_3$ may be expressed in terms of ω by means of the three equations

$$\left.\begin{aligned} \omega_1^2 + \omega_2^2 + \omega_3^2 &= \omega^2, \\ \text{A}\omega_1^2 + \text{B}\omega_2^2 + \text{C}\omega_3^2 &= k^2, \\ \text{A}^2\omega_1^2 + \text{B}^2\omega_2^2 + \text{C}^2\omega_3^2 &= h^2; \end{aligned}\right\} \qquad (108)$$

whence

$$\left.\begin{aligned} (\text{A}-\text{C})(\text{B}-\text{A})\omega_1^2 &= -\{\text{BC}\omega^2 - (\text{B}+\text{C})k^2 + h^2\}, \\ (\text{B}-\text{A})(\text{C}-\text{B})\omega_2^2 &= -\{\text{CA}\omega^2 - (\text{C}+\text{A})k^2 + h^2\}, \\ (\text{C}-\text{B})(\text{A}-\text{C})\omega_3^2 &= -\{\text{AB}\omega^2 - (\text{A}+\text{B})k^2 + h^2\}; \end{aligned}\right\} \qquad (109)$$

and substituting in (107), the right-hand member is expressed in terms of the single variable ω. These expressions may be put into a more convenient form by the following substitutions. Let

$$\frac{(\text{B}+\text{C})k^2 - h^2}{\text{BC}} = \theta_1^2, \quad \frac{(\text{C}+\text{A})k^2 - h^2}{\text{CA}} = \theta_2^2, \quad \frac{(\text{A}+\text{B})k^2 - h^2}{\text{AB}} = \theta_3^2; \quad (110)$$

then

$$\left.\begin{aligned} \omega_1^2 &= -\frac{\text{BC}}{(\text{A}-\text{C})(\text{B}-\text{A})}(\omega^2 - \theta_1^2) = \frac{\text{BC}}{(\text{C}-\text{A})(\text{B}-\text{A})}(\omega^2 - \theta_1^2), \\ \omega_2^2 &= -\frac{\text{CA}}{(\text{B}-\text{A})(\text{C}-\text{B})}(\omega^2 - \theta_2^2), \\ \omega_3^2 &= -\frac{\text{AB}}{(\text{C}-\text{B})(\text{A}-\text{C})}(\omega^2 - \theta_3^2) = \frac{\text{AB}}{(\text{C}-\text{B})(\text{C}-\text{A})}(\omega^2 - \theta_3^2); \end{aligned}\right\} \quad (111)$$

then substituting these in (107), we have

$$\omega \frac{d\omega}{dt} = \{-(\omega^2 - \theta_1^2)(\omega^2 - \theta_2^2)(\omega^2 - \theta_3^2)\}^{\frac{1}{2}}, \qquad (112)$$

which, when integrated, gives ω in terms of t; but being an elliptic function must be dealt with as such; unless in certain cases the initial circumstances or the constitution of the body reduce it to an algebraical or a circular or a logarithmic function.

Now θ_1^2, θ_2^2, θ_3^2 are the roots of the expression in the right-hand member of (112); and are all essentially positive, as may thus be shewn; thus if we take θ_1^2, and replace h^2 and k^2 by their values given in (108),

$$\theta_1^2 = \frac{\text{A}(\text{B}+\text{C}-\text{A})\omega_1^2 + \text{BC}(\omega_2^2+\omega_3^2)}{\text{BC}}, \tag{113}$$

which is positive, as A, B, C are all positive quantities, and B + C is always greater than A. Similarly θ_2^2 and θ_3^2 may be shewn to be positive quantities.

As to the order of magnitude of θ_1^2, θ_2^2, θ_3^2, the following are their differences, viz.

$$\theta_2^2 - \theta_3^2 = \frac{\text{B}-\text{C}}{\text{ABC}}(\text{A}k^2 - h^2), \quad \theta_3^2 - \theta_1^2 = \frac{\text{C}-\text{A}}{\text{ABC}}(\text{B}k^2 - h^2),$$

$$\theta_1^2 - \theta_2^2 = \frac{\text{A}-\text{B}}{\text{ABC}}(\text{C}k^2 - h^2);$$

hence, bearing in mind that $\text{C} > \text{B} > \text{A}$, that $\text{A}k^2 - h^2$ is never positive, that $\text{C}k^2 - h^2$ is never negative, and $\text{B}k^2 - h^2$ is positive or negative according as the polhode is a closed curve about the vertex A or the vertex C, that is according to the initial circumstances of motion and the constitution of the body, it follows that θ_2^2 is the greatest of the three quantities and that θ_1^2 or θ_3^2 is the least, the former or the latter being the case according as the polhode is closed round the vertex C or the vertex A of the momental ellipsoid. If $\text{B}k^2 - h^2$, that is if the polhode is one of the dividing ellipses, $\theta_3 = \theta_1$, and the two lesser roots of the right-hand member of (112) are equal.

It appears also from (112) that of the three factors $\omega^2 - \theta_1^2$, $\omega^2 - \theta_2^2$, $\omega^2 - \theta_3^2$, one must be negative and two positive, or all three must be negative: that is ω^2 must be less than θ_2^2 and greater than either θ_1^2 or θ_3^2, or ω^2 must be less than the least of θ_1^2 and θ_3^2. These latter values are excluded by the values of ω_1^2 and ω_3^2 which are given in (111); so that ω^2 can never be greater than θ_2^2, or less than the greater of θ_1^2 and θ_3^2.

Thus θ_2^2 and the greater of θ_1^2 and θ_3^2 will be the limits of ω^2 corresponding to the time occupied in the motion of the instantaneous axis through one quarter of a polhode, which is a closed curve; and if T is the time through the complete polhode, and θ_1^2 is greater than θ_3^2,

$$\text{T} = 4\int_{\theta_1}^{\theta_2} \frac{\omega\, d\omega}{\{-(\omega^2-\theta_1^2)(\omega^2-\theta_2^2)(\omega^2-\theta_3^2)\}^{\frac{1}{2}}}. \tag{114}$$

309.] To simplify (112) let us suppose θ_1 to be greater than θ_3; so that the corresponding polhode is a closed curve about the vertex C: and let the following substitution be made; viz.

$$\omega^2 = \theta_2^2(\cos u)^2 + \theta_1^2(\sin u)^2; \qquad (115)$$

then $\omega = \theta_1$ when $u = 90°$, and $= \theta_2$ when $u = 0$; and

$$\therefore \quad \omega\, d\omega = -(\theta_2^2 - \theta_1^2)\sin u \cos u\, du,$$

and (112) becomes

$$\frac{du}{dt} = -\{\theta_2^2 - \theta_3^2 - (\theta_2^2 - \theta_1^2)(\sin u)^2\}^{\frac{1}{2}}. \qquad (116)$$

Let $\quad (\theta_2^2 - \theta_3^2)^{\frac{1}{2}} = \left\{-\frac{\text{C}-\text{B}}{\text{ABC}}(\text{A}k^2 - h^2)\right\}^{\frac{1}{2}} = -n$, and

$$\frac{\theta_2^2 - \theta_1^2}{\theta_2^2 - \theta_3^2} = -\frac{\text{B}-\text{A}}{\text{C}-\text{B}}\,\frac{\text{C}k^2 - h^2}{\text{A}k^2 - h^2} = \kappa^2;$$

so that, as θ_1 is greater than θ_3, κ is less than unity; then

$$n\,dt = \frac{du}{\{1-\kappa^2(\sin u)^2\}^{\frac{1}{2}}};$$

and if the limits of integration are such that $u = 0$ when $t = 0$, and u corresponds to t,

$$nt = \int_0^u \frac{du}{\{1-\kappa^2(\sin u)^2\}^{\frac{1}{2}}}; \qquad (117)$$

and this is an elliptic function of the first order; so that if T is the time in which the instantaneous axis moves through the complete circuit of a closed polhode,

$$\text{T} = \frac{4}{n}\int_0^{\frac{\pi}{2}} \frac{du}{\{1-\kappa^2(\sin u)^2\}^{\frac{1}{2}}}.$$

Now introducing the ordinary notation of elliptic functions, we have $u = \text{am}\,(nt)$, $1-\kappa^2(\sin u)^2 = \{\Delta\,\text{am}\,(nt)\}^2$, so that

$$\frac{du}{d.nt} = \Delta\,\text{am}\,(nt);$$

$$\therefore \quad \omega^2 = \theta_2^2\{\cos\text{am}\,(nt)\}^2 + \theta_1^2\{\sin\text{am}\,(nt)\}^2;$$

and hence from (111) we have

$$\left.\begin{aligned}
\omega_1^2 &= \frac{\text{BC}}{(\text{C}-\text{A})(\text{B}-\text{A})}(\omega^2-\theta_1^2) = \frac{\text{C}k^2-h^2}{\text{A}(\text{C}-\text{A})}\{\cos\text{am}\,(nt)\}^2,\\
\omega_2^2 &= -\frac{\text{CA}}{(\text{B}-\text{A})(\text{C}-\text{B})}(\omega^2-\theta_2^2) = \frac{\text{C}k^2-h^2}{\text{B}(\text{C}-\text{B})}\{\sin\text{am}\,(nt)\}^2,\\
\omega_3^2 &= \frac{\text{AB}}{(\text{C}-\text{B})(\text{C}-\text{A})}(\omega^2-\theta_3^2) = -\frac{\text{A}k^2-h^2}{\text{C}(\text{C}-\text{A})}\{\Delta\text{am}(nt)\}^2;
\end{aligned}\right\} \qquad (118)$$

and thus these angular velocities are expressed in terms of t.

Hence also

$$a_3 = \sin\phi \sin\theta = \frac{\text{A}\,\omega_1}{h} = \left\{\frac{\text{A}(\text{C}k^2 - h^2)}{\text{C}-\text{A}}\right\}^{\frac{1}{2}} \frac{\cos \text{am}(nt)}{h},$$

$$b_3 = \cos\phi \sin\theta = \frac{\text{B}\,\omega_2}{h} = \left\{\frac{\text{B}(\text{C}k^2 - h^2)}{\text{C}-\text{B}}\right\}^{\frac{1}{2}} \frac{\sin \text{am}(nt)}{h},$$

$$c_3 = \cos\theta = \frac{\text{C}\,\omega_3}{h} = \left\{\frac{-\text{C}(\text{A}k^2 - h^2)}{\text{C}-\text{A}}\right\}^{\frac{1}{2}} \frac{\Delta \text{am}(nt)}{h},$$

which assign θ and ϕ; and hence also

$$\frac{d\psi}{dt} = \frac{\text{A}\,\omega_1^{\,2} + \text{B}\,\omega_2^{\,2}}{\text{A}^2\,\omega_1^{\,2} + \text{B}^2\omega_2^{\,2}}\, h = \frac{\text{C}-\text{B}+(\text{B}-\text{A})\{\sin \text{am}(nt)\}^2}{\text{A}(\text{C}-\text{B})+\text{C}(\text{B}-\text{A})\{\sin \text{am}(nt)\}^2}\, h;$$

which assigns the precessional velocity in the plane of (x, y) in terms of t.

But what angle does u represent? It has been introduced into the investigation, as a subsidiary angle to simplify expressions, by the equation (115): its geometrical meaning may thus be shewn.

Let the closed polhode about the vertex C be projected on the plane of (x, y); then the projection is the ellipse whose equation is (65), Art. 285, and is

$$\text{A}(\text{C}-\text{A})\,x^2 + \text{B}(\text{C}-\text{B})\,y^2 = \frac{\text{C}k^2 - h^2}{k^2}\,\text{M}g^4;$$

and u is evidently the eccentric angle of this ellipse; for since by (56), Art. 283,

$$\frac{x^2}{\omega_1^{\,2}} = \frac{y^2}{\omega_2^{\,2}} = \frac{\text{M}g^4}{k^2},$$

and from (118),

$$(\cos u)^2 = \frac{\text{A}(\text{C}-\text{A})}{\text{C}k^2 - h^2}\,\omega_1^{\,2} = \frac{\text{A}(\text{C}-\text{A})}{\text{C}k^2 - h^2}\,\frac{k^2}{\text{M}g^4}\,x^2,$$

$$(\sin u)^2 = \frac{\text{B}(\text{C}-\text{B})}{\text{C}k^2 - h^2}\,\omega_2^{\,2} = \frac{\text{B}(\text{C}-\text{B})}{\text{C}k^2 - h^2}\,\frac{k^2}{\text{M}g^4}\,y^2;$$

$$\therefore\quad \text{A}(\text{C}-\text{A})\,x^2 + \text{B}(\text{C}-\text{B})\,y^2 = \frac{\text{C}k^2 - h^2}{k^2}\,\text{M}g^4,$$

which is the equation to the preceding ellipse; consequently $u = \text{am}(nt)$ is the eccentric angle of the ellipse, which is the projection of the polhode on the principal plane (A, B) of the momental ellipsoid.

310.] It is thus evident that ω^2 is a periodic function, of which the maximum value is $\theta_2^{\,2}$, and the minimum is the greater of $\theta_1^{\,2}$ and $\theta_3^{\,2}$; the former or latter being the case according as the polhodes are closed curves about the vertex C or about the vertex A of the momental ellipsoid.

It will also be observed that (118) is an equation of the same form as (43), Art. 245, which determines the time of motion of a heavy body about a fixed horizontal axis.

Also since ω varies as r, see (58), Art. 283, where r is the radius vector of the momental ellipsoid which lies along the instantaneous axis, the critical values of ω and r are simultaneous. Also it follows from Art. 289 that the critical values of the radii vectores of the herpolhode are simultaneous with those of the radii vectores of the polhode. Hence the angular velocity of the body is a maximum when the instantaneous axis lies along a ridge of the herpolhode cone, and is a minimum when it lies along a trough or furrow.

311.] I have chosen the preceding process for expressing the coordinates of position in terms of t, because it follows directly from the equations of motion, and leads to values of ω_1, ω_2, ω_3 in terms of which θ, ϕ, ψ, and consequently the place of the body, are given. The results, however, admit of the following interpretation by means of the polhode and herpolhode curves.

Let (x, y, z) be the instantaneous pole at the time t, so that its locus on the surface of the ellipsoid is the polhode. Then from (56) we have

$$\frac{\omega_1}{x} = \frac{\omega_2}{y} = \frac{\omega_3}{z} = \frac{\omega}{r} = \frac{k}{g^2 \mathrm{M}^{\frac{1}{2}}} = \mu \text{ (say)};$$

then, substituting in (108), we have

$$\left.\begin{aligned} x^2 + y^2 + z^2 &= r^2, \\ \mathrm{A}x^2 + \mathrm{B}y^2 + \mathrm{C}z^2 &= \mathrm{M}g^4, \\ \mathrm{A}^2 x^2 + \mathrm{B}^2 y^2 + \mathrm{C}^2 z^2 &= \frac{h^2 g^4 \mathrm{M}}{k^2}; \end{aligned}\right\} \qquad (119)$$

which are the equations to the polhode, r being the radius vector.

Hence also (112) becomes

$$\frac{r\,dr}{dt} = \mu \left\{ -\left(r^2 - \frac{\theta_1^{\,2}}{\mu^2}\right)\left(r^2 - \frac{\theta_2^{\,2}}{\mu^2}\right)\left(r^2 - \frac{\theta_3^{\,2}}{\mu^2}\right)\right\}^{\frac{1}{2}}, \qquad (120)$$

which assigns the relation between r and t. If, then, T is the time in which the instantaneous axis describes the complete circuit of the polhode cone, and r_1 and r_2 are the maximum and minimum values of r, then

$$\mathrm{T} = 4 \int_{r_1}^{r_2} \frac{r\,dr}{\mu \left\{ -\left(r^2 - \frac{\theta_1^{\,2}}{\mu^2}\right)\left(r^2 - \frac{\theta_2^{\,2}}{\mu^2}\right)\left(r^2 - \frac{\theta_3^{\,2}}{\mu^2}\right)\right\}^{\frac{1}{2}}},$$

where $r_1 = \frac{\theta_2}{\mu}$, and $r_2 = \frac{\theta_1}{\mu}$, or $\frac{\theta_3}{\mu}$ according as the polhode is closed round the vertex C or the vertex A.

T is also the time in which one complete portion of the herpolhode is described.

Hence also the velocity with which the instantaneous pole describes the polhode or the herpolhode may be found. Let ds be the arc of either curve described in the time dt; then

$$\frac{ds^2}{dt^2} = \frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + \frac{dz^2}{dt^2}$$

$$= \frac{1}{\mu^2}\left\{\frac{d\omega_1^2}{dt^2} + \frac{d\omega_2^2}{dt^2} + \frac{d\omega_3^2}{dt^2}\right\}$$

$$= \frac{1}{\mu^2}\left\{\frac{(\text{C}-\text{B})^2}{\text{A}^2}\,\omega_2^2\omega_3^2 + \frac{(\text{A}-\text{C})^2}{\text{B}^2}\,\omega_3^2\omega_1^2 + \frac{(\text{B}-\text{A})^2}{\text{C}^2}\,\omega_1^2\omega_2^2\right\} \quad (121)$$

$$= \frac{1}{\mu^2}\left\{\frac{\text{BC}}{(\text{B}-\text{A})(\text{A}-\text{C})}(\omega^2-\theta_2^2)(\omega^2-\theta_3^2) + \frac{\text{CA}}{(\text{C}-\text{B})(\text{B}-\text{A})}(\omega^2-\theta_3^2)(\omega^2-\theta_1^2)\right.$$
$$\left. + \frac{\text{AB}}{(\text{A}-\text{C})(\text{C}-\text{B})}(\omega^2-\theta_1^2)(\omega^2-\theta_2^2)\right\}$$

$$= \frac{\text{ABC}}{\mu^2(\text{C}-\text{B})(\text{A}-\text{C})(\text{B}-\text{A})}\left\{\frac{\text{C}-\text{B}}{\text{A}}(\mu^2r^2-\theta_2^2)(\mu^2r^2-\theta_3^2)\right.$$
$$\left. + \frac{\text{A}-\text{C}}{\text{B}}(\mu^2r^2-\theta_3^2)(\mu^2r^2-\theta_1^2) + \frac{\text{B}-\text{A}}{\text{C}}(\mu^2r^2-\theta_1^2)(\mu^2r^2-\theta_2^2)\right\}; \quad (122)$$

which gives the velocity in terms of the radius vector.

If dt is eliminated between (120) and (122), we obtain ds in terms of dr, so that by integration s can be expressed in terms of r; and if the limits of integration are r_1 and r_2, the length of the closed polhode curve will be found.

312.] It will have been observed that neither in Poinsot's construction nor in that of Maccullagh is any direct geometrical measure given of the time occupied in the motion of the body from one position to another. This omission has been supplied by Professor Sylvester by the following construction, which is set out at length in Vol. 156 of the Philosophical Transactions, 1866.

At the fixed point O let the momental ellipsoid be constructed, and let $\text{R}_1, \text{R}_2, \text{R}_3$ be its principal semi-axes, so that these quantities vary inversely as the square roots of the principal moments corresponding to them; that is

$$\text{A}\text{R}_1^2 = \text{B}\text{R}_2^2 = \text{C}\text{R}_3^2 = \text{M}g^4; \quad (123)$$

and the equation to the ellipsoid referred to its principal axes is

$$\frac{x^2}{R_1^{\,2}} + \frac{y^2}{R_2^{\,2}} + \frac{z}{R_3^{\,2}} = 1.$$

It is this ellipsoid which rolls on the fixed plane, perpendicular to the invariable axis and at a distance p_0 from the fixed point, and thereby exhibits the motion of the body.

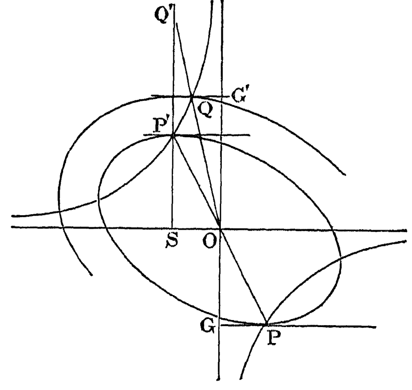
At the fixed point O let another ellipsoid be constructed, confocal with the momental ellipsoid, and let its equation be

$$\frac{x^2}{R_1^{\,2}+\theta} + \frac{y^2}{R_2^{\,2}+\theta} + \frac{z^2}{R_3^{\,2}+\theta} = 1\,; \tag{124}$$

and let p be the distance from O at which the tangent plane to this ellipsoid, which is perpendicular to the invariable axis, cuts that axis; then by the properties of confocal ellipsoids,

$$p^2 = p_0^{\,2} + \theta,$$

so that as p_0 is constant, and θ is constant for the confocal ellipsoid, p is also constant.



Now if P and Q are the points of contact of planes perpendicular to the invariable axis, and consequently parallel, with the momental ellipsoid and the confocal ellipsoid respectively, then, by Art. 28, P, Q, O and the invariable axis are all in one plane, and P and Q lie in an equilateral hyperbola whose centre is O, and of which the invariable axis is an asymptote; the other asymptote consequently lying in the invariable plane.

Let PO be produced to OP′, making OP′ = OP, and through P′ let a plane be drawn perpendicular to the invariable axis; then P′ is on the momental ellipsoid, and is an instantaneous pole exactly like P, the planes passing through P and P′ perpendicular to the invariable axis being planes on which the ellipsoid rolls without sliding; and these are the only planes which have that property. Let OP = OP′ = r, OQ = r', and through P′ draw P′Q′ parallel to the invariable axis and cutting OQ, or OQ produced, in Q′.

Now if ω is the angular velocity of the body about OP, taking its value as given in the preceding Article, we have $\omega = \mu r$.

Let ω', ω'' be the components of this angular velocity about OQ and the invariable axis respectively, so that

$$\frac{\omega}{\text{OP}'} = \frac{\omega'}{\text{OQ}'} = \frac{\omega''}{\text{P}'\text{Q}'};$$

and bearing in mind that P′ and Q are on an equilateral hyperbola of which the invariable axis is one asymptote,

$$\text{OQ}' = \frac{pr'}{p_0}, \quad \text{P}'\text{Q}' = \frac{p^2 - p_0^{\,2}}{p_0} = \frac{\theta}{p_0};$$

$$\therefore \quad \omega' = \frac{hpr'}{\text{M}g^4}, \quad \omega'' = \frac{h\theta}{\text{M}g^4}; \tag{125}$$

so that ω'' is constant for the given confocal ellipsoid in all its positions. Thus the angular velocity ω has been resolved into two angular velocities, one, viz. ω' about OQ, which varies as OQ and is consequently variable; and the other, viz. ω'', about the invariable axis, which is constant. If then the plane which is perpendicular to the invariable axis, and at a distance p from the fixed point O, is perfectly rough and capable of rotating about the invariable axis, the angle through which it rotates in any time, owing to the motion of the ellipsoid which it always touches, varies as the time, and can be read off by means of a dial plate and a fixed arm from which angles can be measured. If this angle is ψ, so that $\frac{d\psi}{dt} = \frac{h\theta}{\text{M}g^4}$, $\psi = \frac{h\theta t}{\text{M}g^4}$, and we have a perfect mechanical contrivance for determining the time occupied in the passage of the body from one position to another. The point Q evidently describes a polhode on the confocal ellipsoid, and a herpolhode on the revolving plate. In reference to the revolving plate, the larger the value of θ, that is, the greater the difference between the distances of the parallel tangent planes from O, the greater will be the angle through which the rotating plate will move in a given time.

313.] It remains for us to consider certain special forms which the preceding expressions take under particular circumstances of the moving body in relation to the constant vis viva and to the constant moment of momentum about the invariable axis.

If $h^2 = \text{A}k^2$, the axis of least moment is the rotation-axis, and is a permanent axis; and consequently $\omega_2 = \omega_3 = 0$; and the angular velocity is constant, and $\text{A}\omega = h$.

Similarly, if $h^2 = \text{C}k^2$, the axis of greatest moment is the rotation-axis and is a permanent axis; and consequently $\omega_1 = \omega_2 = 0$, and the angular velocity is constant, and $\text{C}\omega = h$.

Hence if h is given, or if k is given, the resultant angular velocity will be greatest when the rotation-axis is the axis of least moment, and will be the least when the rotation-axis is the axis of greatest moment.

In these cases of constant angular velocities and permanent rotation-axes, the time varies as the angle described.

If $h^2 = \text{B}k^2$, see Art. 289, equations (28) and (29) become

$$\text{A}\,\omega_1^2 + \text{B}\,\omega_2^2 + \text{C}\,\omega_3^2 = k^2,$$
$$\text{A}^2\omega_1^2 + \text{B}^2\omega_2^2 + \text{C}^2\omega_3^2 = h^2 = \text{B}k^2\,;$$
$$\therefore\quad \frac{\text{A}\,\omega_1^2}{\text{C}-\text{B}} = \frac{k^2 - \text{B}\,\omega_2^2}{\text{C}-\text{A}} = \frac{\text{C}\,\omega_3^2}{\text{B}-\text{A}}. \tag{126}$$

Now substituting in the second equation of (41) the values of ω_1 and ω_3 in terms of ω_2, we have

$$\text{B}^2\frac{d\,\omega_2}{dt} = \left\{\frac{(\text{C}-\text{B})(\text{B}-\text{A})}{\text{A}\,\text{C}}\right\}^{\frac{1}{2}}(h^2 - \text{B}^2\omega_2^2). \tag{127}$$

Let $4h^2(\text{C}-\text{B})(\text{B}-\text{A}) = m^2\text{A}\text{B}^2\text{C}$; then integrating, we have

$$\log \text{K}\,\frac{h+\text{B}\,\omega_2}{h-\text{B}\,\omega_2} = mt, \tag{128}$$

where K is the constant of integration depending on the circumstances of motion at a given time. Thus, suppose that $\omega_2 = \Omega_2$ when $t = 0$, and that the axis of mean moment was then inclined to the invariable axis at an angle 2β, so that $\text{B}\,\Omega_2 = h\cos 2\beta$, then $\text{K} = (\tan\beta)^2$. Retaining however K, we have from the preceding equations

$$\omega_2 = \frac{h}{\text{B}}\,\frac{\epsilon^{mt} - \text{K}}{\epsilon^{mt} + \text{K}}\,;$$
$$\therefore\quad \omega_1^2 = \frac{4\text{K}h^2(\text{C}-\text{B})}{\text{A}\text{B}(\text{C}-\text{A})}\,\frac{\epsilon^{mt}}{(\epsilon^{mt}+\text{K})^2},$$
$$\omega_3^2 = \frac{4\text{K}h^2(\text{B}-\text{A})}{\text{B}\text{C}(\text{C}-\text{A})}\,\frac{\epsilon^{mt}}{(\epsilon^{mt}+\text{K})^2}.$$

These results may also be deduced from (112), since in this case $\theta_1^2 = \theta_3^2$. If we take the invariable plane to be that of (x, y), and the invariable axis for the axis of z, then, as in Art. 309,

$$\cos\theta = \frac{2}{\epsilon^{mt}+\text{K}}\left\{\frac{\text{K}\text{C}(\text{B}-\text{A})\,\epsilon^{mt}}{\text{B}(\text{C}-\text{A})}\right\}^{\frac{1}{2}},$$

$$\tan\phi = \frac{2}{\epsilon^{mt}-\text{K}}\left\{\frac{\text{KA}(\text{C}-\text{B})\epsilon^{mt}}{\text{B}(\text{C}-\text{A})}\right\}^{\frac{1}{2}},$$

$$\frac{d\psi}{dt} = h\frac{(\text{C}-\text{B})(\epsilon^{mt}+\text{K})^2+(\text{B}-\text{A})(\epsilon^{mt}-\text{K})^2}{\text{A}(\text{C}-\text{B})(\epsilon^{mt}+\text{K})^2+\text{C}(\text{B}-\text{A})(\epsilon^{mt}-\text{K})^2};$$

whence, by integration, ψ will be given in terms of t, and the problem will be completely solved.

From (128) it appears that $t = \infty$, when $\text{B}\omega_2 = \pm h$; in which case, as appears from (126), $\omega_1 = \omega_3 = 0$, and consequently the body revolves about the mean axis of the momental ellipsoid, which is a permanent axis. The polhode is the dividing ellipse, along which the instantaneous pole travels, and the rotation-axis ultimately coincides with the mean axis of the ellipsoid. The herpolhode is the spiral which has been described in Art. 289, and the point of contact of the ellipsoid with the fixed plane coincides with the pole of this spiral only when $t = \infty$.

314.] Also other special cases arise when the moving body is of a special form in respect of two or more of the principal moments at the fixed point being equal.

If $\text{B} = \text{A}$, so that the greatest moment is the unequal moment, then, from the last of (41), $\frac{d\omega_3}{dt} = 0$, and consequently ω_3 is a constant $= n$, say; and the first two equations of (41) become

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt}+(\text{C}-\text{A})n\omega_2 &= 0, \\ \text{A}\frac{d\omega_2}{dt}-(\text{C}-\text{A})n\omega_1 &= 0; \end{aligned}\right\} \qquad (129)$$

$$\therefore\quad \omega_1 d\omega_1 + \omega_2 d\omega_2 = 0,$$

$$\omega_1{}^2+\omega_2{}^2 = \text{a constant} = m^2, \text{ say};$$

$$\omega^2 = \omega_1{}^2+\omega_2{}^2+\omega_3{}^2 = m^2+n^2,$$

so that the angular velocity about the instantaneous axis, and also that about the axis of greatest moment is constant; and if γ is the angle between the instantaneous axis and the axis of greatest moment

$$\cos\gamma = \frac{\omega_3}{\omega} = \frac{n}{(m^2+n^2)^{\frac{1}{2}}},$$

which is constant. Also the angle between the invariable axis and the axis of greatest moment is constant: hence the instantaneous axis describes a circular cone around the axis of greatest moment, whose semi-vertical angle is γ.

To express ω_1 and ω_2 in terms of t, from (129) we have

$$\frac{d\omega_1}{(m^2-\omega_1^2)^{\frac{1}{2}}} = \frac{\text{C}-\text{A}}{\text{A}} n\,dt$$
$$= \mu\,dt,$$
$$\text{if} \quad (\text{C}-\text{A})n = \mu \text{A};$$
$$\therefore \quad \omega_1 = m \sin(\mu t + a),$$
$$\omega_2 = m \cos(\mu t + a):$$

where a is a constant introduced in integration. From these values it follows that the instantaneous axis moves over the surface of the cone with a constant angular velocity equal to μ.

If we refer these to the invariable plane as the plane of (x, y), and to the invariable axis as the axis of z, we have, from Art. 282,

$$\cos\theta = \frac{\text{C}n}{h} = \frac{\text{C}n}{(\text{A}^2 m^2 + \text{C}^2 n^2)^{\frac{1}{2}}}; \tag{130}$$
$$\tan\phi = \tan(\mu t + a);$$
$$\therefore \quad \phi = \mu t + a; \tag{131}$$

so that $\phi = a$, when $t = 0$; and thus a is the angle between the line of nodes and the principal axis of ξ, when $t = 0$.

Also
$$\frac{d\psi}{dt} = \frac{h}{\text{A}};$$
$$\therefore \quad \text{A}(\psi - \psi_0) = ht.$$

These equations completely determine the motion of the body, and its position at the time t. From (130) it appears that θ is constant; thus, the axis of greatest moment is always inclined at the same angle to the invariable axis; and therefore describes a right circular cone in space, the axis of which is the invariable axis. And since $\frac{d\psi}{dt}$ is constant, the precessional velocity of the line of intersection of the plane of axes of equal moment with the invariable plane is constant, so that the axis of greatest moment describes the circular cone uniformly. And from (131) it appears that the right ascension of a principal axis of equal moment advances uniformly.

In this case the moving body is such that its momental ellipsoid is an oblate spheroid. All the polhodes are circles parallel to the equator of the spheroid; and all the herpolhodes are also concentric circles whose centre is at G; see Fig. 33. The dividing ellipses in this case unite and become the circle of the equator. Thus, if the instantaneous-axis is ever in the equator, and has a motion along it, it will pass throughout it, and a complete boule-

versement will take place again and again; this will take place when $h^2 = \text{A}k^2$, and the plane on which the spheroid rolls is at a distance from the fixed point equal to the equatorial radius. Although in this case the instantaneous-axis moves in the body through a complete circle, yet in space it is coincident with the invariable axis, and is fixed. Thus, the axis of greatest moment is the only rotation-axis which has stability and permanence. In this case the herpolhode becomes a point.

315.] If the two equal principal axes are the greatest and the mean, that is if $\text{C} = \text{B}$, the results are so exactly analogous to those of the preceding Article that it is unnecessary to explain them at length. In this case however the momental ellipsoid is a prolate spheroid, of which the principal axis of least moment is the axis of revolution. The polhodes are circles in planes perpendicular to this axis, and the herpolhodes are also concentric circles of which G is the centre. The dividing ellipses unite into the circle BCB′C′. If the instantaneous-axis is ever in this circle and has a motion along it, it will move throughout it, and a complete bouleversement will take place; this is the case when $h^2 = \text{C}k^2$: and although the rotation-axis moves in the body through this circle, yet it is fixed in space, being coincident with the invariable axis. The axis OA, which is that of least moment, is the only axis in this system which has stability and permanence.

316.] If $\text{A} = \text{B} = \text{C}$, that is, if the three principal moments are equal,
$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0;$$
$$\therefore \quad \frac{\omega_1}{\Omega_1} = \frac{\omega_2}{\Omega_2} = \frac{\omega_3}{\Omega_3} = \frac{\omega}{\Omega} = 1,$$
and the angular velocity is constant; and the rotation-axis is fixed both relatively to the moving body and in space.

In this case the momental ellipsoid is a sphere, and the distance of the fixed plane on which it rolls is at a distance from its centre equal to its radius. The polhode and herpolhode are only points; and whatever new couple is impressed, and whatever consequently is the displacement of the rotation-axis, the axis in its new place is stable and permanent.

And this case is indeed the only one in which generally it is possible for the angular velocity to be constant throughout the

motion; for, if ω is constant, the solution depends on the three equations

$$\left.\begin{aligned} A^2\omega_1^2+B^2\omega_2^2+C^2\omega_3^2 &= h^2,\\ A\omega_1^2+B\omega_2^2+C\omega_3^2 &= k^2,\\ \omega_1^2+\omega_2^2+\omega_3^2 &= \omega^2; \end{aligned}\right\}$$

whence $\omega_1, \omega_2, \omega_3$ are evidently constant; and consequently

$$\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0;$$

and therefore from (41)

$$(C-B)\omega_2\omega_3 = (A-C)\omega_3\omega_1 = (B-A)\omega_1\omega_2 = 0;$$

which are satisfied (1) when $A = B = C$, that is, when the principal moments are equal, and every axis is permanent and stable; (2) when two of the three quantities $\omega_1, \omega_2, \omega_3$ are equal to zero; that is, when the body rotates about a principal axis; but in this last case on any shifting of the axis a change of angular velocity takes place.

317.] The differential equations (41) are integrable also, at least approximately, when the angle between the rotation-axis of the system and one of the principal axes, say that of the greatest moment, is always small; so that the angles between the rotation-axis and the other principal axes are almost right angles, and thereby their cosines are very small. In this case ω_1 and ω_2 are so small that their squares and their products may be neglected in linear equations which involve their first powers, and ω_3 may be replaced by the resultant instantaneous angular velocity; then, from the third of (41), we have $\frac{d\omega_3}{dt} = 0$; so that $\omega_3 =$ a constant $= n$, (say): then on these conditions the first two equations of (41) become

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}+(C-B)n\omega_2 &= 0,\\ B\frac{d\omega_2}{dt}-(C-A)n\omega_1 &= 0; \end{aligned}\right\} \tag{132}$$

whence we have

$$\left.\begin{aligned} \frac{d^2\omega_1}{dt^2}+\mu^2\omega_1 &= 0,\\ \frac{d^2\omega_2}{dt^2}+\mu^2\omega_2 &= 0, \end{aligned}\right\} \tag{133}$$

if $(C-A)(C-B)n^2 = AB\mu^2$;

$$\therefore \left.\begin{aligned} \omega_1 &= \Omega_1\cos\mu(t-t_0),\\ \omega_2 &= \Omega_2\sin\mu(t-t_0); \end{aligned}\right\} \tag{134}$$

where Ω_1 and Ω_2 are connected by the equation

$$\Omega_1 \{\mathrm{A}(\mathrm{C}-\mathrm{A})\}^{\frac{1}{2}} = \Omega_2 \{\mathrm{B}(\mathrm{C}-\mathrm{B})\}^{\frac{1}{2}},$$

the limits of integration being such that $\omega_1 = \Omega_1$ and $\omega_2 = 0$ when $t = t_0$, and $\omega_1 = 0$ and $\omega_2 = \Omega_2$ when $t = t_0 + \frac{\pi}{2\mu}$: thus the rotation-axis when $t = t_0$ is in the plane of (ξ, ζ); and if Ω is the initial angular velocity

$$\Omega^2 = \Omega_1{}^2 + n^2.$$

Since Ω_1 is very small, (134) shew that ω_1 and ω_2 are always small, so long as $\mathrm{C}-\mathrm{A}$ and $\mathrm{C}-\mathrm{B}$ have the same sign; that is, so long as the principal axis, near to which the rotation-axis is, is the axis of either the greatest or the least moment. If however $\mathrm{C}-\mathrm{A}$ and $\mathrm{C}-\mathrm{B}$ are of different signs, the integrals of (133) involve exponential expressions; and ω_1 and ω_2 will increase indefinitely with the time. Whence we infer that if a body, free from the action of forces producing rotation, rotates at any time about an axis nearly coinciding with the principal axis of greatest or least moment, the rotation-axis will always nearly coincide with that principal axis. But if the principal axis, with which the rotation-axis nearly coincides is the principal axis of mean moment, the rotation-axis will deviate more and more from that axis. Hereby we have another conception of the stability and instability of principal axes; those of greatest and least moment are stable; that of mean moment is unstable.

318.] The problem which I propose next for consideration is that of the motion of a heavy body, having one point fixed which is not its centre of gravity; at which, however, I shall suppose $\mathrm{A} = \mathrm{B}$, each being less than C, which is the other principal moment at the point: I shall also suppose the centre of gravity to be in the C-axis, at a distance h from the fixed point. It is evident that of many systems of particles satisfying these conditions, one is a heavy homogeneous solid of revolution, capable of rotation about an axis passing through a fixed point in its axis of figure: such is a top, the apex of whose peg keeps the same place during the rotation, and the friction at which is neglected.

Let us take the coordinates and all other symbols of Articles 2, 4, and 64 for the purposes of reference, save that we replace $\omega_\xi, \omega_\eta, \omega_\zeta$ by $\omega_1, \omega_2, \omega_3$. Let the fixed point be the origin, and the vertical line through it the axis of z, so that the horizontal plane is the plane of (x, y). Let the axis of unequal moment in

the body be the axis of ζ, the axes of ξ and η being in the plane through the fixed point perpendicular to this axis, and then these three axes make a system of principal axes at the point, and in reference to them the centre of gravity is at the point $(0, 0, h)$. Let m be the mass of the body, so that mg is the weight acting vertically downwards at the centre of gravity, and $mgh \sin\theta$ is the moment of the couple at the time t due to the weight of the body, which tends to increase θ, and the line of nodes in the horizontal plane is the rotation-axis of this couple. Then the equations of motion (27), Art. 275, in reference to the principal axes fixed in the body, are

$$\text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{A})\,\omega_2\omega_3 = mhg\sin\theta\cos\phi, \tag{135}$$

$$\text{A}\frac{d\omega_2}{dt} + (\text{A}-\text{C})\,\omega_3\omega_1 = -mhg\sin\theta\sin\phi, \tag{136}$$

$$\text{C}\frac{d\omega_3}{dt} = 0; \tag{137}$$

from the third we have $\omega_3 =$ a constant $= n$, (say); so that the angular velocity about the axis of unequal moment is constant; and substituting in (135) and (136), we have

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{A})\,n\,\omega_2 &= mhg\sin\theta\cos\phi, \\ \text{A}\frac{d\omega_2}{dt} + (\text{A}-\text{C})\,n\,\omega_1 &= -mhg\sin\theta\sin\phi. \end{aligned}\right\} \tag{138}$$

$$\therefore\ \text{A}\left(\omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt}\right) = mhg\sin\theta\,(\omega_1\cos\phi - \omega_2\sin\phi)$$

$$= mhg\sin\theta\,\frac{d\theta}{dt}, \tag{139}$$

by reason of (123), Art. 64. Consequently, if k^2 is the initial vis viva of the body, and θ_0 is the initial value of θ, integrating, we have

$$\text{A}(\omega_1^2+\omega_2^2) + \text{C}n^2 = k^2 + 2mhg(\cos\theta_0 - \cos\theta); \tag{140}$$

which is indeed the equation of vis viva, and might have been immediately inferred from the principle given in Article 108. Whenever $\theta = \theta_0$, that is, whenever the angle between the vertical line and the axis of unequal moment has its initial value, the vis viva of the system is equal to the initial vis viva; and the vis viva is increased or diminished according as the position of the centre of gravity falls or rises. This is in accordance with the conservation of vis viva.

From (120) and (121), Art. 64,

$$\omega_1^2 + \omega_2^2 = \left(\frac{d\theta}{dt}\right)^2 + \left(\sin\theta \frac{d\psi}{dt}\right)^2; \tag{141}$$

so that (140) becomes

$$\text{A}\left(\frac{d\theta}{dt}\right)^2 + \text{A}(\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2 + \text{C}n^2 = k^2 + 2mhg(\cos\theta_0 - \cos\theta), \tag{142}$$

which is the equation of vis viva in terms of θ, ψ, and t.

Again, from (138), we have

$$\text{A}\left(\sin\phi\frac{d\omega_1}{dt} + \cos\phi\frac{d\omega_2}{dt}\right) + (\text{C}-\text{A})n(\omega_2\sin\phi - \omega_1\cos\phi) = 0; \tag{143}$$

but from (124), Art. 64,

$$\omega_1\sin\phi + \omega_2\cos\phi = \sin\theta\frac{d\psi}{dt};$$

therefore

$$\sin\phi\frac{d\omega_1}{dt} + \cos\phi\frac{d\omega_2}{dt} = \frac{d}{dt}\left(\sin\theta\frac{d\psi}{dt}\right) - (\omega_1\cos\phi - \omega_2\sin\phi)\frac{d\phi}{dt}$$

$$= \frac{d}{dt}\left(\sin\theta\frac{d\psi}{dt}\right) - \frac{d\theta}{dt}\left(n - \cos\theta\frac{d\psi}{dt}\right),$$

by reason of (122) and (123), Art. 64; and substituting these in (143), we have

$$\text{A}\left\{2\cos\theta\frac{d\theta}{dt}\frac{d\psi}{dt} + \sin\theta\frac{d^2\psi}{dt^2}\right\} - \text{C}n\frac{d\theta}{dt} = 0; \tag{144}$$

and multiplying by $\sin\theta$, and integrating, we have

$$\text{A}(\sin\theta)^2\frac{d\psi}{dt} + \text{C}n\cos\theta = h_3; \tag{145}$$

where h_3 is a constant introduced in integration.

Now the left-hand member of this equation is the moment of the momentum of the system at the time t about the axis of z: for as the moments of momenta about the axes of ξ, η, ζ are respectively $\text{A}\omega_1$, $\text{A}\omega_2$, $\text{C}n$, the moment of momentum about the axis of z, see Articles 2 and 4,

$$= \text{A}\omega_1 a_3 + \text{B}\omega_2 b_3 + \text{C}\omega_3 c_3$$

$$= \text{A}\omega_1\sin\theta\sin\phi + \text{A}\omega_2\sin\theta\cos\phi + \text{C}n\cos\theta$$

$$= \text{A}\sin\theta(\omega_1\sin\phi + \omega_2\cos\phi) + \text{C}n\cos\theta$$

$$= \text{A}(\sin\theta)^2\frac{d\psi}{dt} + \text{C}n\cos\theta, \tag{146}$$

so that (145) asserts that this moment of momentum is constant, and is equal to h_3.

This result is in accordance with Art. 93, wherein it is shewn

that although in a problem like the present the moment of momentum is not constant in respect of all axes, yet it is constant for any axis which is parallel to the action-line of the acting forces; and h_3 is the constant moment of momentum of the system for the axis of z. This equation might also have been analogously interpreted as a conservation of areas described on the plane of (x, y).

Also, since $\omega_3 = n$, we have from (122), Art. 64,

$$\frac{d\phi}{dt} = n - \cos\theta \frac{d\psi}{dt}. \tag{147}$$

Now the three equations (142), (145), and (147) are the capital equations of the problem, inasmuch as the first two give $\frac{d\theta}{dt}$ and $\frac{d\psi}{dt}$ in terms of θ, whence θ and ψ can be, theoretically at least, determined in terms of t; and thence ϕ can be determined in terms of t by the last equation: and thus the position of the body will be completely given at any time.

If $\frac{d\psi}{dt}$ is eliminated from (142) by means of (145), we have

$$\text{A}\left(\frac{d\theta}{dt}\right)^2 + \frac{(h_3 - \text{C}n\cos\theta)^2}{\text{A}(\sin\theta)^2} + \text{C}n^2 = k^2 + 2mhg(\cos\theta_0 - \cos\theta); \tag{148}$$

the integral of which is an elliptic transcendent.

If ω is the instantaneous angular velocity,

$$\begin{aligned}\omega^2 &= \omega_1^2 + \omega_2^2 + \omega_3^2 \\ &= \left(\frac{d\theta}{dt}\right)^2 + \left(\sin\theta\frac{d\psi}{dt}\right)^2 + n^2;\end{aligned}$$

and if γ is the angle between the instantaneous axis and the vertical,

$$\begin{aligned}\omega\cos\gamma &= \omega_1 a_3 + \omega_2 b_3 + \omega_3 c_3 \\ &= \sin\theta(\omega_1\sin\phi + \omega_2\cos\phi) + n\cos\theta \\ &= (\sin\theta)^2\frac{d\psi}{dt} + n\cos\theta;\end{aligned}$$

$$\therefore\ \cos\gamma = \frac{h_3 - n(\text{C} - \text{A})\cos\theta}{\text{A}\omega}.$$

319.] The three fundamental equations have been deduced in the preceding Article from Euler's equations of motion by integration: they may, however, be derived directly from the equation of vis viva by Lagrange's process as explained in Section 6,

Chapter III. Let 2T be the vis viva of the body, and let $-\text{V}$ be the force-function; then, denoting time-differentials by accents, as in that section,

$$\begin{aligned} 2\text{T} &= \text{A}\,\omega_1^2 + \text{B}\,\omega_2^2 + \text{C}\,\omega_3^2 \\ &= \text{A}\,(\theta' \cos\phi + \psi' \sin\theta \sin\phi)^2 + \text{A}\,(-\theta' \sin\phi + \psi' \sin\theta\cos\phi)^2 \\ &\qquad\qquad + \text{C}\,(\phi' + \psi'\cos\theta)^2 \\ &= \text{A}\,\{\theta'^2 + \psi'^2 (\sin\theta)^2\} + \text{C}\,(\phi' + \psi'\cos\theta)^2; \qquad (149) \end{aligned}$$

$$\text{V} = mgh\,(\cos\theta_0 - \cos\theta); \qquad (150)$$

therefore

$$\frac{d}{dt}\frac{d\text{T}}{d\theta'} = \text{A}\,\theta''; \qquad \frac{d\text{T}}{d\theta} = \text{A} \sin\theta\cos\theta\,\psi'^2 - \text{C}\,\psi'\,(\phi' + \psi'\cos\theta)\sin\theta,$$

$$\frac{d}{dt}\frac{d\text{T}}{d\psi'} = \frac{d}{dt}\{\text{A}\,\psi'(\sin\theta)^2 + \text{C}\cos\theta\,(\phi' + \psi'\cos\theta)\},$$

$$\frac{d}{dt}\frac{d\text{T}}{d\phi'} = \text{C}\,\frac{d}{dt}(\phi' + \psi'\cos\theta).$$

Also $\left(\frac{d\text{V}}{d\theta}\right) = mgh\sin\theta; \quad \left(\frac{d\text{V}}{d\psi}\right) = 0; \quad \left(\frac{d\text{V}}{d\phi}\right) = 0;$

so that Lagrange's equations are

$$\text{A}\,(\theta'' - \sin\theta\cos\theta\,\psi'^2) + \text{C}\,\psi'(\phi' + \psi'\cos\theta)\sin\theta = mgh\sin\theta; \qquad (151)$$

$$\frac{d}{dt}\{\text{A}\,\psi'(\sin\theta)^2 - \text{C}\cos\theta\,(\phi' + \psi'\cos\theta)\} = 0; \qquad (152)$$

$$\frac{d}{dt}(\phi' + \psi'\cos\theta) = 0. \qquad (153)$$

Hence from (153), $\phi' + \psi'\cos\theta = n = \omega_3;$ (154)

that is, the angular velocity about the axis of ζ is constant.

Also from (152) and (153) we have

$$\text{A}\,\psi'(\sin\theta)^2 + \text{C}\,n\cos\theta = \text{a constant} = h_3, \qquad (155)$$

where h_2 is the moment of the momentum of the system in respect of the axis of z, which is vertical, and consequently this moment is constant.

And (151) becomes

$$\text{A}(\theta'' - \sin\theta\cos\theta\,\psi'^2) + \text{C}\,n\,\psi'\sin\theta = mgh\sin\theta, \qquad (156)$$

and replacing ψ' by its value as given in (155), we have

$$\text{A}\,\frac{d^2\theta}{dt^2} - \frac{(h_3 - \text{C}\,n\cos\theta)(h_3\cos\theta - \text{C}\,n)}{\text{A}\,(\sin\theta)^3} = mgh\sin\theta, \qquad (157)$$

which is a differential equation of the second order, and from which θ is to be determined in terms of t. As this equation is however only the t-differential of (148), it is more convenient to

employ that equation as one of the fundamental equations of the problem.

As these equations involve transcendents of a higher order than algebraical circular and exponential functions, I do not propose to carry further the general investigation, but to consider only one or two special cases which lead to more simple equations.

320.] Let us suppose the axis of the angular velocity which is impressed on the body initially to be the axis of unequal moment; so that initially, $\omega_1 = \omega_2 = 0$ and $\frac{d\theta}{dt} = \frac{d\psi}{dt} = 0$; and therefore $k^2 = \text{C}n^2$, and $h_3 = \text{C}n \cos \theta_0$; thus (142) and (145) become

$$\left.\begin{aligned} \text{A}\left(\frac{d\theta}{dt}\right)^2 + \text{A}(\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2 &= 2mhg(\cos\theta_0 - \cos\theta), \\ \text{A}(\sin\theta)^2\frac{d\psi}{dt} &= \text{C}n(\cos\theta_0 - \cos\theta); \end{aligned}\right\} \quad (158)$$

the former of these shews that h and $(\cos\theta_0 - \cos\theta)$ are always of the same sign; and that consequently if h is positive, θ is greater than θ_0; and if h is negative, θ is less than θ_0; thus, the inclination of the axis of unequal moment to the vertical increases or decreases according as the centre of gravity is initially above or below the fixed point. And from the latter it appears that the precessional motion, of which $\frac{d\psi}{dt}$ is the symbol and the measure, is direct or retrograde, according as the centre of gravity is initially above or below the fixed point.

To fix our thoughts on the interpretation and discussion of these equations, I will take h to be positive and θ_0 to be an acute angle, so that the centre of gravity of the body, is initially above the horizontal plane of (x, y) drawn through the fixed point, and at a distance equal to $h \cos\theta_0$ from it. In this case then θ is never less than θ_0, and the centre of gravity never rises to a position higher than its initial position. Also, since $\cos\theta_0 - \cos\theta$ is always positive, $\frac{d\psi}{dt}$ has the same sign as n; that is, the rotation of the meridian plane in which the axis of greatest moment lies about the vertical is in the same direction as the rotation of the body about its axis of greatest moment. I shall take n to be positive, and consequently $\frac{d\psi}{dt}$ is also positive.

If we eliminate $\frac{d\psi}{dt}$ from the two equations (158), we have

$$dt = \frac{\text{A} \sin\theta\, d\theta}{(\cos\theta_0 - \cos\theta)^{\frac{1}{2}}\{2mhg(\text{A}\sin\theta)^2 - \text{C}^2n^2(\cos\theta_0 - \cos\theta)\}^{\frac{1}{2}}}, \quad (159)$$

the positive sign being taken, because I will consider the motion at the instant when θ is increasing with the time; hence θ will continue to increase until it attains to a value which is a root of the quadratic factor in the denominator of (159), when $\frac{d\theta}{dt} = 0$; let this root be θ_1, and it lies between θ_0 and π, because the quadratic factor is positive when $\theta = \theta_0$ and is negative when $\theta = \pi$, so that a root lies between these limits. Let the other root of this factor be θ_2 the value of which we shall consider hereafter. Then (159) takes the form

$$dt = \frac{\text{A} \sin\theta\, d\theta}{(2mhg\text{A})^{\frac{1}{2}}\{(\cos\theta - \cos\theta_0)(\cos\theta - \cos\theta_1)(\cos\theta - \cos\theta_2)\}^{\frac{1}{2}}}, \quad (160)$$

so that $\frac{d\theta}{dt} = 0$ for three values of θ, viz. θ_0, θ_1, θ_2; let us consider them in order; when $\theta = \theta_0$, we have the initial position of the body, and θ increases until $\theta = \theta_1$, when $\frac{d\theta}{dt} = 0$, and θ becomes a maximum, then θ decreases from θ_1 to θ_0, when $\frac{d\theta}{dt}$ again vanishes, and θ has its least real value, as shewn by the first equation of (158). But $\frac{d\theta}{dt} = 0$ also when $\theta = \theta_2$; and what is the meaning of this factor? Let us consider the motion from the following point of view.

About the fixed point O as a centre describe a sphere with the radius h, so that the centre of gravity of the body moves on the surface of this sphere. Let N and P be the points where the vertical through O and the axis of the body respectively intersect the sphere. Then, if P is (x, y, z), we have

$$z = h\cos\theta, \quad z_0 = h\cos\theta_0, \quad z_1 = h\cos\theta_1, \quad z_2 = h\cos\theta_2.$$

Also let A and C be replaced by their values ma^2 and mc^2, where a and c are the principal radii of gyration of the body at O; then (160) takes the form

$$dt = \frac{-a\,dz}{(2g)^{\frac{1}{2}}\{(z - z_0)(z - z_1)(z - z_2)\}^{\frac{1}{2}}}, \quad (161)$$

and z_1 and z_2 are the roots of the quadratic factor in (159), so that

$$z^2-h^2-\frac{c^4 n^2}{2a^2 g}(z-z_0) = (z-z_1)(z-z_2). \qquad (162)$$

Now this quantity is positive if $z = \infty$, is negative if $z = h$, is negative also if $z = z_0$, is positive if $z = -h$, and is positive also if $z = -\infty$; consequently it has two real roots, one of which is greater than h, and the other lies between z_0 and $-h$; the latter root is z_1, to which the real angle θ_1 corresponds; the former root is the quantity which we have denoted by z_2 and to which we have assumed θ_2 to be the corresponding angle: but as z_2 is greater than h, there is no real corresponding angle, and consequently the axis of the body has no place in its motion corresponding to z_2 or θ_2, and the factor $z-z_2$ is extraneous to the circumstances of the problem, and is negative for all possible values of θ. Now for the reality of (161) it is necessary, that either all three factors under the radical sign should be positive, or that one should be positive and two negative. The first condition cannot be satisfied because $z-z_2$ is always negative; hence it is necessary that one other factor, and only one, should be negative. This condition is satisfied whenever z is not greater than z_0 and not less than z_1: hence all possible values of z lie between these limits, and P lies on some part of the spherical surface which is at a height above the horizontal plane not less than z_1 and not greater than z_0; that is, P is always found at some point of a zone between two parallels of latitude whose angular distances from N are θ_0 and θ_1, the angular velocity with which it moves in the meridian plane being given by (160) or (161).

321.] This meridian plane however revolves with an angular velocity $\frac{d\psi}{dt}$ about the vertical axis ON; but from (158)

$$\frac{d\psi}{dt} = \frac{\text{C}n(\cos\theta_0-\cos\theta)}{\text{A}(\sin\theta)^2}, \qquad (163)$$

so that this angular velocity is not uniform, but depends on θ; and the variations of it are periodic, the period being the same as that of the variation of P. This precessional velocity vanishes when $\theta = \theta_0$, and attains its maximum value, viz. $\frac{2hg}{c^2 n}$ when $\theta = \theta_1$, and afterwards decreases, and vanishes again when $\theta = \theta_0$.

The combination of these motions, viz. the nutational motion given by (159) and the precessional motion given by (163), plainly indicates the character of the path described by P on the surface

of the sphere, and the fluted form of the surface of the cone described by the axis of greatest moment of the moving body: viz. how the path of P consists of a series of arcs, like the outlines of the petals of a dahlia, lying between two parallels of latitude corresponding to θ_1 and θ_0; and how these arcs touch the exterior parallel and meet the interior at right angles and thus form cusps on that parallel, so that the path has points of discontinuity; it does not however fall into our province here to consider the path beyond these points and how P_2 is a point on it. If we project on the horizontal plane of (x, y) the curve described by P it is evidently contained between two concentric circles whose centres are at O, and whose radii are $h \sin\theta_1$ and $h \sin\theta_0$; it consists of a series of arcs, like, as above mentioned, the round petals of a dahlia, which touch the outer circle and are at right angles to the inner circle and form cusps at the points where they meet it. These curves are delineated in Fig. 35.

The periodicity of the motion and the symmetry of the recurring arcs is shewn by the form which the definite integral for the time takes as an elliptic function. Let T be the time in which P passes from a cusp to its point of contact with the θ_1-circle; then from (161),

$$T = \int_{z_0}^{z_1} \frac{-a\,dz}{(2g)^{\frac{1}{2}}\{(z-z_0)(z-z_1)(z-z_2)\}^{\frac{1}{2}}}.$$

Let $z = z_0(\sin\chi)^2 + z_1(\cos\chi)^2$, so that $z = z_1$ when $\chi = 0$, and $z = z_0$ when $\chi = \frac{\pi}{2}$: and let $z_0 - z_1 = k^2(z_2 - z_1)$; then making these substitutions,

$$T = \frac{a\,2^{\frac{1}{2}}}{\{g(z_2-z_1)\}^{\frac{1}{2}}} \int_0^{\frac{\pi}{2}} \frac{d\chi}{\{1-k^2(\sin\chi)^2\}^{\frac{1}{2}}},$$

which is an elliptic function of the first kind; and as it involves only $\sin\chi$, it is evidently a finite quantity, and is periodical and symmetrical in its successive equal values. Hence the motion of the axis of greatest moment is periodic, and the alternate falling and rising of that axis which is observed is fully accounted for. If the arc AA′, see Fig. 35, is an aliquot part of $2\pi h \sin\theta_0$, the axis of greatest moment returns periodically to a position which it has previously occupied; and if AA′ and $2\pi h \sin\theta_0$ are incommensurable, the axis never occupies the same position more than once.

If ω is the instantaneous angular velocity,

$$\omega^2 = \omega_1^2 + \omega_2^2 + n^2;$$

and ω is a minimum when $\omega_1^2+\omega_2^2 = 0$, that is, when $t = 0$ and when $\theta = \theta_0$; and ω is a maximum when $\omega_1^2+\omega_2^2$ is a maximum; and since

$$\omega_1^2+\omega_2^2 = \left(\frac{d\theta}{dt}\right)^2+(\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2$$
$$= 2mhg(\cos\theta_0-\cos\theta), \quad (164)$$

the angular velocity is a maximum when θ has its greatest value; that is, when $\theta = \theta_1$, and when the angle of inclination of the axis of unequal moment to the vertical is the greatest. This is evidently the case by reason of the principle of vis viva.

If h is negative, similar results follow, except that θ_1 will be less than θ_0; thus, the principal axis of unequal moment will come nearer to the vertical line than it was in its initial position, and will make periodical ascents and descents.

322.] Let us assume the initial angular velocity of the body, viz. n, about its axis of unequal moment to be very great, then since θ_1 is a finite value of θ determined by the equation

$$2mh\text{A}g(\sin\theta)^2-\text{C}^2n^2(\cos\theta_0-\cos\theta) = 0,$$

this condition can be satisfied when $\cos\theta_0-\cos\theta$ is a small quantity; that is, when θ_1 is a very little greater than θ_0, and the two parallels of latitude within which the path of the pole P is contained are very near together. In this case then we may assume $\theta = \theta_0+u$, where u is a small angle of which the cubes and higher powers may be neglected; then

$$\cos\theta_0-\cos\theta = \cos\theta_0-\cos(\theta_0+u)$$
$$= 2\sin\frac{u}{2}\sin\frac{2\theta_0+u}{2} = u\sin\theta_0+\frac{u^2\cos\theta_0}{2}; \quad (165)$$

$$\sin\theta = \sin(\theta_0+u) = \sin\theta_0+u\cos\theta_0; \quad (166)$$

so that (159), omitting $2mh\text{A}g\cos\theta_0$ when added to $-\text{C}^2n^2$, becomes

$$dt = \frac{\text{A}\,du}{\{2mh\text{A}g\sin\theta_0 u-\text{C}^2n^2u^2\}^{\frac{1}{2}}}. \quad (167)$$

Let $\dfrac{mh\text{A}g\sin\theta_0}{\text{C}^2n^2} = a$; so that a is a small quantity; then

$$dt = \frac{\text{A}}{\text{C}n}\frac{du}{(2au-u^2)^{\frac{1}{2}}};$$

$$t = \frac{\text{A}}{\text{C}n}\text{versin}^{-1}\frac{u}{a}, \quad (168)$$

$$u = a\,\text{versin}\frac{\text{C}nt}{\text{A}},$$

the limits of integration being such that $u = 0$ when $t = 0$;

$$\therefore \quad \theta = \theta_0 + a \operatorname{versin} \frac{\text{C}nt}{\text{A}},$$

$$= \theta_0 + a - a \cos \frac{\text{C}nt}{\text{A}}, \tag{169}$$

so that θ varies from its least value θ_0 to its greatest value $\theta_0 + 2a$, this last value occurring whenever $t = (2\lambda + 1)\frac{\pi \text{A}}{\text{C}n}$, where λ is any integer; and the periodic time between two successive equal and similar values of θ is $\frac{2\pi \text{A}}{\text{C}n}$; the greater therefore n is, the less is this period.

A similar result may be arrived at from the equation (161), if we substitute $z = z_0 - \zeta$, where ζ is a small decrement of z, and proceed with the integration.

Also from (163), if we replace $\cos\theta_0 - \cos\theta$ by its value given in (165), and retain only terms involving the first power of u, we have

$$\frac{d\psi}{dt} = \frac{\text{C}nu}{\text{A}\sin\theta_0} = \frac{mhg}{\text{C}n}\left\{1 - \cos\frac{\text{C}nt}{\text{A}}\right\}, \tag{170}$$

$$\because \quad \psi = \frac{mhg}{\text{C}n}\left\{t - \frac{\text{A}}{\text{C}n}\sin\frac{\text{C}nt}{\text{A}}\right\},$$

$$= \frac{mhgt}{\text{C}n} - \frac{mh\text{A}g}{\text{C}^2 n^2}\sin\frac{\text{C}nt}{\text{A}}, \tag{171}$$

the limits being such that $\psi = 0$ when $t = 0$. Hence ψ consists of two terms of which the former is non-periodic and the latter is periodic. The former increases with t at an uniform rate, and the latter varies from its greatest value $\frac{mh\text{A}g}{\text{C}^2 n^2}$ to its least value $-\frac{mh\text{A}g}{\text{C}^2 n^2}$ periodically, the period being $\frac{2\pi\text{A}}{\text{C}n}$; $\frac{d\psi}{dt} = 0$ when $u = 0$, and has its greatest value when $u = \theta_1 - \theta_0$.

Also from (147),

$$\frac{d\phi}{dt} = n - \cos\theta\frac{d\psi}{dt}$$

$$= n - \frac{mhg\cos\theta_0}{\text{C}n}\left(1 - \cos\frac{\text{C}nt}{\text{A}}\right), \tag{172}$$

$$\therefore \quad \phi = \left(n - \frac{mhg\cos\theta_0}{\text{C}n}\right)t + \frac{mh\text{A}g\cos\theta_0}{\text{C}^2 n^2}\sin\frac{\text{C}nt}{\text{A}}, \tag{173}$$

the limits being such that $\phi = 0$, when $t = 0$: the explanation of this expression is similar to that of (171).

The three equations (169), (171), and (173) give the complete solution of the problem, and supply the means of indicating the motion of the axis of unequal moment with great geometrical exactness.

323.] From the fixed point O as centre, as in Art. 320, let a sphere be described whose radius is h, so that the centre of gravity of the body always is on the surface of this sphere: let N and P be the points where the vertical through O and the axis of unequal moment intersect the sphere; then the preceding equations determine the motion of P on the sphere in the following manner. In reference to N as pole let two parallels of latitude be described at angular distances θ_0 and $\theta_0 + 2a$ from N; then these circles are near to each other and P is never found outside them; and it describes an undulating line which at regular and equal distances touches the outside circle and meets the inside circle in a cusp at right angles. Let another parallel of latitude be described at an angular distance from N equal to $\theta_0 + a$, so that this lies midway between the preceding parallels. Now imagine a point P_0 to travel uniformly along this intermediate circle with an angular velocity about ON equal to $\frac{mhg}{Cn}$, and let us call P_0 the mean pole and OP_0 the mean axis; then (171) shews that the true pole P is in a meridian plane which is sometimes in advance of, and sometimes behind, the meridian plane of the mean pole, the amplitude of this deviation being $\frac{mhAg}{C^2n^2}$; and (169) shews that the true pole is sometimes below and sometimes above the intermediate circle, the angular amplitude being evidently a. These deviations are periodic, and the period is evidently $\frac{2\pi A}{Cn}$; as n is large all these deviations and periods are very small.

As P_0 moves uniformly along the intermediate parallel of latitude, P describes a curve about it: and as the deviations of P from P_0 are all small, we may without sensible error suppose them to take place in the tangent plane of the sphere at P_0; and denote the deviations along the meridian plane and the tangent to the parallel of latitude respectively by y and x; then the former is evidently the periodic part of $h\theta$, and the latter is the periodic part of $h \sin\theta_0 \psi$, since $h \sin\theta_0$ is approximately the

radius of the parallel of latitude on which the true pole is at the time t. So that

$$y = \frac{m h^2 \text{A} g \sin \theta_0}{\text{C}^2 n^2} \cos \frac{\text{C} n t}{\text{A}}, \qquad x = \frac{m h^2 \text{A} g \sin \theta_0}{\text{C}^2 n^2} \sin \frac{\text{C} n t}{\text{A}};$$

$$\therefore \quad x^2 + y^2 = \frac{(m h^2 \text{A} g \sin \theta_0)^2}{\text{C}^4 n^4}; \qquad (174)$$

which is the equation to a circle; consequently the real pole describes a circle about the mean pole with an uniform angular velocity $= \frac{\text{C} n}{\text{A}}$. Thus the mean pole moves uniformly along the intermediate parallel of latitude, and the true pole describes about it as centre the circle whose equation is (174); and the true pole never coincides with the mean pole. The uniform motion of the mean pole along the parallel of latitude is called the Precession; and is measured by the angle $\frac{m h g}{\text{C} n}$: the periodic parts of θ and of ψ are called the Nutations, the former being the nutation in latitude and the latter the nutation in longitude.

The true pole has its greatest velocity at the point where its path touches the lower parallel of latitude, because at that point the nutation in longitude acts parallel to and in the same direction with the precession; and the true pole becomes stationary at the point where it meets the upper parallel of latitude, because at that point the nutation of longitude is equal and parallel to the precession and acts in a contrary direction: hence arises the cusp. If the distance between two successive cusps is an aliquot part of the upper parallel of latitude, the path of the true pole is a re-entering curve; but if it is incommensurable with that parallel, the series of arches in the successive periodic revolutions is not re-entrant, and the true pole rever returns to a place which it had previously occupied.

The curves described by the mean and the true poles define precisely the cones which are described by the mean and the true axis of greatest moment of the body in its motion, inasmuch as they are the director-curves of the cones. Thus the cone described by the mean axis is a circular cone about the vertical as its axis, having its semivertical angle equal to $\theta_0 + \alpha$; and the true axis of greatest moment describes a cone whose surface is fluted and intersects at equal intervals the cone of the mean axis, the external ridges being curved and the internal depressions lying

along straight lines which are pole-lines converging to the vertex of the cone.

The motion in right ascension is given by (173).

The angular velocity of the body and the place of the instantaneous axis at the time t may thus be found. By Art. 64 we have,

$$\left.\begin{aligned}\omega_1 &= \frac{d\theta}{dt}\cos\phi + \frac{d\psi}{dt}\sin\theta\sin\phi,\\ \omega_2 &= -\frac{d\theta}{dt}\sin\phi + \frac{d\psi}{dt}\sin\theta\cos\phi,\\ \omega_3 &= \frac{d\phi}{dt} + \frac{d\psi}{dt}\cos\theta;\end{aligned}\right\} \qquad (175)$$

now replacing the right-hand members by their values which are given in (169), (171) and (173), and retaining the capital terms which are of lowest dimensions, we have

$$\left.\begin{aligned}\omega_1 &= \frac{mhg\sin\theta_0}{\mathrm{C}n}\left\{\sin nt + \sin\frac{\mathrm{C}-\mathrm{A}}{\mathrm{A}}nt\right\},\\ \omega_2 &= \frac{mhg\sin\theta_0}{\mathrm{C}n}\left\{\cos nt - \cos\frac{\mathrm{C}-\mathrm{A}}{\mathrm{A}}nt\right\},\\ \omega_3 &= n;\end{aligned}\right\} \qquad (176)$$

$$\therefore\quad \omega^2 = \frac{4m^2h^2g^2(\sin\theta_0)^2}{\mathrm{C}^2n^2}\left(\sin\frac{\mathrm{C}nt}{2\mathrm{A}}\right)^2 + n^2. \qquad (177)$$

Also from these values the position of the instantaneous axis at any time may be determined.

From (177) it follows that ω is a periodic quantity, of which the period is $\frac{2\pi\mathrm{A}}{\mathrm{C}n}$; that its least value is n, and that this occurs whenever $\frac{\mathrm{C}nt}{2\pi}$ is zero or a multiple of π; and that it reaches its greatest value whenever $\frac{\mathrm{C}nt}{2\mathrm{A}}$ is an odd multiple of $\frac{\pi}{2}$.

324.] Let us next consider the case wherein the inclination of the axis of greatest moment to the vertical is constant throughout the motion; that being the case wherein the two parallels of latitude within which that axis moves are coincident, and the axis consequently describes a circular cone whose axis of figure is the vertical line through the fixed point. In this case, as the centre of gravity of the body is always in the same horizontal plane, no work is done on the system, and the vis viva is constant. Let θ_0 be the constant value, then as $\frac{d\theta}{dt} = 0$,

(142) becomes
$$\text{A}(\sin\theta_0)^2\left(\frac{d\psi}{dt}\right)^2+\text{C}n^2=k^2;$$
but if Ω_1, Ω_2 and n are the initial angular velocities about the principal axes, from (140)
$$k^2=\text{A}(\Omega_1^2+\Omega_2^2)+\text{C}n^2;$$
$$\therefore\quad \frac{d\psi}{dt}=\pm(\Omega_1^2+\Omega_2^2)^{\frac{1}{2}}\operatorname{cosec}\theta_0=a\ (\text{say}); \qquad (178)$$
so that the angular velocity of precession about the vertical is constant, and is a; consequently if $\psi=0$, when $t=0$,
$$\psi=a\,t. \qquad (179)$$

Thus the right circular cone is described uniformly by the axis of greatest moment about the vertical through O.

Also from (147)
$$\frac{d\phi}{dt}=n-a\cos\theta_0=\beta\ (\text{say}), \qquad (180)$$
consequently if $\phi=0$, when $t=0$,
$$\phi=\beta t, \qquad (181)$$
and the right ascension of the ξ-axis increases uniformly with the time, and is positive or negative according as $a\cos\theta_0$ is less than or greater than n.

Hence also from the values given in (175)
$$\omega_1=a\sin\theta_0\sin\beta t,\quad \omega_2=a\sin\theta_0\cos\beta t,\quad \omega_3=n; \qquad (182)$$
$$\therefore\quad \omega^2=(a\sin\theta_0)^2+n^2=\Omega_1^2+\Omega_2^2+n^2, \qquad (183)$$
and is constant throughout the motion; and hereby the position of the instantaneous axis of rotation at any time is given.

325.] But under what circumstances does this invariability of inclination occur? This question is of considerable interest, because it bears on phaenomena which are presented by certain machines, and are at first sight inconsistent with mechanical principles. The equations (182) are the three final integrals of (135), (136) and (137), and must consequently satisfy them. Hence substituting in the first two we have
$$\left.\begin{aligned}\sin\theta_0\{\text{A}a^2\cos\theta_0-\text{C}na+mhg\}\cos\beta t&=0,\\ \sin\theta_0\{\text{A}a^2\cos\theta_0-\text{C}na+mhg\}\sin\beta t&=0;\end{aligned}\right\} \qquad (184)$$
and as these conditions are to be satisfied for all values of t, it is necessary that
$$\sin\theta_0\{\text{A}a^2\cos\theta_0-\text{C}na+mhg\}=0. \qquad (185)$$
This condition is satisfied,

(1) When $\sin\theta_0 = 0$; that is when $\theta_0 = 0$, and when $\theta_0 = \pi$. In both cases the axis of unequal moment is vertical, in the former case the centre of gravity being above the fixed point, and in the latter case below it. In both cases $\omega_1 = \omega_2 = 0$, $\omega_3 = \omega = n$, the rotation-axis is vertical, being a principal permanent axis, and the angular velocity of the body is constant.

(2) When
$$\text{A}\,a^2\cos\theta_0 - \text{C}\,n\,a + m\,h\,g = 0, \tag{186}$$
this condition depending on the constitution of the body, and the initial circumstances of rotation. This equation being a quadratic in terms of a, which is the angular velocity of precession, has two roots and gives the following values for a, viz.,
$$2\text{A}\,a\cos\theta_0 = \text{C}\,n \pm (\text{C}^2n^2 - 4\text{A}\,h\,m\,g\cos\theta_0)^{\frac{1}{2}}. \tag{187}$$
Now these roots are real and unequal, or are equal, or are impossible according as C^2n^2 is greater than, equal to, or less than $4\text{A}\,h\,m\,g\cos\theta_0$; in order, therefore, if $h\cos\theta_0$ is a positive quantity, that motion may be possible with invariable inclination, C^2n^2 must not be less than $4\text{A}\,h\,m\,g\cos\theta_0$, and if $\text{C}^2n^2 = 4\text{A}\,h\,m\,g\cos\theta_0$, a has only one value, viz., $\dfrac{\text{C}\,n}{2\text{A}\cos\theta_0}$. If $h\cos\theta_0$ is negative, there are always two real values of a, one of which is positive and the other is negative, and this is always the case when θ_0 is greater than 90°.

If a very large angular velocity is initially given to the body about its axis of unequal moment, so that n is very large in comparison of the other constants, we may expand the radical in (187), and obtain the following values of a; viz.
$$\begin{aligned}2\text{A}\,a\cos\theta_0 &= \text{C}n \pm \text{C}n\left\{1 - \frac{4\text{A}\,h\,m\,g\cos\theta_0}{\text{C}^2n^2}\right\}^{\frac{1}{2}} \\ &= \text{C}n \pm \text{C}n\left\{1 - \frac{2\text{A}\,h\,m\,g\cos\theta_0}{\text{C}^2n^2}\right\} \\ &= 2\text{C}n, \text{ or } = \frac{2\text{A}\,h\,m\,g\cos\theta_0}{\text{C}n};\end{aligned}$$
$$\therefore\ a = \frac{\text{C}n}{\text{A}\cos\theta_0}, \text{ or } = \frac{m\,g\,h}{\text{C}n}: \tag{188}$$
thus there are two values of a, of which the former is very large and the latter is very small.

326.] If $\cos\theta_0 = 0$, that is, if $\theta_0 = 90°$, (186) shews that there is only one value of a, viz. $\dfrac{m\,h\,g}{\text{C}n}$: in this case the axis

of greatest moment is horizontal, and revolves uniformly with that angular velocity in the horizontal plane which passes through the fixed point: and since a has the same sign as n, the direction of rotation of the axis is the same as that of the body about its axis of greatest moment. As the precessional velocity varies inversely as n, it becomes very small if n is large; consequently if a rapid rotation is given to the body about its axis of unequal moment, that axis revolves with a slow angular velocity in the horizontal plane.

Now this is a remarkable case, and seems at first sight inconsistent with the principles of mechanics. A heavy body is rotating about a fixed point which is not its centre of gravity, and the line joining the fixed point and the centre of gravity moves in a horizontal plane with a constant angular velocity about the fixed point, so that the weight of the body apparently is without effect on the axis. Let us then consider the question in the light of first principles, and apart from the equations of motion.

Let us take the notation which has been employed in the preceding articles; so that n is the initial angular velocity of the body about its axis of greatest moment which is horizontal; and as no moment of a force, either impressed or centrifugal, acts to change this angular velocity, it is constant throughout the motion. As this axis is horizontal, the plane in which the other two principal axes at the fixed point lie is vertical: let the axis of ω_1 in this plane be inclined at an angle ϕ to the horizontal plane, the line of intersection of these two planes being the line of nodes; then $\frac{d\psi}{dt}$ is the angular velocity of this line of nodes about the vertical through the fixed point. Let Ω_1 and Ω_2 be the initial values of ω_1 and ω_2. As the axis of greatest moment and the line of nodes are both in the same horizontal plane and are at right angles to each other, there is no angular velocity of the body about the lines of nodes, and consequently,

$$\omega_1 \cos\phi + \omega_2 \cos(90^\circ + \phi) = 0;$$

$$\therefore \quad \omega_1 \cos\phi - \omega_2 \sin\phi = 0.$$

Also, as rotation about the vertical is due to ω_1 and ω_2 only,

$$\frac{d\psi}{dt} = \omega_1 \sin\phi + \omega_2 \cos\phi;$$

and therefore
$$\omega_1 = \frac{d\psi}{dt}\sin\phi, \qquad \omega_2 = \frac{d\psi}{dt}\cos\phi, \tag{189}$$
$$\frac{d\psi^2}{dt^2} = \omega_1^2 + \omega_2^2. \tag{190}$$

As the only forces acting on the body are the reaction at the fixed point and gravity, and as no work is done by either of these forces, the vis viva is constant: therefore
$$A\omega_1^2 + A\omega_2^2 + Cn^2 = A\Omega_2^2 + A\Omega_2^2 + Cn^2;$$
$$\therefore \quad \omega_1^2 + \omega_2^2 = \Omega_1^2 + \Omega_2^2;$$
$$\therefore \text{ from (190)}, \quad \frac{d\psi^2}{dt^2} = \Omega_1^2 + \Omega_2^2 = a^2 \text{ (say)};$$
and thus the angular velocity of the revolving line of nodes in the horizontal plane is constant. Hence if $\psi = 0$ when $t = 0$,
$$\psi = at.$$
Hence also
$$\frac{d\phi}{dt} = n;$$
that is, the angular velocity of the right ascension is constant, and is the same as that of the body about its axis of unequal moment; and if $\phi = 0$ when $t = 0$, as may be the case, since the position of the axes of ξ and η in the vertical plane is indeterminate,
$$\phi = nt.$$
Hence, then, $\quad \omega_1 = a\sin nt, \qquad \omega_2 = a\cos nt.$

The constants a and n are not independent. This is evident, for the relation of these quantities to each other must change with the mass of the body and the distance of its centre of gravity from the fixed point. This relation may be determined from the equations of motion, of which the solution is
$$\omega_1 = a\sin nt, \qquad \omega_2 = a\cos nt, \qquad \omega_3 = n;$$
substituting these in either
$$A\frac{d\omega_1}{dt} + (C-A)\,\omega_2\omega_3 = mgh\cos\phi,$$
or
$$A\frac{d\omega_2}{dt} - (C-A)\,\omega_3\omega_1 = -mgh\sin\phi,$$
we have
$$Can = mgh;$$
thus the product of a and n is constant, each varying inversely as the other. The same result as is given by (186).

327.] Another special case which deserves consideration is that

wherein (1) A = B; (2) the inclination of the C-axis to the vertical is constant throughout the motion, so that $\theta = \theta_0$; (3) there is no motion in right ascension, so that ϕ is constant. This is the case of the ordinary compound conical pendulum.

Since A = B, and the centre of gravity is in the C-axis, $\omega_3 = n$. Also since θ is constant, $\frac{d\theta}{dt} = 0$, and consequently

$$\omega_1 \cos\phi - \omega_2 \sin\phi = 0.$$

Also since ϕ is constant and the position of the A-axis in the plane of (ξ, η) is arbitrary, we may so take it that $\phi = 0$; hence from the preceding condition, $\omega_1 = 0$ throughout the motion, and $\frac{d\omega_1}{dt} = 0$. Hence from the equations of motion (138),

$$(\text{C}-\text{A})\, n\, \omega_2 = mgh \sin\theta_0;$$

$$\therefore \quad \omega_2 = \frac{mgh \sin\theta_0}{(\text{C}-\text{A})\, n}. \tag{191}$$

Also since

$$\sin\theta_0 \frac{d\psi}{dt} = \omega_1 \sin\phi + \omega_2 \cos\phi,$$

$$\therefore \quad \frac{d\psi}{dt} = \frac{mgh}{(\text{C}-\text{A})\, n}, \tag{192}$$

which gives the constant angular velocity about the vertical through the fixed point with which the vertical plane containing the C-axis revolves, and thus determines the motion in azimuth.

Also since ϕ is constant, $\frac{d\phi}{dt} = 0$; and consequently from (147)

$$n = \cos\theta_0 \frac{d\psi}{dt} = \frac{mgh \cos\theta_0}{(\text{C}-\text{A})\, n};$$

$$\therefore \quad n = \left\{\frac{mgh \cos\theta_0}{\text{C}-\text{A}}\right\}^{\frac{1}{2}} = \omega_3, \tag{193}$$

which gives the angular velocity of the body about the C-axis, and assigns the relation between n and θ_0, which the conditions of the problem require.

From (192) and (193) we have

$$\frac{d\psi}{dt} = \pm\left\{\frac{mgh}{(\text{C}-\text{A}) \cos\theta_0}\right\}^{\frac{1}{2}};$$

and consequently the periodic time, or the time of a complete revolution in azimuth, is

$$= 2\pi \left\{\frac{(\text{C}-\text{A}) \cos\theta_0}{mgh}\right\}^{\frac{1}{2}}. \tag{194}$$

Hence θ_0 must be less or greater than 90°, according as C is greater or less than A.

In the case of a particle m placed at the end of a thin rod at a distance h from the fixed point, C will be very small in comparison of A and may be neglected in the preceding expressions, and $\mathrm{A} = mh^2$; and if a is the angle between the rod and the vertical taken downwards, then

$$\frac{d\psi}{dt} = \Big(\frac{g}{h\cos a}\Big)^{\frac{1}{2}}, \text{ and the periodic time } = 2\pi\Big(\frac{h\cos a}{g}\Big)^{\frac{1}{2}}. \quad (195)$$

This is the case of the simple conical pendulum.

Hence if l is the distance along the C-axis from the fixed point at which a particle, whose mass is equal to that of the moving body, has to be placed, so that the periodic times in azimuth of the moving body and of the particle should be the same, then from (194) and (195) we have

$$l = \frac{\mathrm{A}-\mathrm{C}}{mh} = \frac{\mathrm{A}}{mh}, \text{ if C is omitted.}$$

The angular velocity of the body and the position of the rotation-axis may thus be found;

$$\omega_1 = 0, \quad \omega_2 = \Big\{\frac{mgh}{(\mathrm{C}-\mathrm{A})\cos\theta_0}\Big\}^{\frac{1}{2}}\sin\theta_0, \quad \omega_3 = n = \Big\{\frac{mgh\cos\theta_0}{\mathrm{C}-\mathrm{A}}\Big\}^{\frac{1}{2}};$$

$$\therefore \quad \omega^2 = \frac{mgh}{(\mathrm{C}-\mathrm{A})\cos\theta_0}.$$

Thus ω is equal to the velocity in azimuth; and as

$$\frac{\omega_1}{\omega} = 0, \quad \frac{\omega_2}{\omega} = \sin\theta_0, \quad \frac{\omega_3}{\omega} = \cos\theta_0,$$

it follows that the vertical line through the fixed point is the axis of rotation of the body.

328.] Many machines have been devised for the purpose of exhibiting the phaenomena which are expressed in the preceding equations; the construction of some is so curious that they are for the most part found only in collections of mechanical apparatus; others are so simple in form that they are the toys of children. Of the latter kind is the common spinning-top, of the several motions of which the preceding Articles give explanations, provided that the point of its peg continues in the same place, and the friction of the point is neglected. Of the former kind is, in the first place, Bohnenberger's machine, which we have already described in Art. 28; it is delineated in Fig. 8, and the first account of it was given in 1817 in the Tübinger Blätter für Naturwissenschaft, Tome III; it is also described in Gilbert's

Annalen, Bande 60, Leipzig, 1819. The rotating body in the middle is in our figure a sphere, but any other body may be substituted for that; and if the centre of gravity of it coincides with the centre of the three several rings, then, according to the notation in the preceding Articles, $h = 0$. Let us suppose the central body whether it is a sphere, an oblate spheroid, a cone, a cylinder, or any other body such that $A = B$, which rotates about the axis AA', to be capable of removal from the ring $ABA'B'$; and to it when so removed let a rapid rotation be given by means of a suitable machine; let it be replaced with its pivots in the holes at A and A'; then the construction of the machine allows the several movements consequent on the rotation of the body to be exhibited, when $h = 0$; for throughout, the centre of gravity will remain in the centre of the rings, and be unmoved. And if the pivots at B and B' are fastened so that no rotation takes place about the axis BB', the inclination of the axis AA' to the vertical CC' is constant throughout: this is the case wherein $\theta = \theta_0$ and $h = 0$, &c. Thus, if the central body is a sphere of radius a,

$$A = B = C = \frac{8\pi\rho a^5}{15},$$

and therefore, from (191), $\alpha = 0$, and $a = \infty$; and therefore the equatorial plane of the sphere always intersects the horizontal plane along the same line; and no rotation can be given to the sphere whereby its axis will describe a conical surface about the vertical CC'.

Another machine of the latter kind is that devised by Fessel, which is described in Poggendorff's Annalen, Bande 90, Leipzig, 1853, and which is delineated in Fig. 37. Q is a heavy fixed stand, the vertical shaft of which is a cylinder bored smoothly, in which works a vertical rod CC', as far as possible without friction, carrying at its upper end a small frame BB'. In BB' a horizontal axis works, at right angles to which is a small cylinder D, with a tightening screw H, through which passes a long rod GG', to one end of which is affixed a large ring AA', and along which slides a small cylinder carrying a weight W, which is capable of being fixed at any point of the rod; and so that it may act as a counterpoise to the ring, or to the ring and any weight attached to it. An axis AA' works on pivots in the ring, in the same straight line with GG'; to AA' a disc, or sphere, or cone, or any other body can be attached, and thus can rotate about

AA′ as its axis; to the body thus attached to AA′ a rapid rotation can be given, either by means of a string wound round AA′, or by a machine contrived for the purpose when AA′ and its attached body are applied to it. It is evident that the counterpoise W can be so adjusted that the centre of gravity of the rod, the ring, the attached body, and the counterpoise, should be in the axis BB′; or at any point on either side of it; that is, h may be positive, or be equal to 0, or may be negative. Also by fixing BB′ in the arm of CC′ which carries it, the inclination of the rod GG′ to the vertical may be made constant, that is, θ may be equal to θ_0 throughout the motion. When the counterpoise is so adjusted that the centre of gravity of the rod GG′ and its appendages is in CC′, then $h = 0$, or, what is equivalent, $mg = 0$.

If the counterpoise is adjusted so that the centre of gravity of the rod GG′, of the ring, and of W, without AA′ and its attached body, is in BB′, then the weight of the body will produce its full effect, and the results indicated in the foregoing Articles will be exhibited.

329.] In application of the general equations of rotatory motion we may here insert another problem which is of great interest and importance, although perhaps it more properly comes into the following Chapter.

When a body has motion of both translation and rotation, the investigation into these several motions may be conducted separately, by virtue of those fundamental theorems which have been proved in Section 2 of Chap. III, and the rotation may be considered relative to the centre of gravity and an axis passing through it; just as if the centre of gravity was a fixed point and had no motion of translation. This is precisely what I propose to do now: I propose to consider the rotatory phaenomena of the earth, having its centre of gravity fixed at least hypothetically, under the action of the attracting forces of the sun and the moon; I shall indeed consider it as merely a mathematical problem; but it will have its application to these three bodies: and as the resulting differential equations will not admit of integration exactly in their general form, I shall make those hypotheses as to small quantities which are given to us by the circumstances of these bodies. Our inquiry too will be general, and will include the action of all bodies by which the rotation

of the earth is affected; that is, of not only the sun and the moon, if there are others whose influence affects the earth's motion of rotation. The law of action of these bodies on the earth is of course that of gravitation. The attraction varies directly as the product of the masses, and inversely as the square of the distance.

Let m be the mass of the body whose action on the earth we are considering; let the centre of gravity of the earth be the origin, and let the central principal axes of the earth be, as heretofore, the axes of ξ, η, ζ; the ζ-axis being the geometrical polar axis; and let A, B, C be the central principal moments of the earth relative to these axes respectively; let dm' be a mass-element of the earth, of which the density is ρ, and let its place be (ξ, η, ζ), then ρ is a function of these coordinates; let (x, y, z) be the centre of gravity of m; $r' =$ the distance of (x, y, z) from (ξ, η, ζ); and let r be the distance of m from the centre of gravity of the earth; and let the attraction which two unit-particles at an unit-distance exert on each other be the unit of attraction, and be unity; then

$$\left.\begin{aligned} r^2 &= x^2+y^2+z^2, \\ r'^2 &= (x-\xi)^2+(y-\eta)^2+(z-\zeta)^2. \end{aligned}\right\} \qquad (196)$$

Now, for two reasons, we consider the attraction of m on the earth to be the same as if m were condensed into a particle of mass m at its centre of gravity; (1) because the distance between m and the earth is supposed to be very great, and consequently the theorem proved in Art. 223, Vol. III, is applicable to its action; (2) because the bounding surface of m is nearly spherical, and m is supposed to consist of a series of concentric spherical shells, the attraction of each of which on an external particle dm' is the same as if it were condensed into its centre of gravity.

Let X, Y, Z be, relatively to the earth's principal axes, the axial components of the attraction of m on the earth; then

$$\left.\begin{aligned} \text{X} &= m\iiint\frac{(x-\xi)\,\rho\, d\xi\, d\eta\, d\zeta}{r'^3}, \\ \text{Y} &= m\iiint\frac{(y-\eta)\,\rho\, d\xi\, d\eta\, d\zeta}{r'^3}, \\ \text{Z} &= m\iiint\frac{(z-\zeta)\,\rho\, d\xi\, d\eta\, d\zeta}{r'^3}; \end{aligned}\right\} \qquad (197)$$

$$\left.\begin{aligned} \therefore\quad L = Zy - Yz &= m\iiint \frac{\rho\,(z\eta - y\zeta)\,d\xi\,d\eta\,d\zeta}{r'^3},\\ M = Xz - Zx &= m\iiint \frac{\rho\,(x\zeta - z\xi)\,d\xi\,d\eta\,d\zeta}{r'^3},\\ N = Yx - Xy &= m\iiint \frac{\rho\,(y\xi - x\eta)\,d\xi\,d\eta\,d\zeta}{r'^3}; \end{aligned}\right\} \quad (198)$$

the integrations in each equation being such that all the elements of the earth are included.

As the distance of the centre of m from the centre of the earth is very great in comparison with the mean radius of the earth, and consequently with the coordinates of any element of the earth, even when m is the moon; so the quantities under the signs of integration in the right-hand members of the preceding equations may be expressed as series of terms rapidly converging in powers of $\frac{\xi}{r}, \frac{\eta}{r}, \frac{\zeta}{r}$: the greatest value of either of these quantities is, in the case of the moon, $\frac{1}{59.96}$; and in the case of the sun, $\frac{1}{23984}$; in the following expansion therefore I shall omit all powers of these quantities above the second.

For the effect of subsequent terms in the series, the student may consult a Memoir, having for its title, "Théorie du mouvement de la Terre autour de son Centre de Gravité," by M. J. A. Serret; and contained in Vol. V of "Annales de l'Observatoire Impérial de Paris," 1859. He will there find the mode of calculating the terms which arise on the hypothesis, that the oblateness of the northern and southern hemispheres of the earth is different; and on the hypothesis, that the earth is not symmetrical relatively to the polar axis of figure.

Now, from (196),

$$\begin{aligned} \frac{1}{r'^3} &= \{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2\}^{-\frac{3}{2}} \\ &= \{x^2 + y^2 + z^2 - 2\,(x\xi + y\eta + z\zeta) + \xi^2 + \eta^2 + \zeta^2\}^{-\frac{3}{2}} \end{aligned}$$

$$\begin{aligned} \frac{1}{r'^3} &= \frac{1}{r^3}\left\{1 - \frac{2\,(x\xi + y\eta + z\zeta)}{r^2} + \frac{\xi^2 + \eta^2 + \zeta^2}{r^2}\right\}^{-\frac{3}{2}} \\ &= \frac{1}{r^3}\left\{1 + \frac{3\,(x\xi + y\eta + z\zeta)}{r^2} - \frac{3}{2}\,\frac{\xi^2 + \eta^2 + \zeta^2}{r^2} \right. \\ &\qquad\qquad \left. + \frac{15}{2}\,\frac{(x\xi + y\eta + z\zeta)^2}{r^4}\right\}; \end{aligned} \quad (199)$$

let us substitute this value in (198); then, since the centre of gravity of the earth is the origin, and the central principal axes are the coordinate axes,

$$\iiint \rho\xi\, d\xi\, d\eta\, d\zeta = \iiint \rho\eta\, d\xi\, d\eta\, d\zeta = \iiint \rho\zeta\, d\xi\, d\eta\, d\zeta = 0; \quad (200)$$

$$\iiint \rho\eta\zeta\, d\xi\, d\eta\, d\zeta = \iiint \rho\zeta\xi\, d\xi\, d\eta\, d\zeta = \iiint \rho\xi\eta\, d\xi\, d\eta\, d\zeta = 0; \quad (201)$$

and consequently, omitting all powers of small quantities above the second, (198) become

$$\left.\begin{aligned} \text{L} &= 3m(\text{C}-\text{B})\frac{yz}{r^5}, \\ \text{M} &= 3m(\text{A}-\text{C})\frac{zx}{r^5}, \\ \text{N} &= 3m(\text{B}-\text{A})\frac{xy}{r^5}. \end{aligned}\right\} \quad (202)$$

With regard to the last two terms of (199) which do not appear in these equations, having been omitted on account of the smallness of the quantities, I would observe, that they disappear of themselves in the integration, if the earth is supposed to be symmetrical in the distribution of its elements in the northern and southern hemispheres, and with respect to its polar axis of figure. So that under this hypothesis the equations (202) are much more approximate than they appear to be at first sight.

Since $\frac{x}{r}, \frac{y}{r}, \frac{z}{r}$ are the direction-cosines of the line joining the centres of the earth and the attracting body, it appears that L, M, and N vary directly as the mass of the attracting body, and inversely as the cube of the distance of its centre from the centre of the earth. Hence, if we calculate, from a synoptic table of the elements of the moon and of the planets, this quantity, it will at once be seen that the sun and the moon are the only bodies which produce any sensible effect on the rotation of the earth; the effect of the sun is due to its very large mass; and the effect of the moon, which is much greater, to its nearer distance.

330.] Equations (202) admit of further simplification; and let us first consider them with respect to the sun.

Let $n' =$ the mean angular velocity of the earth about the

sun; let E = the mass of the earth; then, as the eccentricity of the earth's orbit is very small, r may be taken as the mean distance of the earth from the sun; and, equating the earth's periodic time in terms of n' with that given in Vol. III, Art. 404, (154), we have

$$\frac{2\pi}{n'} = \frac{2\pi r^{\frac{3}{2}}}{(m+E)^{\frac{1}{2}}}; \tag{203}$$

now $\frac{E}{m} = \frac{1}{389551}$, according to Encke, quoted by sir John Herschel; and this quantity being small may be neglected, so that

$$\frac{m}{r^3} = n'^2; \tag{204}$$

and therefore for the action of the sun, the equations (202) become

$$\left.\begin{aligned} L &= 3n'^2(C-B)\frac{yz}{r^2}, \\ M &= 3n'^2(A-C)\frac{zx}{r^2}, \\ N &= 3n'^2(B-A)\frac{xy}{r^2}. \end{aligned}\right\} \tag{205}$$

Again, let us consider (202) with respect to the moon; and let all the quantities which refer to the sun receive an accent, and thus refer to the moon.

Let n'' be the mean angular velocity of the moon about the earth; then, if we neglect the eccentricity of the moon's orbit, and take r' to be the mean distance, by the same theorem as that which we have just now applied to the sun,

$$\frac{2\pi}{n''} = \frac{2\pi r'^{\frac{3}{2}}}{(m'+E)^{\frac{1}{2}}}; \tag{206}$$

but $\frac{E}{m'} = 81.84$ nearly, $= e$ (say); so that

$$\frac{m'}{r'^3} = \frac{n''^2}{1+e}; \tag{207}$$

and substituting this value in (202) we have

$$\left.\begin{aligned} L' &= \frac{3n''^2}{1+e}(C-B)\frac{y'z'}{r'^2}, \\ M' &= \frac{3n''^2}{1+e}(A-C)\frac{z'x'}{r'^2}, \\ N' &= \frac{3n''^2}{1+e}(B-A)\frac{x'y'}{r'^2}; \end{aligned}\right\} \tag{208}$$

and the equations of rotation of the earth become

$$\left.\begin{aligned}
\frac{d\omega_1}{dt}+\frac{\text{C}-\text{B}}{\text{A}}\omega_2\omega_3 &= \frac{3(\text{C}-\text{B})}{\text{A}}\left\{\frac{n'^2yz}{r^2}+\frac{n''^2y'z'}{(1+e)r'^2}\right\},\\
\frac{d\omega_2}{dt}+\frac{\text{A}-\text{C}}{\text{B}}\omega_3\omega_1 &= \frac{3(\text{A}-\text{C})}{\text{B}}\left\{\frac{n'^2zx}{r^2}+\frac{n''^2z'x'}{(1+e)r'^2}\right\},\\
\frac{d\omega_3}{dt}+\frac{\text{B}-\text{A}}{\text{C}}\omega_1\omega_2 &= \frac{3(\text{B}-\text{A})}{\text{C}}\left\{\frac{n'^2xy}{r^2}+\frac{n''^2x'y'}{(1+e)r'^2}\right\}.
\end{aligned}\right\} \qquad (209)$$

The complete integration of these equations is beyond the power of analysis; and we are obliged to have recourse to methods of approximation, taking advantage of those circumstances as to small quantities which the relations of the sun, earth, and moon offer to us: these we proceed to explain.

331.] In the first place, geodetic measurements shew that the figure of the earth is nearly that of a solid of revolution, whose axis is the polar axis of figure; and as there is no reason to suppose any great want of symmetry in the distribution of the material elements in the interior of the earth, we may suppose the two principal moments in the plane of the equator, and consequently all the moments of inertia in that plane, to be equal; thus, $\text{B} = \text{A}$; and this equality exists whatever are the positions of the axes of x and y in the plane of the equator.

I may moreover observe, that the most profound calculations*, based on the hypothesis of an unsymmetrical distribution of material within the earth, lead to the conclusion that ω_3 is constant to a first approximation, and that consequently $\text{B} = \text{A}$; this result follows from the fact that the action of the sun and moon is very small in comparison of the actual vis viva of the earth.

Observations made with the pendulum are in accordance with direct measurement, and shew the earth to be a solid of revolution, whose polar axis is shorter than the equatorial; and that its figure is approximately an oblate spheroid; and thus C, which is the central principal moment relative to the axis of revolution, is the greatest of all moments. Now, putting $\text{B} = \text{A}$ in (209), it is plain that C and A enter into the equations of motion only in the form $\frac{\text{C}-\text{A}}{\text{A}}$: the value of this quantity cannot be determined by direct observation, because we are ignorant of the law of density of the matter of the earth, and we are obliged to have

* See the Memoir of Serret quoted above; also Le Verrier, Annales de l'Observatoire Impérial de Paris, Tome II.

recourse to indirect methods. The observed values of precession and nutation give it a value of nearly $\frac{1}{306}$*, which is beyond doubt almost correct; also a hypothesis of Laplace, discussed in the Mécanique Céleste, Livre XI, gives a result nearly identical; this value we shall take. Since the physical constitution of the earth enters into the equations of motion only by means of these quantities, it is evident that the phaenomena of precession and nutation would be the same, whatever change took place in the earth, so long as the ratio $\frac{\text{C}-\text{A}}{\text{A}}$ was unaltered.

Again, the actual axis of rotation of the earth is almost fixed in it, and is almost identical with the axis of figure; that is, the poles of the earth are almost fixed points on its surface. Were they not so, geographical latitudes would vary from time to time; whereas no variation has been indicated by observation, so far as I know. Moreover, as the true rotation-axis of the earth in all its positions nearly coincides with the axis of figure, the true angular velocity ω, which is the resultant of ω_1, ω_2, ω_3, is nearly equal to ω_3, which is the angular velocity about the earth's axis of figure, and is constant; and thus ω_1 and ω_2 are very small quantities. Thus, if we image the actual rotation of the earth by the rolling of one cone on another, that cone which the earth's axis describes in itself has a very small vertical angle, the cone fixed in space having a vertical angle a little greater than 46° 55′. See Art. 302.

This is the information which observation gives as to the circumstances of the constitution and the figure of the earth, and as to the approximate invariability of its angular velocity, and of the position of its rotation-axis.

Under these circumstances the equations of motion become

$$\left.\begin{aligned}
\frac{d\omega_1}{dt}+\frac{\text{C}-\text{A}}{\text{A}}\omega_2\omega_3 &= \frac{3(\text{C}-\text{A})}{\text{A}}\left\{\frac{n'^2yz}{r^2}+\frac{n''^2y'z'}{(1+e)r'^2}\right\}, \\
\frac{d\omega_2}{dt}-\frac{\text{C}-\text{A}}{\text{A}}\omega_3\omega_1 &= -\frac{3(\text{C}-\text{A})}{\text{A}}\left\{\frac{n'^2zx}{r^2}+\frac{n''^2z'x}{(1+e)r'^2}\right\}, \\
\frac{d\omega_3}{dt} &= 0;
\end{aligned}\right\} \tag{210}$$

from the third of these

$$\omega_3 = n, \tag{211}$$

* See the Memoirs of Serret and Le Verrier.

if n is the angular velocity of the earth about its polar axis of figure.

Also, for convenience of expression, let

$$\frac{\text{C}-\text{A}}{\text{A}} = a;$$

then the first two of (210) become

$$\left.\begin{aligned} \frac{d\omega_1}{dt} + an\omega_2 &= 3a\left\{\frac{n'^2 yz}{r^2} + \frac{n''^2 y'z'}{(1+e)r'^2}\right\}, \\ \frac{d\omega_2}{dt} + an\omega_1 &= -3a\left\{\frac{n'^2 zx}{r^2} + \frac{n''^2 z'x'}{(1+e)r'^2}\right\}; \end{aligned}\right\} \quad (212)$$

and from these equations all the phaenomena of the rotation are to be deduced.

Equations (210) shew that the action of both the sun and the moon on the earth is due to the physical constitution of the earth itself. If $\text{C} = \text{A}$, that is, if all the principal central moments of the earth were equal, $\frac{d\omega_1}{dt} = \frac{d\omega_2}{dt} = \frac{d\omega_3}{dt} = 0$; and thus the angular velocity would be constant, and the earth's rotation-axis would be fixed in itself, and would be absolutely fixed in space; the protuberant matter at the earth's equator, which causes the inequality of the central principal moments, is thus the indirect cause of the peculiar motion of the earth's rotation-axis, which we are about to investigate.

332.] The arrangement of the bodies which is convenient for our system of symbols and equations is exhibited in Fig. 38.

O is the centre of the earth; and the plane XOY is the fixed plane of the ecliptic; OX being the line of the vernal equinox when $t = 0$. About O as a centre a sphere is described whose radius is equal to unity; and the several curved lines of the figure are the intersections of the surface of this sphere by various planes and lines drawn through O, and all refer to the configuration of the system at the time t; yxN is the plane of the earth's equator, so that ON is the line of the vernal equinox, and XON is the precession; Oz is the earth's polar axis about which the angular velocity is n, and Ox and Oy are the earth's principal axes in the plane of the equator, Ox being so chosen that it coincides at the same time with OX, OS, and ON; OS is the radius vector of the sun, which is always in the plane of the ecliptic; OM is the radius vector of the moon, MN′I being the plane of the moon's orbit; ON′ is the line of intersection of that plane with the plane

of the ecliptic, and is the line of the moon's nodes; OI is the line of intersection of the plane of the moon's orbit with the plane of the earth's equator. Let i be the angle of inclination of the plane of the moon's orbit to the ecliptic; then i is nearly constant, and has a mean value of $5^\circ\, 8'\, 48''$; we shall take it to be constant. Now the line of nodes of the moon revolves in the plane of the ecliptic, and performs a complete revolution in about 6793 days. Thus ON′ revolves about OZ; let β be its angular velocity; then, if $n = 1$, we have approximately,

$$n' = \frac{1}{365.25}, \quad n'' = \frac{1}{27.32}, \quad \beta = \frac{1}{6793};$$

so that β is much less than the other quantities; the small fraction a is also a factor of all the terms into which these quantities enter.

As the angular motion of the line of equinoxes is very small, the angle XON is very small compared with NOx, or XOS; so that approximately NOS $=$ XOS $= n't$; and NO$x = nt$. We shall also in calculating small terms neglect variations of θ. From this arrangement we have

$$\left.\begin{aligned}
\frac{x}{r} &= \cos x\text{OS} = \cos nt \cos n't + \sin nt \sin n't \cos\theta, \\
\frac{y}{r} &= \cos y\text{OS} = -\sin nt \cos n't + \cos nt \sin n't \cos\theta, \\
\frac{z}{r} &= \cos z\text{OS} = -\sin n't \sin\theta;
\end{aligned}\right\} \qquad (213)$$

which are thus expressed in terms of t and of constants.

Again, as to the moon; let us in the first place refer it to the ecliptic; then, if ψ'_0 is the longitude of the moon's node at the vernal equinox, that is, when $t = 0$, NON′ $= \psi'_0 + \beta t = \psi'$, say; and if ϕ'_0 is the moon's right ascension at the vernal equinox, N′OM $= \phi'_0 + n''t = \phi'$, say; then, as i is very small, N′OM and its projection on the plane of the ecliptic may be considered to be equal; so that the longitude of M is $\phi' + \psi'$; and, if we replace $\sin i$ by i, $\cos$ MOZ $= i \sin\phi'$;

$$\left.\begin{aligned}
\frac{x'}{r'} &= \cos x\text{OM} = \{\sin(\phi'+\psi')\cos\theta + i\sin\phi'\sin\theta\}\sin nt + \cos(\phi'+\psi')\cos nt, \\
\frac{y'}{r'} &= \cos y\text{OM} = \{\sin(\phi'+\psi')\cos\theta + i\sin\phi'\sin\theta\}\cos nt - \cos(\phi'+\psi')\sin nt, \\
\frac{z'}{r'} &= \cos z\text{OM} = i\sin\phi'\cos\theta - \sin(\phi'+\psi')\sin\theta;
\end{aligned}\right\} (
$$

and as $\phi'+\psi' = \phi'_0+\psi'_0+(\beta+n'')t$, these quantities are expressed in terms of t, and of known quantities; they are to be substituted in the equations (212); which are then to be integrated.

The linear form of the equations (212) shews that the effects of the action of the sun and the moon may be calculated separately; and that the whole effect is the sum of the two separate effects. We shall consequently calculate each by itself. Instead of determining ω_1 and ω_2 by means of these equations, it will be more convenient to calculate θ and ψ directly, as the position of the earth will hereby be determined with reference to fixed lines.

The equations which connect these angles with the principal angular velocities are given in Art. 64; and are

$$\frac{d\theta}{dt} = \omega_1 \cos\phi - \omega_2 \sin\phi,$$

$$\sin\theta \frac{d\psi}{dt} = \omega_1 \sin\phi + \omega_2 \cos\phi.$$

333.] Our object in this inquiry is not to calculate accurately the motion of the earth's rotation-axis, and the earth's angular velocity which determines the length of a day; but to trace roughly, and to indicate in their salient points, the results of the action of the sun and moon. We shall therefore retain only the larger quantities, and small quantities of the first order; and we shall only notice the kind of change which is produced, with a view rather to the general effect of such action than to numerical calculations.

We will first consider the terms in (212) which refer to the sun, and which will be replaced by their values in (213).

Now the earth's axis is inclined to the normal of the ecliptic at an angle which is nearly constant; let I be its mean value, which is about 23°27′32″; this angle being that between the earth's equator and the ecliptic is called the obliquity of the ecliptic. It is the mean value of θ according to our arrangement, and we shall replace θ by it in terms involving small quantities. Also, as the earth rotates uniformly about its polar axis with the angular velocity n, and as the angular velocity of ON is very small, $\phi = nt$, omitting small quantities; and thus the equations of the last Article become

$$\left.\begin{aligned}\frac{d\theta}{dt} &= \omega_1 \cos nt - \omega_2 \sin nt, \\ \sin \text{I} \frac{d\psi}{dt} &= \omega_1 \sin nt + \omega_2 \cos nt;\end{aligned}\right\} \tag{215}$$

$$\therefore \frac{d^2\theta}{dt^2} = \cos nt \frac{d\omega_1}{dt} - \sin nt \frac{d\omega_2}{dt} - n(\omega_1 \sin nt + \omega_2 \cos nt)$$

$$= \cos nt \frac{d\omega_1}{dt} - \sin nt \frac{d\omega_2}{dt} - n \sin \text{I} \frac{d\psi}{dt}; \tag{216}$$

$$\sin \text{I} \frac{d^2\psi}{dt^2} = \sin nt \frac{d\omega_1}{dt} + \cos nt \frac{d\omega_2}{dt} + n \frac{d\theta}{dt}; \tag{217}$$

substituting for $\frac{d\omega_1}{dt}$ and $\frac{d\omega_2}{dt}$ their values given in (212), taking only the terms which depend on the sun's action, we have

$$\frac{d^2\theta}{dt^2} + (1+a)n \sin \text{I} \frac{d\psi}{dt} + 3an'^2(\sin n't)^2 \sin \text{I} \cos \text{I} = 0; \tag{218}$$

$$\sin \text{I} \frac{d^2\psi}{dt^2} - (1+a)n \frac{d\theta}{dt} - 3an'^2 \sin n't \cos n't \sin \text{I} = 0. \tag{219}$$

Integrating (218) we have

$$\frac{d\theta}{dt} + (1+a)n \sin \text{I}\, \psi + \frac{3an'^2}{2} \sin \text{I} \cos \text{I} \left(t - \frac{\sin 2n't}{2n'}\right) = 0; \tag{220}$$

no constant being added, because if no disturbing force acts, that is, if $n'=0$, $\frac{d\theta}{dt} = 0$; and $\psi = 0$, when $t=0$. Substitute for $\frac{d\theta}{dt}$ in (219), and we have

$$\frac{d^2\psi}{dt^2} + (1+a)^2 n^2 \psi = -\frac{3a(1+a)n'^2 n \cos \text{I}}{2} t$$

$$+ \frac{3an'}{4} \{(1+a)n \cos \text{I} + 2n'\} \sin 2n't; \tag{221}$$

$$\therefore \psi = -\frac{3an'^2 \cos \text{I}}{2(1+a)n} t + \frac{3an'}{4} \frac{(1+a)n \cos \text{I} + 2n'}{(1+a)^2 n^2 - 4n'^2} \sin 2n't$$

$$+ \text{C}' \sin\{(1+a)nt + \gamma\}; \tag{222}$$

where C′ and γ are constants introduced in integration; but since $\psi = 0$ when $t = 0$, and $\frac{d\psi}{dt}$ is independent of t when $n' = 0$, C′ $= 0$, and $\gamma = 0$.

Also, in the coefficient of $\sin 2n't$ we may omit $\frac{n'^2}{n^2}$, on account

of its smallness; and, since $a = \frac{C-A}{A}$,

$$\frac{a}{1+a} = \frac{C-A}{C}; \tag{223}$$

and thus (222) becomes

$$\psi = -\frac{3n'^2 \cos I}{2n}\frac{C-A}{C}t + \frac{3n' \cos I}{4n}\frac{C-A}{C}\sin 2n't. \tag{224}$$

Replacing ψ in (220) by this value we have an identity; which shews that the terms herein retained destroy each other in the variation of θ, although they give a finite result in the value of ψ. We must therefore replace ψ by the value which it has before small terms are omitted; that is, we must substitute for ψ the value given in (222), putting however $C' = 0$; then (220) becomes

$$\frac{d\theta}{dt} = -\frac{3n'^2 \sin I}{2n}\frac{C-A}{C}\sin 2n't; \tag{225}$$

$$\therefore \quad \theta = I + \frac{3n' \sin I}{4n}\frac{C-A}{C}\cos 2n't; \tag{226}$$

where I is the constant of integration and is the mean value of θ.

Equations (224) and (226) exhibit the effects of the sun's action on the rotation of the earth. ψ is the angle through which the line of equinoxes, ON in Fig. 38, moves in the time t, and is called the Solar Precession of the equinoxes; (224) shews that it consists of two terms, from the former of which it appears that ψ increases directly as the time; and from the latter, that this continual motion is accompanied by a periodical variation, of which the periodic time is $\frac{\pi}{n'}$, that is, is half a year. This periodical quantity is called the Solar Nutation of the Earth's Axis in Longitude, or, the Nutation of the Equinoxes. Thus, the line of equinoxes is sometimes a little in advance of, and sometimes a little behind, its mean place; and coincides with its mean place every half year; but as the coefficient of this periodical part is very small, so does the term scarcely ever acquire a sensible magnitude.

From (226) it appears that θ has a mean value I; but that the earth's axis has a small oscillatory motion, depending on the second term of which the period is also half a year; and this second term is always very small because its coefficient is small. It is called the Solar Nutation of the Earth's Axis in Latitude,

or, the Nutation of the Obliquity. Thus, the mean rotation-axis of the earth would have a very slow progressive motion in space, inclined at a constant angle I to the normal of the ecliptic, if it were disturbed by only the sun's action.

334.] The effect of the moon on the rotation of the earth is expressed by the latter terms in the right-hand members of (212); these we now proceed to inquire into, and by a process similar to that by which we have investigated the action of the sun.

For abridgment of notation let the moon's longitude $=\mu+\nu t$; so that

$$\mu = \phi'_0+\psi'_0; \quad \nu = \beta+n''; \tag{227}$$

and let us replace θ by I; (214) become

$$\left.\begin{aligned}
\frac{x'}{r'} &= \{\sin(\mu+\nu t)\cos\text{I}+i\sin(\phi'_0+n''t)\sin\text{I}\}\sin nt+\cos(\mu+\nu t)\cos nt,\\
\frac{y'}{r'} &= \{\sin(\mu+\nu t)\cos\text{I}+i\sin(\phi'_0+n''t)\sin\text{I}\}\cos nt-\cos(\mu+\nu t)\sin nt,\\
\frac{z'}{r'} &= i\sin(\phi'_0+n''t)\cos\text{I}-\sin(\mu+\nu t)\sin\text{I};
\end{aligned}\right\} \tag{228}$$

as i is a small angle, the squares and higher powers of it will be omitted. Substituting these quantities in (212) and in (216), we have

$$\begin{aligned}
\frac{d^2\theta}{dt^2} &= -(1+a)n\sin\text{I}\,\frac{d\psi}{dt}+\frac{3an''^2}{1+e}\frac{z'}{r'}\{\sin(\mu+\nu t)\cos\text{I}+i\sin(\phi'_0+n''t)\sin\text{I}\}\\
&= -(1+a)n\sin\text{I}\,\frac{d\psi}{dt}+\frac{3an''^2}{1+e}\Big\{-\frac{\sin 2\text{I}}{4}\{1-\cos 2(\mu+\nu t)\}\\
&\quad+\frac{i\cos 2\text{I}}{2}(\cos\{\psi'_0+\beta t\}-\cos\{2\phi'_0+\psi'_0+(\beta+2n'')t\})\Big\};
\end{aligned} \tag{229}$$

and substituting in (217),

$$\begin{aligned}
\sin\text{I}\,\frac{d^2\psi}{dt^2} &= (1+a)n\frac{d\theta}{dt}+\frac{3an''^2}{1+e}\Big\{\frac{\sin\text{I}}{2}\sin 2(\mu+\nu t)\\
&\quad+\frac{i\cos\text{I}}{2}(\sin\{\psi'_0+\beta t\}-\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\})\Big\}.
\end{aligned} \tag{230}$$

From (229) we have

$$\begin{aligned}
\frac{d\theta}{dt} &= -(1+a)n\sin\text{I}\,\psi+\frac{3an''^2}{1+e}\Big\{-\frac{\sin 2\text{I}}{4}\Big(t-\frac{\sin 2(\mu+\nu t)}{2\nu}\Big)\\
&\quad+\frac{i\cos 2\text{I}}{2}\Big(\frac{\sin(\psi'_0+\beta t)}{\beta}-\frac{\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}}{\beta+2n''}\Big)\Big\}.
\end{aligned} \tag{231}$$

Now substituting this in (230), we have

$$\frac{d^2\psi}{dt^2}+(1+a)^2n^2\psi=-\frac{3an''^2(1+a)n\cos\text{I}}{2(1+e)}t$$
$$+\frac{3an''^2}{1+e}\left\{\frac{(1+a)n\cos\text{I}+2\nu}{4\nu}\sin 2(\mu+\nu t)\right.$$
$$+\frac{(1+a)n\cos 2\text{I}+\beta\cos\text{I}}{2\beta\sin\text{I}}i\sin(\psi'_0+\beta t)$$
$$\left.-\frac{(1+a)n\cos 2\text{I}+(\beta+2n'')\cos\text{I}}{2(\beta+2n'')\sin\text{I}}i\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}\right\}; \quad (232)$$

$$\therefore\ \psi=-\frac{3an''^2\cos\text{I}}{2(1+e)(1+a)n}t+\frac{3an''^2}{1+e}\left\{\frac{(1+a)n\cos\text{I}+2\nu}{4\nu\{(1+a)^2n^2-4\nu^2\}}\sin 2(\mu+\nu t)\right.$$
$$+\frac{(1+a)n\cos 2\text{I}+\beta\cos\text{I}}{2\beta\sin\text{I}\{(1+a)^2n^2-\beta^2\}}i\sin(\psi'_0+\beta t)$$
$$\left.-\frac{(1+a)n\cos 2\text{I}+(\beta+2n'')\cos\text{I}}{2(\beta+2n'')\sin\text{I}\{(1+a)^2n^2-(\beta+2n'')^2\}}i\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}\right\}; \quad (233)$$

the constants being omitted for the same reason as they are omitted in (222).

Now β is very small compared with n'', and thus ν may be replaced by n''; and the squares and higher powers of $\frac{n''}{n}$ may be omitted; so that after all reductions (233) becomes

$$\psi=-\frac{3an''^2\cos\text{I}}{2(1+e)(1+a)n}t+\frac{3an''\cos\text{I}}{4n(1+e)(1+a)}\sin 2\{\phi'_0+\psi'_0+(\beta+n'')t\}$$
$$+\frac{3an''^2\cos 2\text{I}}{2\beta n(1+e)(1+a)\sin\text{I}}i\sin(\psi'_0+\beta t)$$
$$-\frac{3an''\cos 2\text{I}}{4n(1+a)(1+e)\sin\text{I}}i\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}. \quad (234)$$

If we substitute this value for ψ in (231) it leads to an identity, and thus it appears that the terms which are herein retained cancel each other in the variation of θ; we must therefore replace ψ in (231) by its more approximate value, which is given in 33); and we have

$$\frac{d\theta}{dt}=\frac{3an''^2}{(1+e)(1+a)n}\left\{\frac{-\sin\text{I}}{2}\sin 2(\mu+\nu t)-\frac{\cos\text{I}}{2}i\sin(\psi'_0+\beta t)\right.$$
$$\left.+\frac{\cos\text{I}}{2}i\sin\{2\phi'_0+\psi'_0+(\beta+2n'')t\}\right\}; \quad (235)$$

$$\theta=\text{I}+\frac{3an''^2}{2(1+e)(1+a)n}\left\{\frac{\sin\text{I}}{2n''}\cos 2\{\phi'_0+\psi'_0+(\beta+n'')t\}\right.$$
$$\left.+\frac{\cos\text{I}}{\beta}i\cos(\psi'_0+\beta t)-\frac{\cos\text{I}}{2n''}i\cos\{2\phi'_0+\psi'_0+(\beta+2n'')t\}\right\}; \quad (236)$$

where I is the mean value of θ, and is the constant introduced in integration.

In (234) and (236) the last terms which involve the angle $2\phi'_0+\psi'_0+(\beta+2n'')t$ are to be omitted, because of the smallness of the coefficient in which i is a factor; the next preceding terms in each however must be retained, because β, which is a very small quantity, is in the denominator of the coefficient, and this brings it into importance, although it contains i as a factor; thus, if we replace $\frac{a}{1+a}$ by its value, given in (223), we have from (234) and (236),

$$\psi = -\frac{3n''^2 \cos \text{I}}{2(1+e)n}\frac{\text{C}-\text{A}}{\text{C}}t + \frac{3n''\cos \text{I}}{4n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}\Big\{\sin 2\{\phi'_0+\psi'_0+(\beta+n'')t\} + \frac{4n''\cos 2\text{I}}{\sin 2\text{I}}\frac{i}{\beta}\sin(\psi'_0+\beta t)\Big\}; \quad (237)$$

$$\theta = \text{I} + \frac{3n''}{2n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}\Big\{\frac{\sin \text{I}}{2}\cos 2\{\phi'_0+\psi'_0+(\beta+n'')t\} + n''\cos \text{I}\frac{i}{\beta}\cos(\psi'_0+\beta t)\Big\}. \quad (238)$$

On comparing these values with (224) and (226), which express the sun's action, it is evident that they produce effects on the earth's axis of precisely the same kind; so that what has there been said of solar precession and nutation, may here, mutatis mutandis, be said of lunar precession and nutation; but the effect of the terms in these latter expressions is much greater than that of those in the former, because n'' is much greater than n'.

335.] The whole precession and nutation is the sum of the two separate effects; but before we add, we must make a remark or two on the signs of our quantities. We have taken all the angular velocities to be positive; that is, we have supposed the bodies to revolve from the axis of X towards that of Y in Fig. 38; and this hypothesis is in accordance with the convention of signs which has been adopted throughout the volume; it is not however necessarily that of the actual motion of the earth and moon, of the moon's line of nodes, and of the apparent motion of the sun: let OX be east on the ecliptic, and let OZ be the normal to the ecliptic towards the north: now all the bodies revolve in their orbits, as well as about their axes, from west to east; so that the signs of n and of n'' are to be changed; that

of n' is correct, because the sun's motion is apparent only, being due to the actual motion of the earth. The line of the moon's nodes also retrogrades, that is, goes from east to west, so that the sign of β is correct.

Also let ☊ be the longitude of the moon's line of nodes at the time; let ☉ and ☾ be the longitudes of the sun and moon respectively*; then

$$☊ = \psi'_0 + \beta t, \qquad ☉ = n't, \qquad ☾ = \phi'_0 + \psi'_0 + (\beta - n'')t; \quad (239)$$

so that for the whole precession and nutation we have

$$\psi = \frac{3\cos\text{I}}{2n}\frac{\text{C}-\text{A}}{\text{C}}\left(n'^2 + \frac{n''^2}{1+e}\right)t$$
$$- \frac{3\cos\text{I}}{4n}\frac{\text{C}-\text{A}}{\text{C}}\left\{\frac{4n''^2\cos 2\text{I}}{(1+e)\sin 2\text{I}}\frac{i}{\beta}\sin ☊ - \frac{n''}{1+e}\sin 2☾ - n'\sin 2☉\right\}; \quad (240)$$

$$\theta = \text{I} - \frac{3}{2n}\frac{\text{C}-\text{A}}{\text{C}}\left\{\frac{n''^2\cos\text{I}}{1+e}\frac{i}{\beta}\cos ☊ - \frac{n''\sin\text{I}}{2(1+e)}\cos 2☾ - \frac{n'\sin\text{I}}{2}\cos 2☉\right\}; \quad (241)$$

the terms involving n'' and n' arise from the action of the moon and sun respectively.

The second of these equations shews that the earth's axis is inclined to the normal of the ecliptic at an angle which is nearly constant; yet that there are small variations of the angle which are expressed by the latter terms of (241); these terms are periodic, and are very small because their coefficients are small; they depend on the longitude of the moon's ascending node on the ecliptic, on the longitude of the sun, and on the longitude of the moon; they constitute the luni-solar nutation in latitude or in obliquity.

Equation (240) shews that the line of equinoxes has a general retrograde motion along the ecliptic, with an angular velocity

$$= \frac{3\cos\text{I}}{2n}\frac{\text{C}-\text{A}}{\text{C}}\left(n'^2 + \frac{n''^2}{1+e}\right) = \Psi, \text{ say}; \quad (242)$$

this quantity is called the luni-solar precession of the equinoxes; yet that this retrograde motion is not uniform, but is subject to slight variations, which are periodic, and are expressed by the last three terms of the right-hand member of (240); that these periodical quantities are very small, because their coefficients are small: they likewise depend on the longitude of the moon's line of nodes, and on the longitudes of the sun and the moon; and they constitute the luni-solar nutation in longitude.

* It will be observed that no distinction has been made between true and mean longitude, true and mean ecliptic, &c.; our calculations have not been carried far enough for such accurate positions.

The motion therefore of the earth's axis in space will be well represented by the Fig. 36; in which O is the centre of the earth and is supposed to be fixed, and the radius of the sphere is unity. The axis whose motion is defined by the equations

$$\theta = \text{I}, \qquad \psi = \Psi t, \tag{243}$$

may be called the mean axis of the earth, I and Ψt being respectively the mean obliquity and the mean precession. And the axis which is defined by the complete expressions (240) and (241) will be the true axis. Let $\text{COR} = \text{I}$; then the circle RS will be that along which the mean axis will intersect the surface of the sphere; and if OQ is the mean axis at the time t, and OP is the true axis, the angle POQ will be small; and as t varies OP will be sometimes before, and sometimes behind OQ; and sometimes nearer to, and sometimes farther from the pole of the ecliptic. Thus, the true axis of the earth will intersect the sphere in a wavy line contained between two parallels of latitude of the sphere at distances from RS, determined by the greatest positive and negative values of the periodic terms of θ given in (241).

The motion of the true axis relatively to the mean axis may, as to its principal and its most important terms, be exhibited in the following way. Suppose the point of intersection of the mean axis with the sphere to be an origin, at which two axes originate, say of ξ and η, in the plane touching the sphere; that of η being a tangent to the parallel along which the mean axis moves, and the ξ-axis being perpendicular to it, and thus being a tangent to the meridian through the place of the mean axis. Now the most important periodic terms in (240) and (241) are those which depend on the longitude of the moon's ascending node, on account of the smallness of β, as we have before observed; let these principal terms in the directions of the two axes of ξ and η be represented by ξ and η, so that

$$\xi = \frac{3}{2n}\frac{\text{C}-\text{A}}{\text{C}}\frac{n''^2 \cos \text{I}}{1+e}\frac{i}{\beta}\cos \Omega,$$

$$\eta = \frac{3}{2n}\frac{\text{C}-\text{A}}{\text{C}}\frac{n''^2 \cos 2\text{I}}{1+e}\frac{i}{\beta}\sin \Omega;$$

$$\therefore \quad \frac{\xi^2}{(\cos \text{I})^2} + \frac{\eta^2}{(\cos 2\text{I})^2} = \Big(\frac{3}{2n}\frac{\text{C}-\text{A}}{\text{C}}\frac{n''^2}{1+e}\frac{i}{\beta}\Big)^2; \tag{244}$$

which is the equation to an ellipse whose axes are in the ratio

of $\cos \text{I}$ to $\cos 2\text{I}$, and of which that directed towards the pole of the ecliptic is the greatest; and thus it follows that so far as the most important terms affect the motion, the true axis describes a small ellipse on the surface of the sphere relatively to the mean axis which passes through the centre of the ellipse. This ellipse is called the Ellipse of Nutation.

In the preceding image of the motion of the earth's axis, we have assumed the earth's centre to be fixed, and the radius of the sphere, to whose surface we have referred the motion of the axis, has also been assumed to be unity. The earth's centre however is not fixed; yet the image is a correct representation of the facts, because we refer the motion of the earth's axis to the sidereal vault, of which we may say the radius is so great that, in comparison of it, the distance through which the earth's centre moves is infinitesimal. Thus, the mean axis describes a circle about the pole of the ecliptic, the angular radius of which circle is $23° 27' 32''$; and the true axis describes a wavy line symmetrically situated with reference to this circle; and if the mean axis is considered fixed, the true axis describes an ellipse on the sidereal vault, the centre of which is the place where it is pierced by the mean axis.

The periodic time in which the mean pole describes its circle is $\frac{2\pi}{\Psi}$; and the true pole will describe its ellipse about the mean pole in the same time as that in which the moon's line of nodes describes a complete revolution.

The value of the annual luni-solar precession is found as follows:

$$\Psi = \frac{3\cos \text{I}}{2n}\frac{\text{C}-\text{A}}{\text{C}}\left(n'^2 + \frac{n''^2}{1+e}\right)$$

$$= 3\cos \text{I}\frac{\text{C}-\text{A}}{\text{C}}\frac{n'}{n}\left\{1 + \frac{n''^2}{(1+e)n'^2}\right\}\frac{n'}{2};$$

so that the annual luni-solar precession

$$= 3\cos \text{I}\frac{\text{C}-\text{A}}{\text{C}}\frac{n'}{n}\left\{1 + \frac{n''^2}{(1+e)n'^2}\right\}180°.$$

Now if we take the epoch to be Jan. 1, 1850*, $\text{I} = 23° 27' 32''$,

$$\frac{\text{C}-\text{A}}{\text{C}} = \frac{1}{306}, \quad \frac{n'}{n} = \frac{1}{365{\cdot}25}, \quad \frac{n''}{n'} = \frac{365{\cdot}25}{27{\cdot}32}, \text{ and } e = 81{\cdot}84;$$

* See the Memoir of M. Serret in Vol. V of "Annales de l'Observatoire Impérial de Paris," page 321.

and therefore
$$1+\frac{n''^2}{(1+e)n'^2} = 3\cdot 15764;$$
also
$$\cos 23^\circ 27' 32'' = \cdot 91735;$$
therefore the annual luni-solar precession
$$= \frac{\cdot 91735}{102} \times \frac{315\cdot 764}{36525} \times 180 \times 60 \times 60'',$$
$$= 50''\cdot 3828. \qquad (245)$$

The observed value of the luni-solar precession is $50''\cdot 37140$; so that our result is very nearly correct, although it is only approximate. I may* in passing remark, that the coefficients of $\sin \Omega$ and of $\cos \Omega$ in (240) and (241) respectively, are $-17''\cdot 251$, and $+9''\cdot 223$; the former being the largest value of the principal term of the nutation of the equinoxes, and the latter being the largest value of the principal term of the nutation of the obliquity. Also the mean axis describes a complete circle in the heavens in 25724 years.

336.] Of the problem of precession and nutation an approximate solution has also been given by M. Poinsot in the Additions to the "Connaissance des Temps" for 1858. The principal terms only are found by it; but it exhibits the problem in such an elementary form, and dissects the results of the action of the sun and moon into the several phaenomena so distinctly, that it is peculiarly fitted for a didactic treatise. We shall employ the symbols of the preceding Articles, and shall make use of the couples of the impressed momenta which have been therein determined.

We consider all quantities at the time t, and investigate the effects which accrue during the infinitesimal time dt. If n is the angular velocity of the earth about its rotation-axis, and G is the moment of the effective couple, and C is the moment of inertia relative to that axis; then
$$G = nC. \qquad (246)$$
Let L and M be the moments of the impressed couples relative to the axes of x and y in the plane of the earth's equator; as the

* On this subject see the Memoir entitled "Numerus constans Nutationis ex ascensionibus rectis stellæ polaris in Speculâ Dorpatensi annis 1822 ad 1838 observatis deductus," by C. F. Peters, and contained in "Mémoires de l'Académie Impériale des Sciences de Saint Pétersbourg, 6e série, première partie, Sciences mathématiques et physiques, Tome III, Saint Pétersbourg, 1844."

position of these axes in that plane is indeterminate, and as we shall consider the effects for only the time dt, we may suppose the axis of x to lie along the line of equinoxes, and the axis of y to be perpendicular to it. Thus (213) and (214) become

$$\frac{x}{r} = \cos n't, \quad \frac{y}{r} = \cos \text{I} \sin n't, \quad \frac{z}{r} = -\sin \text{I} \sin n't; \quad (247)$$

$$\left.\begin{aligned} \frac{x'}{r'} &= \cos(\mu+\nu t), \\ \frac{y'}{r'} &= \cos \text{I} \sin(\mu+\nu t) + i \sin \text{I} \sin(\phi'_0 + n''t), \\ \frac{z'}{r'} &= -\sin \text{I} \sin(\mu+\nu t) + i \cos \text{I} \sin(\phi'_0 + n''t); \end{aligned}\right\} \quad (248)$$

and therefore

$$\text{L} = \frac{-3n'^2}{2}(\text{C}-\text{A}) \sin \text{I} \cos \text{I} (1 - \cos 2n't), \quad (249)$$

$$\text{M} = \frac{3n'^2}{2}(\text{C}-\text{A}) \sin \text{I} \sin 2n't; \quad (250)$$

and if we omit the terms in L' and M', which involve the angle $2\phi'_0 + \psi'_0 + (\beta + 2n'')t$, because the coefficient is small and does not rise into importance in the subsequent integration,

$$\text{L}' = \frac{3n''^2(\text{C}-\text{A})}{2(1+e)}\Big\{-\sin \text{I} \cos \text{I}\{1 - \cos 2(\mu+\nu t)\} + i \cos 2\text{I} \cos(\psi'_0 + \beta t)\Big\}, \quad (251)$$

$$\text{M}' = \frac{3n''^2(\text{C}-\text{A})}{2(1+e)}\Big\{\sin \text{I} \sin 2(\mu+\nu t) + i \cos \text{I} \sin(\psi'_0 + \beta t)\Big\}; \quad (252)$$

the moments of these couples in the time dt are severally $\text{L}dt$, $\text{M}dt$, $\text{L}'dt$, and $\text{M}'dt$; and besides them we have also the couple G. Their effects are to be considered.

The axis of $\text{L}dt$ is the line of equinoxes, and as the axis of G is perpendicular (approximately) to the plane of the equator, it is perpendicular to this line. Consequently the axis of the resultant of $\text{L}dt$ and G is the diagonal of the rectangle, of which the line-representatives meeting at the earth's centre are the adjacent sides: let OG be this diagonal; and let $d\lambda$ be the angle at which it is inclined to the axis of G, $d\lambda$ being necessarily infinitesimal because $\text{L}dt$ is infinitesimal and G is finite. And thus

$$\begin{aligned} d\lambda &= \frac{\text{L}dt}{\text{G}} \\ &= -\frac{3n'^2}{2n}\frac{\text{C}-\text{A}}{\text{C}} \sin \text{I} \cos \text{I} (1 - \cos 2n't)dt. \end{aligned} \quad (253)$$

As $d\lambda$ lies in the plane which contains the axes of L and G, the axis about which the body revolves through $d\lambda$ is the line in the plane of the equator perpendicular to the line of equinoxes; this infinitesimal rotation therefore will not produce an appreciable change of obliquity, but only a change of position of the line of equinoxes; and if $d\psi$ is that angle,

$$d\psi = \frac{d\lambda}{\sin \text{I}}$$

$$= -\frac{3n'^2}{2n}\frac{\text{C}-\text{A}}{\text{C}}\cos \text{I}\{1-\cos 2n't\}dt; \qquad (254)$$

$$\therefore \quad \psi = -\frac{3n'^2}{2n}\frac{\text{C}-\text{A}}{\text{C}}\cos \text{I}\left\{t-\frac{\sin 2n't}{2n'}\right\}; \qquad (255)$$

which result is the same as (224).

As the axis of L′ is the same as that of L, it may be treated in the same way relatively to G; thus, if ψ' is the angle of precession due to the effect of $\text{L}'dt$, from (251) we have

$$\psi' = \frac{3n''^2}{2n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}\left\{-\cos \text{I}\,t+\frac{\cos \text{I}}{2\nu}\sin 2(\mu+\nu t)+\frac{i\cos 2\text{I}}{\beta \sin \text{I}}\sin(\psi'_0+\beta t)\right\}; \quad (256)$$

and replacing μ and ν by their values, and omitting small quantities, this becomes

$$\psi' = -\frac{3n''^2\cos \text{I}}{2n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}t+\frac{3n''\cos \text{I}}{4n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}\left\{\sin 2\{\phi'_0+\psi'_0+(\beta+n'')t\}\right.$$
$$\left.+\frac{4n''\cos 2\text{I}}{\sin 2\text{I}}\frac{i}{\beta}\sin(\psi'_0+\beta t)\right\}; \quad (257)$$

which is precisely the same result as (237). The sum of (255) and (257) is the total luni-solar precession, and nutation of the equinoxes.

Next let us consider the effects of M and M′. Since the axis of M is in the plane of the equator, and perpendicular to the line of equinoxes, the rotation-axis of the resultant of $\text{M}dt$ and G is in the plane perpendicular to the line of equinoxes; and if $d\mu$ is the angle at which the axis of this new couple is inclined to that of G,

$$d\mu = \frac{\text{M}dt}{\text{G}}; \qquad (258)$$

as $d\mu$ lies in the plane of the axes of G and $\text{M}dt$, this shifting of the rotation-axis is equivalent to a rotation of the body through a small angle $d\mu$ about the line of equinoxes; but hereby θ will be diminished by $d\theta$; so that $d\mu = -d\theta$;

$$\therefore \quad d\theta = -\frac{\text{M}\,dt}{\text{G}}$$

$$= -\frac{3n'^2}{2n}\frac{\text{C}-\text{A}}{\text{C}}\sin\text{I}\sin 2n't\,dt\,;$$

$$\therefore \quad \theta = \text{I} + \frac{3n'\sin\text{I}}{4n}\frac{\text{C}-\text{A}}{\text{C}}\cos 2n't, \tag{259}$$

which is the same result as (226).

As the axis of M′ is the same as that of M, it may be combined in the same way with G; and if θ' is the obliquity due to the action of $\text{M}'dt$,

$$\theta' = \text{I} + \frac{3n''^2}{2n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}\left\{\sin\text{I}\frac{\cos 2(\mu+\nu t)}{2\nu} + i\cos\text{I}\frac{\cos(\psi'_0+\beta t)}{\beta}\right\}$$

$$= \text{I} + \frac{3n''}{2n(1+e)}\frac{\text{C}-\text{A}}{\text{C}}\left\{\frac{\sin\text{I}}{2}\cos 2\{\phi'_0+\psi'_0+(\beta+n'')t\} + n''\cos\text{I}\frac{i}{\beta}\cos(\psi'_0+\beta t)\right\}; \tag{260}$$

which is the same result as (238). And thus the addition of (255) and (257) will give (240); and that of (259) and (260) will give (241). But it is of course unnecessary to repeat them.

An account, with great exactness, of the effects of all the terms in the lunar and solar precession and nutation, will be found in the Memoir of M. Poinsot; but it would be out of place to insert it here.

337.] It remains for us still to examine the pressure borne by the fixed point of the body through which the rotation-axis always passes.

The pressure P, as well as the direction-cosines of its line of action, are to be determined by means of equations (24) or (26), Art. 275.

Let us refer the line of pressure to the principal axes fixed in the moving body; let $\overline{\text{M}}$ be the mass of the body, and $(\bar{x}, \bar{y}, \bar{z})$ the place of its mass-centre at the time t; then from (26), Art. 275,

$$\left.\begin{aligned}
\text{P}\cos\lambda &= \Sigma.m\text{X} - \overline{\text{M}}\left\{\bar{z}\frac{d\omega_2}{dt} - \bar{y}\frac{d\omega_3}{dt}\right\} - \overline{\text{M}}\omega_1(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + \overline{\text{M}}\omega^2\bar{x},\\
\text{P}\cos\mu &= \Sigma.m\text{Y} - \overline{\text{M}}\left\{\bar{x}\frac{d\omega_3}{dt} - \bar{z}\frac{d\omega_1}{dt}\right\} - \overline{\text{M}}\omega_2(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + \overline{\text{M}}\omega^2\bar{y},\\
\text{P}\cos\nu &= \Sigma.m\text{Z} - \overline{\text{M}}\left\{\bar{y}\frac{d\omega_1}{dt} - \bar{x}\frac{d\omega_2}{dt}\right\} - \overline{\text{M}}\omega_3(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z}) + \overline{\text{M}}\omega^2\bar{z}\,;
\end{aligned}\right\} \tag{261}$$

of these equations we have the following particular results.

If the mass-centre is the origin, $\bar{x} = \bar{y} = \bar{z} = 0$; therefore

$$P\cos\lambda = \Sigma . m\mathrm{X}, \qquad P\cos\mu = \Sigma . m\mathrm{Y}, \qquad P\cos\nu = \Sigma . m\mathrm{Z}; \qquad (262)$$

that is, the pressure at the origin is due to the impressed forces only.

And if the body is not subject to the action of any force, then $P = 0$, and there is no pressure at the fixed point. Under these circumstances the resultant of the centrifugal forces developed by the motion is zero.

If no forces act, so that $\Sigma . m\mathrm{X} = \Sigma . m\mathrm{Y} = \Sigma . m\mathrm{Z} = 0$; and consequently $L = M = N = 0$; then, replacing $\frac{d\omega_1}{dt}, \frac{d\omega_2}{dt}, \frac{d\omega_3}{dt}$ by their values in (41), Art. 278, we have

$$\left.\begin{aligned} P\cos\lambda &= M\bar{x}(\omega_2^2+\omega_3^2) - M(B+C-A)\,\omega_1\left\{\frac{\omega_2\bar{y}}{C}+\frac{\omega_3\bar{z}}{B}\right\},\\ P\cos\mu &= M\bar{y}(\omega_3^2+\omega_1^2) - M(C+A-B)\,\omega_2\left\{\frac{\omega_3\bar{z}}{A}+\frac{\omega_1\bar{x}}{C}\right\},\\ P\cos\nu &= M\bar{z}(\omega_1^2+\omega_3^2) - M(A+B-C)\,\omega_3\left\{\frac{\omega_1\bar{x}}{B}+\frac{\omega_2\bar{y}}{A}\right\}. \end{aligned}\right\} \qquad (263)$$

Also if no forces act, and there is no pressure at the fixed point, multiplying (261) severally by $\bar{x}$, $\bar{y}$, $\bar{z}$, and adding, we have

$$(\omega_1\bar{x}+\omega_2\bar{y}+\omega_3\bar{z})^2 - \omega^2(\bar{x}^2+\bar{y}^2+\bar{z}^2) = 0, \qquad (264)$$

$$(\omega_3\bar{y}-\omega_2\bar{z})^2+(\omega_1\bar{z}-\omega_3\bar{x})^2+(\omega_2\bar{x}-\omega_1\bar{y})^2 = 0; \qquad (265)$$

$$\therefore \quad \frac{\bar{x}}{\omega_1} = \frac{\bar{y}}{\omega_2} = \frac{\bar{z}}{\omega_3}; \qquad (266)$$

and replacing ω_1, ω_2, ω_3 in (263) by the proportionals given in these last equations, we have

$$\frac{A\bar{x}^2}{B-C} = \frac{B\bar{y}^2}{C-A} = \frac{C\bar{z}^2}{A-B}; \qquad (267)$$

which can only be satisfied, if $\bar{x} = \bar{y} = \bar{z} = 0$; so that whenever the equations (263) are true, the mass-centre is the origin.

CHAPTER VII.

THE MOTION OF A RIGID BODY, OR OF AN INVARIABLE MATERIAL SYSTEM, FREE FROM ALL CONSTRAINT.

SECTION 1.—*Motion of a free invariable system under the action of instantaneous forces.*

338.] As our inquiry proceeds, our problem becomes more general; and the conditions of constraint become fewer. The subject of motion is still a rigid body, or a system of particles of invariable form; and thus all the internal forces, which enter into the equations of motion in the most general problem, in this disappear, because they are introduced in pairs neutralizing each other. We return therefore to the equations which are given in Chap. III, Art. 73; viz., (34) and (35) which are applicable to instantaneous forces; and (37) and (38) which express the action of finite accelerating forces.

In the solution of the problem we shall find it convenient to employ the principle of the independence of the motion of translation of the mass-centre, and of the rotation about an axis passing through it, which has been proved in Section 2 of Chapter III. For we shall thereby resolve into two distinct parts complicated motion which arises from the action of given forces; we shall consider the forces as they produce either simple translation or rotation, and shall investigate their effects; and the whole motion will be the result of these two separate motions. And the process too is most convenient for the course taken in our treatise; for the motion of a free invariable system is thus resolved into that of simple translation of a particle at its mass-centre, and that of rotation about an axis passing through the mass-centre considered as a fixed point; motion of the former kind has been completely discussed in Vol. III; and that of the latter, as far as is possible, in the Chapter preceding the present; and we have investigated these as the effects of forces similar to those which we have now to consider.

This mode of resolution is most convenient for a dynamical reason also: because all the forces which act on the several

particles of the system may be transferred, each in its own line of action, direction, and intensity, to the mass-centre; and may there act on a particle of mass equal to that of the whole system; and the motion of the mass-centre will be that of the particle under the action of the forces thus transferred; and because the rotation relative to the axis passing through the mass-centre, which we may suppose to be a fixed point, is the effect of the forces, as they act at their several points of application. At no point, except the mass-centre, are these dynamical propositions true. And an examination of what has preceded shews the reason of this. The centrifugal forces generated in the motion neutralise themselves at the mass-centre; they produce thereon no pressure; and thus cause no acceleration or retardation of it: whatever is the pressure at, or the motion of, the mass-centre, this is due to the impressed forces, and to them alone. This fact has been presented to us again and again in the course of our work.

Kinematically indeed other modes of estimating motion might have been taken. In Chapter II it has been proved that whatever is the motion of a body, it always consists of a motion of translation of *any* particle of it along a definite path, and of a motion of rotation about an axis passing through that particle; and the choice of the particle whose motion of translation is considered is arbitrary. And when force acts on the body, the effect of it, in combination with the centrifugal forces developed in the motion, will be to change the line of motion and velocity of the particle, the rotation-axis of the body passing through that particle, and its angular velocity about that axis; and the equations of motion will be formed in a manner which indicates these several changes: these we shall hereafter exhibit. Or, again, the motion which takes place in an infinitesimal time-element always consists of a rotation about the central axis, and of a sliding or of a motion of translation along that axis; and the effects of the impressed forces and of the centrifugal forces developed in the motion will be a shifting of the central axis, a change of velocity along it, and a change of angular velocity about it; and as the shifting of the central axis may take place in the most general way possible, so will it consist of both a displacement of translation and of a subsequent rotation about one of its points, so that the central axes in the new and the

old positions do not intersect each other; and these four separate effects will be produced by the acting forces; they will be exhibited therefore in the equations of motion, which will evidently be greatly complicated; and the acting forces will have to be resolved along lines, the position of which is continually changing.

It is right to say thus much as to other modes of considering the motion of a free invariable system; although we shall for the most part confine ourselves to that motion of the mass-centre, and of rotation about it, to which our equations of motion most conveniently adapt themselves.

339.] In this Section I shall consider the action of instantaneous or impulsive forces on a body or invariable system of particles, free from all constraint; and with this object I shall assume the truth of the theorems proved in Section 2 of Chap. III, shewing the independence of the motion of the whole mass collected into its mass-centre, and of the rotation of the system about an axis passing through that centre.

Let us take a point fixed in space as the origin, and let coordinate axes originate at it parallel to the principal axes of the body at the instant when the force acts; in reference to these axes let $(\bar{x}, \bar{y}, \bar{z})$ be the place of the mass-centre, and let (x, y, z) be the place of m; also in reference to the principal central axes of the body let (x', y', z') be the place of m, so that

$$x = \bar{x}+x', \quad y = \bar{y}+y', \quad z = \bar{z}+z'.$$

Let X, Y, Z be the axial components of the momentum of an impulsive force or blow acting at the point (ξ, η, ζ) in reference to the principal central axes, and let it be operated on so as to produce an impulse on the mass at the mass-centre of which the axial components are X, Y, Z, and of which the moments of the couples about the respective axes of x, y, z are $\eta \text{Z}-\zeta \text{Y}$, $\zeta \text{X}-\xi \text{Z}$, $\xi \text{Y}-\eta \text{X}$. Let all the impulses be operated on in the same way, and let $\Sigma.\text{X}$, $\Sigma.\text{Y}$, $\Sigma.\text{Z}$ be the sum of the momenta acting on $\Sigma.m$ at the origin, and let

$$\Sigma.(\eta \text{Z}-\zeta \text{Y}) = \text{L}, \quad \Sigma.(\zeta \text{X}-\xi \text{Z}) = \text{M}, \quad \Sigma.(\xi \text{Y}-\eta \text{X}) = \text{N}$$

be the moments of the couples about the three axes. Let u, v, w be the components of the resulting velocity of the mass-centre; then by (60), Art. 82,

$$u = \frac{\Sigma.\text{X}}{\Sigma.m}, \quad v = \frac{\Sigma.\text{Y}}{\Sigma.m}, \quad w = \frac{\Sigma.\text{Z}}{\Sigma.m}. \tag{1}$$

Let Ω be the angular velocity of the body about an axis passing through the mass-centre due to the action of L, M, N, and let $\Omega_1, \Omega_2, \Omega_3$ be its axial components; then, as the axes are principal axes,

$$\Omega_1 = \frac{\mathrm{L}}{\mathrm{A}}, \quad \Omega_2 = \frac{\mathrm{M}}{\mathrm{B}}, \quad \Omega_3 = \frac{\mathrm{N}}{\mathrm{C}}. \tag{2}$$

Let v be the velocity of m, in reference to the fixed origin and axes, and let $\mathrm{v}_x, \mathrm{v}_y, \mathrm{v}_z$ be its axial components; then

$$\left.\begin{aligned} \mathrm{v}_x &= u + z'\Omega_2 - y'\Omega_3 = u + \frac{\mathrm{M}z'}{\mathrm{B}} - \frac{\mathrm{N}y'}{\mathrm{C}}, \\ \mathrm{v}_y &= v + x'\Omega_3 - z'\Omega_1 = v + \frac{\mathrm{N}x'}{\mathrm{C}} - \frac{\mathrm{L}z'}{\mathrm{A}}, \\ \mathrm{v}_z &= w + y'\Omega_1 - x'\Omega_2 = w + \frac{\mathrm{L}y'}{\mathrm{A}} - \frac{\mathrm{M}x'}{\mathrm{B}}; \end{aligned}\right\} \tag{3}$$

which equations determine the velocity and the line of motion of m.

340.] These equations bring us to the same circumstances of motion as those exhibited from a kinematical point of view in Article 61, (105), and these are (1) a motion of translation whereby every particle of the system describes in dt an equal and parallel path the direction-cosines of the line of which are proportional to u, v, w; and (2) a rotation about an axis passing through a point, which is in this case the mass-centre, the direction-cosines of which are proportional to $\Omega_1, \Omega_2, \Omega_3$.

If the motion of the body is reduced to a motion of translation along a certain line, and a rotation about the same line as its rotation-axis, this line is the central axis; and when the motion has been arranged in this form, all particles which are at the same distance from the central axis move with the same velocity, and consequently v is the same and is constant for all. Hence if (x, y, z) is a point on this locus,

$$(u + z\Omega_2 - y\Omega_3)^2 + (v + x\Omega_3 - z\Omega_1)^2 + (w + y\Omega_1 - x\Omega_2)^2 = \mathrm{v}^2, \tag{4}$$

where v is constant. This is evidently the equation to a right circular cylinder, the equations to the axis of which are

$$\frac{\Omega^2 x + v\Omega_3 - w\Omega_2}{\Omega_1} = \frac{\Omega^2 y + w\Omega_1 - u\Omega_3}{\Omega_2} = \frac{\Omega^2 z + u\Omega_2 - v\Omega_1}{\Omega_3}, \tag{5}$$

and

$$\text{the radius of a transverse section} = \frac{\mathrm{v}}{\Omega}. \tag{6}$$

Thus the motion is what is technically called a screw, see Art. 50;

and the radius of the transverse section is equal to the pitch of the screw.

Equations (5) are evidently those of the central axis of the system, being the line such that all particles on it have only motion of translation along that line, all other particles having a motion of rotation about the line together with a motion of translation along a line parallel to it.

It will be observed that (3) are subject to the condition

$$\mathrm{V}_x \Omega_1 + \mathrm{V}_y \Omega_2 + \mathrm{V}_z \Omega_3 = u \mathrm{V}_x + v \mathrm{V}_y + w \mathrm{V}_z, \tag{7}$$

which expresses the fact that the velocity of a particle along the rotation-axis is the same as that of the mass-centre along the same line; that is, all particles have an equal velocity of translation along lines parallel to the central axis.

341.] If the action of the impulsive forces is such as to produce a motion of rotation only without any motion of translation, then all the points on that axis are at rest, and we have for these points the following equations, viz.

$$\left.\begin{aligned} u + z\Omega_2 - y\Omega_3 &= 0, \\ v + x\Omega_3 - z\Omega_1 &= 0, \\ w + y\Omega_1 - x\Omega_2 &= 0; \end{aligned}\right\} \tag{8}$$

but these are the equations to the central axis, and may be expressed in the form (5). This axis is in this case called the spontaneous axis of rotation. It passes through the mass-centre when $u = v = w = 0$, that is when $\Sigma.\mathrm{X} = \Sigma.\mathrm{Y} = \Sigma.\mathrm{Z}$.

Equations (8) are subject to the condition

$$u\Omega_1 + v\Omega_2 + w\Omega_3 = 0;$$

$$\therefore \quad \Omega_1 \Sigma.\mathrm{X} + \Omega_2 \Sigma.\mathrm{Y} + \Omega_3 \Sigma.\mathrm{Z} = 0; \tag{9}$$

which shews that the line of action of the resultant of the impressed momenta is perpendicular to the instantaneous rotation-axis, that is, to the spontaneous axis.

If $\Omega_1, \Omega_2, \Omega_3$ are replaced by their values which are given in (2), the condition for the existence of a spontaneous axis becomes

$$\frac{\mathrm{L}\,\Sigma.\mathrm{X}}{\mathrm{A}} + \frac{\mathrm{M}\,\Sigma.\mathrm{Y}}{\mathrm{B}} + \frac{\mathrm{N}\,\Sigma.\mathrm{Z}}{\mathrm{C}} = 0. \tag{10}$$

If the motion is due to a single blow applied at the point (ξ, η, ζ) whose momentum is Q, of which X, Y, Z are the axial components, then

$$\mathrm{L} = \eta\mathrm{Z} - \zeta\mathrm{Y}, \quad \mathrm{M} = \zeta\mathrm{X} - \xi\mathrm{Z}, \quad \mathrm{N} = \xi\mathrm{Y} - \eta\mathrm{X};$$

and (10) becomes

$$\mathrm{A}(\mathrm{B}-\mathrm{C})\,\xi\mathrm{YZ} + \mathrm{B}(\mathrm{C}-\mathrm{A})\,\eta\mathrm{ZX} + \mathrm{C}(\mathrm{A}-\mathrm{B})\,\zeta\mathrm{XY} = 0. \tag{11}$$

The condition is evidently satisfied, when the line of the blow is parallel to one of the principal axes, say to the axis of z; because in that case $X = Y = 0$: also (10) is satisfied because

$$\Sigma . X = \Sigma . Y = 0, \text{ and } N = 0.$$

Since the resulting motion is only one of rotation, and condition (9) shews that its axis is perpendicular to the line of the impressed momentum, that is to the plane of the impressed couple, the rotation-axis is a principal axis of the momental ellipsoid at the point where it is intersected by the plane of the couple. The spontaneous axis is the central axis of the system at the instant after the blow has been struck.

These theorems will be applied in subsequent articles.

342.] The spontaneous axis has the following important property, which shews that the blow has produced the greatest effect or worked to the greatest advantage on the system, because the consequent energy or vis viva of the system is greater as to rotation in respect of it than it would be of any other rotation-axis.

Let T be the kinetic energy due to the impulsive forces; then

$$2T = \Sigma . m(v_x^2 + v_y^2 + v_z^2), \tag{12}$$

$$DT = \Sigma . m(v_x dv_x + v_y dv_y + v_z dv_z); \tag{13}$$

and as the variations are to be due to changes only in Ω_1, Ω_2, and Ω_3, which are the same for all the particles of the system, we have

$$\left.\begin{aligned} dv_x &= z d\Omega_2 - y d\Omega_3, \\ dv_y &= x d\Omega_3 - z d\Omega_1, \\ dv_z &= y d\Omega_1 - x d\Omega_2. \end{aligned}\right\} \tag{14}$$

Now the equations of rotation as given in (35), Art. 73, are

$$\left.\begin{aligned} \Sigma . m(y v_z - z v_y) &= \Sigma(yZ - zY), \\ \Sigma . m(z v_x - x v_z) &= \Sigma(zX - xZ), \\ \Sigma . m(x v_y - y v_x) &= \Sigma(xY - yX); \end{aligned}\right\} \tag{15}$$

and if we multiply these by $d\Omega_1$, $d\Omega_2$, $d\Omega_3$ respectively and add, we have by reason of (14),

$$\Sigma . m(v_x dv_x + v_y dv_y + v_z dv_z) = \Sigma(X dv_x + Y dv_y + Z dv_z). \tag{16}$$

But from the principle of virtual velocities the equation of motion is

$$\Sigma . \{(X - m v_x)\delta x + (Y - m v_y)\delta y + (Z - m v_z)\delta z\} = 0, \tag{17}$$

and if δx, δy, δz are replaced by the actual velocities v_x, v_y, v_z, we have

$$\Sigma . m(v_x^2 + v_y^2 + v_z^2) = \Sigma(X v_x + Y v_y + Z v_z), \tag{18}$$

whence differentiating

$$2\Sigma . m(v_x dv_x + v_y dv_y + v_z dv_z) = \Sigma(X dv_x + Y dv_y + Z dv_z), \tag{19}$$

whence substituting from (13) and (16) we have $2\,\mathrm{DT} = \mathrm{DT}$, and therefore $\mathrm{DT} = 0$, and T is either a maximum or a minimum; that is, the kinetic energy due to the impulse is a maximum or a minimum when the rotation takes place about the spontaneous axis.

And it is evidently a maximum: for if we give to $\mathrm{v}_x, \mathrm{v}_y, \mathrm{v}_z$ increments $d\mathrm{v}_x, d\mathrm{v}_y, d\mathrm{v}_z$ of such magnitudes that their squares are not to be neglected, then the right-hand member of (13) becomes

$$2\,\Sigma.m(\mathrm{v}_x d\mathrm{v}_x + \mathrm{v}_y d\mathrm{v}_y + \mathrm{v}_z d\mathrm{v}_z) + \Sigma.m\{(d\mathrm{v}_x)^2 + (d\mathrm{v}_y)^2 + (d\mathrm{v}_z)^2\}; \quad (20)$$

and we shall eventually have

$$\Sigma.m(\mathrm{v}_x d\mathrm{v}_x + \mathrm{v}_y d\mathrm{v}_y + \mathrm{v}_z d\mathrm{v}_z) + \Sigma.m\{(d\mathrm{v}_x)^2 + (d\mathrm{v}_y)^2 + (d\mathrm{v}_z)^2\} = 0; \quad (21)$$

that is, the increment of the vis viva for the finite variation is less than it is for the infinitesimal variation by

$$\Sigma.m\{(dv_x)^2 + (dv_y)^2 + (dv_z)^2\}; \quad (22)$$

that is, by the sum of the vires vivae due to the velocities lost by the different particles of the system; and consequently the vis viva determined as above is a maximum.

The preceding proof of this theorem is due to Lagrange *; and the proof that the vis viva corresponding to the spontaneous axis is a maximum is due to his editor, M. Bertrand. The theorem was originally discovered by Euler †, and restated by Lagrange; and although the proof given by the latter holds true for a material system of invariable form, yet his mode of expression is so obscure, that it is almost impossible to understand his meaning when it is applied to a system of variable form. The following proof is given by M. Delaunay ‡, and this is sufficient for all material systems.

343.] Let $2\,\mathrm{T}$ be the vis viva of the system arising from the angular velocities due to the impulsive forces, then

$$2\mathrm{T} = \mathrm{A}\,\omega_1^2 + \mathrm{B}\,\omega_2^2 + \mathrm{C}\,\omega_3^2. \quad (23)$$

And if l, m, n are the direction-cosines of the undetermined axis, this becomes

$$2\mathrm{T} = (\mathrm{A}l^2 + \mathrm{B}m^2 + \mathrm{C}n^2)\,\omega^2;$$
$$= \frac{(\mathrm{L}l + \mathrm{M}m + \mathrm{N}n)^2}{\mathrm{A}l^2 + \mathrm{B}m^2 + \mathrm{C}n^2}; \quad (24)$$

* See Mécanique Analytique, tome I, p. 271, ed. 3, par M. J. Bertrand, Paris, 1853.

† Theoria Motus corporum solidorum, cap. IX, Theorema 8 (Art. 637.) Gryphiswaldiæ, 1790.

‡ Liouville's Journal, tome V, p. 255.

and equating to zero the total differential of this, we have

$$(\text{B}\,\text{L}\,m^2+\text{C}\,\text{L}\,n^2-\text{A}\,\text{M}\,lm-\text{A}\,\text{N}\,ln)\,dl+\ldots+\ldots=0;$$

also,
$$l\,dl+m\,dm+n\,dn=0;$$

whence we have
$$\frac{\text{L}}{\text{A}l}=\frac{\text{M}}{\text{B}m}=\frac{\text{N}}{\text{C}n}; \tag{25}$$

and therefore by means of (6)

$$\frac{l}{\Omega_1}=\frac{m}{\Omega_2}=\frac{n}{\Omega_3}; \tag{26}$$

so that the vis viva is a maximum or a minimum when the rotation-axis through the mass-centre is parallel to the spontaneous axis.

This theorem may also be inferred from that given in Art. 123, in which it is proved that in the motion of a system of material particles, subject to restraints, every change of restraint brings as a consequence loss of vis viva. Hence if a system moves about a spontaneous axis, any change of axis being of the nature of a restraint will bring with it loss of vis viva; and consequently the vis viva about the spontaneous axis is a maximum.

344.] The following are cases in which the principles and equations of the preceding Articles are applied.

A body of mass m is struck by a blow whose momentum is Q and line of action is parallel to a central principal axis and is in a central principal plane. It is required to determine the circumstances of motion.

Let G, see Fig. 39, be the mass-centre, and let the principal central axes be the coordinate axes: let CQ be the line of the blow, being in the principal plane of (x, y) and parallel to the axis of y, and let it cut the axis of x in C, and let GC $= h$. Now as $\text{X}=\text{Z}=0$, $\text{Y}=\text{Q}$, the condition (11) for a spontaneous axis is satisfied. Also $\Omega_1=\Omega_2=0$, $\Omega_3=\dfrac{\text{Q}h}{\text{C}}=\Omega$, where $\text{C}=mk^2$, say, and is the principal moment about the axis of z:

$$v=\frac{\text{Q}}{m}, \quad u=w=0.$$

Thus the equations to the spontaneous axis (8) or (5) are

$$x=-\frac{\text{C}}{mh}=-\frac{k^2}{h}, \quad y=0, \quad z=\frac{0}{0}, \tag{27}$$

which represent a line parallel to the axis of z, and intersecting the axis of x at a distance $=\dfrac{k^2}{h}$ from the origin on the negative

side; the line OR in Fig. 39 is this line, and is thus the spontaneous axis. Let GO $= h'$;

$$\therefore \quad h' = \frac{k^2}{h}, \quad \text{and } hh' = k^2. \tag{28}$$

The line CO, which $= h + h'$, is perpendicular to the line of blow and to the spontaneous axis; it is thus the shortest distance between them, and it passes through the mass-centre. In reference to this construction C and O are called respectively centre of percussion and centre of spontaneous rotation. Thus the terms are analogous to those used in Art. 238. Hence the effect of the blow is to cause the body to rotate about the line OR with the angular velocity $\frac{Qh}{C} = \frac{Qh}{mk^2}$.

This result is also thus evident. Let two equal and opposite momenta Q acting along the axis of y be introduced at the origin which is the mass-centre; one of these will be equal and parallel to the momentum of the blow, and will cause the whole body to move parallel to the axis of y with the same velocity $\frac{Q}{m}$, all particles describing equal and parallel paths; the other Q will with the momentum of the blow form a couple about the axis of z, whose moment $= Qh$. Thus if O is *any* point on the axis of x, in the time dt it will describe a line parallel to the axis of y, the positive path being $\frac{Q}{m}\,dt$, and the negative path being

$$\Omega \times \text{GO} = \frac{Qh}{mk^2} \times \text{GO}\,dt;$$

consequently the path $= \frac{Q\,dt}{mk^2}(k^2 - h \times \text{GO})$; consequently if the paths *are* equal to each other, O remains at rest, and

$$\text{GO} = \frac{k^2}{h}. \tag{29}$$

Since $hh' = k^2$, as in Art. 237, it follows that if C is the centre of percussion, O is the centre of spontaneous rotation; and if O is the centre of percussion, C is the centre of spontaneous rotation. Thus the centres of spontaneous rotation and of percussion are reciprocal.

Also, since the product hh' is constant, it follows that the smaller h is, the greater is h', and vice versâ. If $h = 0, h' = \infty$; that is, if the blow is given at the mass-centre, the axis of

spontaneous rotation is at an infinite distance, so that the body has only a motion of translation. If the axis of spontaneous rotation passes through the mass-centre, $h' = 0$, and consequently $h = \infty$; which indicates that the blow must be given at an infinite distance from G, or that the impressed force must be a couple.

Let $\text{OC} = l = h + h'$; so that

$$l = h + \frac{k^2}{h}; \tag{30}$$

and consequently, corresponding to a variation of h, l is a minimum, when $h = k$; in which case $\text{OG} = k = \text{GC}$, and $\text{OC} = 2k$; and this is the shortest possible distance between the centres of percussion and of spontaneous rotation.

In all these expressions occurs k, which is a central principal radius of gyration; of this there are generally three different values, corresponding to the three central moments of inertia; of which the greatest and least are those corresponding to the greatest and least moments of inertia, and the mean corresponds to the mean moment of inertia. Thus l is the minimum minimorum when k is the least; and is the maximum minimorum when k is the greatest.

The velocity of the centre of percussion after the blow

$$= (h + h')\,\Omega = \frac{\text{Q}}{m}\left(1 + \frac{h}{h'}\right). \tag{31}$$

345.] The preceding investigation leads to this result: when a body is rotating freely about an axis parallel to one of its central principal axes and lying in one of its central principal planes, the whole momentum of the body may be considered to be due to a single blow impressed on it in a line parallel to the central principal axis which is perpendicular to the former principal plane, and lying in the principal plane which is perpendicular to the former principal axis. And consequently, if the body at that instant met with a fixed obstacle at the point where the blow acted, the whole momentum would be taken from the body, and the body would be brought to rest, the fixed obstacle being struck with a momentum equal to that which was originally imparted to the body. Now, in reference to the given rotation-axis considered as a spontaneous axis, the centre of percussion would be the position of the fixed obstacle, and the momentum of the blow which it would receive would evidently be Q. Is,

however, the point C thus determined the position of the fixed obstacle against which the body m would impinge with the greatest momentum? Let us consider this question.

Suppose the body to impinge on an obstacle fixed at C′, whose distance from G $= x$, see Fig. 40, and suppose the momentum which the obstacle receives to be P; let O′ be the point reciprocal to C′; that is, O′ is the spontaneous centre of rotation, when C′ is the centre of percussion; and thus

$$\text{GO}' = \frac{k^2}{x}. \tag{32}$$

At the instant of impact of the body on C′, Q is the whole momentum of the body, and its line of action is CQ; let us suppose it to be resolved into two parts P and P′, acting at C′ and O′ with lines of action parallel to CQ; then, by the laws of composition of parallel forces,

$$\text{Q} = \text{P} + \text{P}'; \tag{33}$$

$$\left.\begin{aligned} \text{Q} \times \text{O}'\text{C} &= \text{P} \times \text{O}'\text{P}', \\ \text{Q} \times \text{CC}' &= \text{P}' \times \text{O}'\text{C}'; \end{aligned}\right\} \tag{34}$$

so that

$$\text{P} = \text{Q}\frac{k^2 + hx}{k^2 + x^2}; \tag{35}$$

$$\text{P}' = \text{Q}\frac{x^2 - hx}{k^2 + x^2}. \tag{36}$$

As O′ is reciprocal to C′, P′ produces no momentum at C′; so that P is the only part of Q which affects the obstacle at C′.

If $x = -\frac{k^2}{h} = -h' = -$GO, $\text{P} = 0$; that is, an obstacle placed at the spontaneous centre receives no blow.

If $x = 0$, $\text{P} = \text{Q}$; that is, an obstacle placed at the mass-centre receives a blow equal to the whole momentum.

If $x = h$, $\text{P} = \text{Q}$; that is, an obstacle placed at the centre of percussion receives a blow equal to the whole momentum.

Thus, a body strikes with the same momentum at its mass-centre and at its centre of percussion; but with this difference; when it strikes an obstacle placed at its centre of percussion, it is brought to rest; when it strikes an obstacle placed at its mass-centre, its angular velocity continues what it was before impact.

If $x = \infty$, $\text{P} = 0$, $\text{P}' = \text{Q}$; and $\text{GO}' = 0$, so that the body strikes at its mass-centre with a momentum $\text{P}' = \text{Q}$.

If x is negative, and less than $\frac{k^2}{h}$ or h', P is still positive; but

if x is negatively greater than h', P is negative. In this case, C′ falls on the negative side of O, and the body strikes an obstacle in a direction contrary to that for all points on the right-hand side of O; and thus the obstacle must be placed on the opposite side of the line OGC.

346.] To determine the position of the obstacle, when the momentum of the blow with which the body strikes it is a maximum or minimum, we must take the x-differential of (35) and equate it to zero. Thus,

$$\frac{d\text{P}}{dx} = \text{Q}\,\frac{-hx^2 - 2k^2 x + hk^2}{(k^2+x^2)^2} = 0, \text{ if}$$

$$x = -h' \pm (h'^2 + k^2)^{\frac{1}{2}}; \tag{37}$$

and changes sign from + to − for the upper sign, and from − to + for the lower sign. Hence we have two critical values of P, which are respectively a maximum and a minimum: let these be T and T′, where T is the maximum corresponding to $x = -h' + (h'^2 + k^2)^{\frac{1}{2}}$; in which case

$$\text{T} = \text{Q}\,\frac{(k^2+h'^2)^{\frac{1}{2}} + h'}{2h'} = \frac{\text{Q}}{2}\Big\{1 + \Big(1 + \frac{h}{h'}\Big)^{\frac{1}{2}}\Big\}, \tag{38}$$

which is manifestly greater than Q.

And corresponding to $x = -h' - (h'^2 + k^2)^{\frac{1}{2}}$,

$$\text{T}' = -\text{Q}\,\frac{(k^2+h'^2)^{\frac{1}{2}} - h'}{2h'} = -\frac{\text{Q}}{2}\Big\{\Big(1 + \frac{h}{h'}\Big)^{\frac{1}{2}} - 1\Big\}, \tag{39}$$

which is evidently negative, and acts in a direction opposite to that of T; and thus satisfying the criteria of a minimum, it is indeed the greatest negative value.

The former result is apparently paradoxical; for as T is greater than Q, the momentum of the blow with which the obstacle is struck is greater than that of the whole moving body; a momentum therefore is extracted from the body greater than that which it has. The explanation of the seeming paradox is, that an opposite momentum, viz. T′, has been generated; and T + T′ = Q; so that the sum of the two resulting momenta is equal to the whole momentum of the body; and the principle of the conservation of momentum still rules this case: more however will be said on this subject hereafter.

347.] Let the points of application of T and T′ be R and R′; see Fig. 41; these are called the centres of greatest percussion; they

are evidently reciprocal to each other, as centres of percussion and of spontaneous rotation. Also, since

$$x+h' = \pm (h'^2+k^2)^{\frac{1}{2}}; \tag{40}$$

$$\begin{aligned} \text{OR} = \text{OR}' &= (h'^2+hh')^{\frac{1}{2}}, \\ &= (\text{GO} \times \text{OC})^{\frac{1}{2}}; \end{aligned} \tag{41}$$

so that the two centres of maximum percussion are equally distant from the spontaneous centre; and the distance is a mean proportional between the distances of the mass-centre and of the centre of percussion from that same centre. This property gives an easy geometrical construction for the determination of the centres. Also this distance is equal to the radius of gyration of the body about the spontaneous axis; because k is the radius of gyration about the axis through the mass-centre parallel to the spontaneous axis.

If $h = 0$, that is, if the original blow Q is given at the mass-centre, so that the spontaneous centre is at an infinite distance, and the body has only a motion of translation, then

$$\text{P} = \text{Q}\, \frac{k^2}{k^2+x^2}; \tag{42}$$

$$\therefore \quad \frac{d\text{P}}{dx} = -\frac{2\,\text{Q}k^2x}{(k^2+x^2)^2} = 0, \tag{43}$$

if $x = 0$, and changes sign from $+$ to $-$; and the greatest value of P is Q; that is, the greatest blow which the body is capable of giving is that at its mass-centre.

348.] If the body is originally put into motion by a couple whose moment is N, so that the body has only a motion of rotation about an axis passing through its mass-centre, then in (35), $\text{Q} = 0$, $h = \infty$, and $\text{Q}h = \text{N}$; so that

$$\text{P} = \frac{\text{N}x}{k^2+x^2}; \tag{44}$$

$$\therefore \quad \frac{d\text{P}}{dx} = \frac{\text{N}(k^2-x^2)}{(k^2+x^2)^2} = 0, \tag{45}$$

if $x = \pm k$; and P has two corresponding values, which are respectively positive and negative; each of which $= \frac{\text{N}}{2k}$; and their lines of action are equidistant from the mass-centre, the distance being equal to the central radius of gyration.

Thus, if a sphere of radius a rotates about a vertical diameter, the greatest blow will be given on an obstacle at a distance $= a\,(.4)^{\frac{1}{2}}$ from the centre of the sphere.

If a circular plate of radius a revolves about an axis through its centre perpendicular to its plane, it will strike an obstacle in its plane with the greatest effect when that obstacle is at a distance $= a\,(.5)^{\frac{1}{2}}$ from the centre.

If Ω is the angular velocity of the body, $\text{N} = mk^2\Omega$; and therefore from (44),

$$\text{P} = \Omega x \frac{mk^2}{k^2+x^2}. \tag{46}$$

Now the velocity of a point in the body at a distance $= x$ from the rotation-axis through the mass-centre is Ωx; and since momentum is equal to the product of the mass and the velocity, a mass $= \dfrac{mk^2}{k^2+x^2}$ moving with the velocity with which the body impinges on the obstacle at its point of impact would produce a blow of equal momentum. And since, when $x = k$, this mass $= \dfrac{m}{2}$, it follows, that when the body impinges on the obstacle with the greatest effect, the momentum of the blow is the same as that of a particle of half the mass of the body, moving directly with a velocity equal to that of the corresponding centre of maximum percussion.

A similar result is true for the centre of maximum percussion corresponding to $x = -k$.

349.] The subject from this point of view requires more consideration. For suppose the body to impinge against, not a fixed obstacle, but a finite moveable mass m', then the velocities after impact, both of the body and of m', depend on the mass of m', and on the mass which, moving with the velocity of impact, would have the same momentum as the blow P due to the moving body.

In the general expressions for P and P$'$ given in (35) and (36), let Q be replaced by its value $mh'\Omega$ given in (30); and let us inquire what masses moving with the velocities at the points of impact of P and P$'$ respectively, will produce the momenta P and P$'$; let M and M$'$ be the masses required; then, since the velocities of the points of impact are respectively

$$(h'+x)\Omega \text{ and } \left(h'-\frac{k^2}{x}\right)\Omega,$$

$$\therefore \quad \text{M}\,(h'+x)\Omega = \text{Q}\,\frac{k^2+hx}{k^2+x^2}; \tag{47}$$

$$M(h'+x)\Omega = mh'\Omega\frac{hh'+hx}{k^2+x^2}; \tag{48}$$

$$\therefore \quad M = \frac{mk^2}{k^2+x^2}; \tag{49}$$

similarly
$$M' = \frac{mx^2}{k^2+x^2}: \tag{50}$$

these equations assign the fractions of m, which, moving with the velocities of the body at C' and O', would produce momenta equal to P and P' respectively.

In reference to these values, let it be observed, that

(1) $M+M' = m$; so that the sum of the two masses is equal to that of the whole moving body.

(2) $Mx = M'\frac{k^2}{x}$; so that the two masses statically equilibrate about G, the mass-centre of the body; and thus M and M' have the same mass-centre as m.

And thus the masses, which, placed at two reciprocal centres, may equivalently replace a body so far as impact at these centres is concerned, are equal to the whole body, and are to each other inversely as the distances from the mass-centre.

(3) $Mx^2+M'\frac{k^4}{x^2} = mk^2$; so that the moment of inertia of the two masses relative to the central principal axis, which is perpendicular to the line of blow, is equal to that of the body; and consequently the moment of the two relative to the spontaneous axis is equal to that of the body relative to the same axis.

In all these respects, then, a rigid inflexible straight bar, whose mass must be neglected, of any length $= x+\frac{k^2}{x}$, with masses equal to M and M' at its two ends, will equivalently replace a body; it will have the same mass, the same mass-centre, and the same moment of inertia; and when it is charged with the same momentum of impulsion, it will have the same spontaneous axis, and the same percussion at a corresponding point.

From (49) it appears, that $M = m$ only when $x = 0$; the mass-centre therefore is the only point at which the momentum of the blow is the same as that of the whole body.

If $x = h$, $M = \frac{mh'}{l}$, which is only a fraction of m. Now we have already shewn that $P = Q$, both when the point of impact

is at the centre of percussion and at the mass-centre; in the latter case the momentum is due to the mass m, moving with the velocity $\frac{\text{Q}}{m}$; in the former it is due to the mass $\frac{m h'}{l}$, moving with a velocity $\frac{\text{Q} l}{m h'}$; that is, in the former case we have a smaller mass and a greater velocity. Although the effects will be the same when the impact takes place against a fixed obstacle, yet they will not be the same when the object impinged upon is a moveable finite mass, say m'. Thus, if m' is at rest, when the body strikes it, and its elasticity is e, then, if V' is the velocity of m' after impact, we have, from Vol. III, Art. 263, the following values:

If the mass-centre is the point of impact,

$$\text{V}' = \frac{\text{Q}(1+e)}{m+m'}. \tag{51}$$

And if the centre of percussion is the point of impact,

$$\text{V}' = \frac{\text{Q}(1+e)l}{m h' + m' l}, \tag{52}$$

and therefore is greater in the latter case than in the former.

350.] Let us however investigate the position of m', when the velocity communicated to it at rest by the impact of the body is a maximum.

Let the distance of m' from the mass-centre $= x$; then P, the momentum of impact, is given by (35), and the mass corresponding to the momentum of the blow is given by (49); so that by (9), Art. 263, Vol. III,

$$\text{V}' = \frac{\text{Q}(1+e)h(h'+x)}{(m+m')k^2 + m' x^2}. \tag{53}$$

Of this quantity let the x-differential be taken and equated to zero; thence we have

$$x^2 + 2h'x - \left(1 + \frac{m}{m'}\right)k^2 = 0; \tag{54}$$

$$\therefore \quad x + h' = \pm\left\{h'^2 + \left(1 + \frac{m}{m'}\right)k^2\right\}^{\frac{1}{2}}; \tag{55}$$

so that, as in the case of greatest percussion against a fixed obstacle, two points give critical effects; to one of which corresponds a positive, and to the other a negative maximum: these points are equidistant from the spontaneous centre; and the distance of each from that centre

$$=\left\{h'^2+\left(1+\frac{m}{m'}\right)k^2\right\}^{\frac{1}{2}};$$

this distance depends on m', the mass of the particle impinged upon, and is less the greater m' is. The points determined will coincide with the centres of greatest percussion only when $m'=\infty$; which is a result in accordance with the fact, that a fixed obstacle is nothing else than a particle or body of infinite mass. Thus we have arrived at two new points; which, however, are not reciprocal to each other, as the centres of greatest percussion are.

Also, corresponding to the values of x, given in (55),

$$\text{V}'=\frac{\text{Q}(1+e)}{2(m+m')}\left[\pm\left\{1+\left(1+\frac{m}{m'}\right)\frac{h}{h'}\right\}^{\frac{1}{2}}+1\right]; \tag{56}$$

which are the greatest values of the velocity with which the particle m' can be projected after impact by the body.

When the body impinges on m' at rest at a distance x from the mass-centre, the velocity of m' is given in (53); if V is the velocity of the impinging point of the body after collision, then, by (8), Art. 263, Vol. III,

$$\text{V}=\frac{(m-em')k^2-em'x^2}{(m+m')k^2+m'x^2}(h'+x)\frac{\text{Q}}{mh'}; \tag{57}$$

and the momentum at that point after collision

$$=\frac{(m-em')k^2-em'x^2}{(m+m')k^2+m'x^2}\frac{(h'+x)h\text{Q}}{k^2+x^2}. \tag{58}$$

If we take the x-differential of (57) and equate it to zero, the point will be determined at which the body must impinge on m', and continue to proceed with the greatest velocity.

Thus, if a body, which has been put into motion by a blow whose momentum is Q, impinges on a particle m' at a distance $=x$ from its mass-centre, under the circumstances of the preceding Articles so that the momentum of a blow given against a fixed obstacle at the point $=\text{Q}\dfrac{k^2+hx}{k^2+x^2}$; then after impact on m' the momentum is given by (58). Now as the point at which P′, see equation (36), acts is the point reciprocal to that at which P acts, so P′ will not affect the momentum given by (58), and consequently the spontaneous axis is not altered by the collision. But, if Ω' is the angular velocity about the spontaneous axis after the collision,

$$\Omega' = \frac{(m - em')k^2 - em'x^2}{(m+m')k^2 + m'x^2} \frac{(h'x + x^2)\Omega}{k^2 + x^2}. \tag{59}$$

If $m' = \infty$, then the momentum of the blow which the body is capable of at a distance $= x$ from the mass-centre after impact

$$= -e\frac{k^2 + hx}{k^2 + x^2} \mathrm{Q}$$

$$= -e\mathrm{P}; \tag{60}$$

that is, is e times the momentum before impact, and acts in an opposite direction. And

$$\Omega' = -e\frac{h'x + x^2}{k^2 + x^2}\Omega. \tag{61}$$

In this case, if $x = h$, $\Omega' = -e\Omega$; and the effect of the impact is to change the direction of the angular velocity about the spontaneous axis, and to diminish it in the ratio of $e : 1$.

351.] Suppose however that, when the body impinges on the fixed obstacle at a distance $= x$ from the mass-centre, the point of impact is brought to rest, and has no further motion; that is, suppose that in the preceding Article $m' = \infty$ and $e = 0$; then the momentum of the body is reduced to a quantity P', whose value is given in (42), and which acts at a distance $= -\dfrac{k^2}{x}$ from the mass-centre. Let us consider the effect of this.

Since
$$\mathrm{P}' = \mathrm{Q}\frac{x^2 - hx}{k^2 + x^2},$$

the velocity of the mass-centre

$$= \frac{\mathrm{Q}}{m}\frac{x^2 - hx}{k^2 + x^2} = u' \text{ (say)}; \tag{62}$$

and the angular velocity about an axis passing through the mass-centre, which is also the angular velocity about the spontaneous axis which passes through the fixed point,

$$= \frac{\mathrm{Q}}{m}\frac{h - x}{k^2 + x^2} = \Omega' \text{ (say)}. \tag{63}$$

If $x = h$, $u' = 0$, and $\Omega' = 0$: in this case the fixed obstacle is at the centre of percussion, and the body is brought entirely to rest.

From these values however of u' and Ω', some interesting questions arise: we can determine the values of x, which will render u' a maximum; or will give it a given value; say, will make it equal to the original velocity of the mass-centre,

and in an opposite direction. Similar values too may be found for Ω'.

Let us first take u'; if u' is a maximum, then

$$x+h' = \pm (h'^2+k^2)^{\frac{1}{2}};$$

which give the two centres of greatest percussion. In reference to this property M. Poinsot has called these points the centres of greatest reflexion. One will be a centre of reflexion in a direction opposite to that in which the mass-centre was moving previously to the impact; and the other will be the centre of reflexion in the same direction.

For if we take the upper sign,

$$u' = \frac{Q}{2mh'}\{h'-(h'^2+k^2)^{\frac{1}{2}}\}$$

$$= -\frac{\Omega}{2}\{(h'^2+k^2)^{\frac{1}{2}}-h'\}; \tag{64}$$

which is evidently negative; and therefore the mass-centre of the body moves after the impact in a direction contrary to that of its former motion; and thus has undergone a true reflexion.

If we take the lower sign,

$$u' = \frac{Q}{2mh'}\{h'+(h'^2+k^2)^{\frac{1}{2}}\}$$

$$= \frac{\Omega}{2}\{h'+(h'^2+k^2)^{\frac{1}{2}}\}; \tag{65}$$

in which case the mass-centre of the body moves in the same direction as before the impact, and with an increased velocity.

If the velocity of the mass-centre after the impact is the same as it was before the impact, but in an opposite direction; then, from (62),

$$\frac{Q}{m}\frac{x^2-hx}{k^2+x^2} = -\frac{Q}{m};$$

$$\therefore \quad x = \frac{h}{4} \pm \frac{1}{4}(h^2-8k^2)^{\frac{1}{2}}; \tag{66}$$

these two points have been called by Poinsot centres of perfect reflexion. They are however only possible when h^2 is not less than $8k^2$; that is, only when the original blow Q has been given at a greater distance from the mass-centre than the limits thus assigned.

352.] Next let us consider the value of Ω', given in (63); Ω' has a maximum value when the x-differential of it $=0$; in which case

$$x = h \pm (h^2+k^2)^{\frac{1}{2}}; \tag{67}$$

which are two values always real, one being positive and the other negative. These points are situated at equal distances from the original centre of percussion; and the distance is equal to the radius of gyration of the body about an axis passing through the centre of percussion, and parallel to the spontaneous axis. These points have been named by Poinsot centres of greatest conversion. On comparing the values of x which assign these centres with those which assign the centres of greatest reflexion, it is evident that these bear the same relation to the centre of percussion as those do to the spontaneous centre. So that the centres of greatest conversion in a body become the centres of greatest reflexion, and vice versâ, if the centre of percussion and the spontaneous centre are interchanged.

If in the value for Ω' given in (63) we substitute for x the value given in (67) with the upper sign,

$$\Omega' = -\frac{\Omega}{2h}\{(h^2+k^2)^{\frac{1}{2}}-h\}; \tag{68}$$

which is negative, and thus indicates that for this centre of conversion the angular velocity of the body is in a direction the contrary of what it was before the impact.

If we take the lower sign in (67),

$$\Omega' = \frac{\Omega}{2h}\{(h^2+k^2)^{\frac{1}{2}}+h\}, \tag{69}$$

which is positive; and this shews that the direction of the angular velocity for this centre of greatest conversion is the same as that of the body before impact.

If the angular velocity of the body after impact is the same as before impact, but in an opposite direction, then $\Omega' = -\Omega$; and

$$\frac{Q}{m}\,\frac{h-x}{k^2-x^2} = -\,\frac{Qh}{mk^2};$$

$$\therefore\quad x = \frac{h'}{2} \pm \frac{1}{2}(h'^2-8k^2)^{\frac{1}{2}}; \tag{70}$$

which gives possible values only, provided that h'^2 is not less than $8k^2$. These two points have been called by Poinsot centres of perfect conversion.

If the angular velocity after impact is to have a given value; say, if the angular velocity after impact $= n$ times the angular velocity before impact, it is only necessary to equate the value of Ω' given in (63) to $n\Omega$, and the resulting quadratic equation will give the positions of the corresponding points of impact, and will,

by the nature of its roots, also assign the limits of possibility of the problem.

I may in conclusion observe, that in the Memoir by M. Poinsot a geometrical construction is given whereby the several centres may be determined.

353.] Let us now consider a problem of the same kind, though somewhat less special, in which the condition (10), Art. 341, necessary for the existence of a spontaneous axis, is also satisfied; that, namely, in which the line of action of the impulsive blow is parallel to a central principal axis, although it does not, as in the problem just discussed, lie in a central principal plane.

Let the line of the blow be parallel to the central principal axis of z; and let the point of impact be (x_0, y_0) in the principal plane of (x, y); let Q = the momentum of the blow; and let all these circumstances be delineated in Fig. 42; wherein G, the mass-centre, is the origin, $\mathrm{G}x$, $\mathrm{G}y$, $\mathrm{G}z$ are the three central principal axes, relative to which the moments of inertia of the body are A, B, C respectively, and the corresponding radii of gyration are severally a, b, c. Let C be the point (x_0, y_0) in the plane (x, y), whereat the blow, whose moment is Q, strikes the body; the point C will be called the centre of impulsion. Now, in this case,

$$\mathrm{X} = \mathrm{Y} = 0; \qquad \mathrm{Z} = \mathrm{Q}; \tag{71}$$

$$\Omega_1 = \frac{\mathrm{Q}y_0}{\mathrm{A}}, \qquad \Omega_2 = -\frac{\mathrm{Q}x_0}{\mathrm{B}}, \qquad \Omega_3 = 0. \tag{72}$$

Let the mass of the body $= m$, and let x, y, z be the current coordinates of the spontaneous axis; then its equations, which are given in (8), Art. 341, become

$$z = 0,$$

$$\frac{\mathrm{Q}}{m} + \frac{\mathrm{Q}}{\mathrm{A}} yy_0 + \frac{\mathrm{Q}}{\mathrm{B}} xx_0 = 0; \tag{73}$$

which are the equations to a line in the plane of (x, y). Let A and B be replaced respectively by their equivalents ma^2 and mb^2, then the equation to the spontaneous axis in the plane of (x, y) becomes

$$\frac{xx_0}{b^2} + \frac{yy_0}{a^2} + 1 = 0; \tag{74}$$

which is the line EF in Fig. 42: thus the body by reason of the blow Q begins to revolve about a line which is in the central principal plane perpendicular to the line of blow.

Let CG be produced, and cut the spontaneous axis in the point O; of which let the coordinates be x'_0, y'_0; then

$$x'_0 = -\frac{a^2 b^2 x_0}{a^2 x_0^2 + b^2 y_0^2}, \quad y'_0 = -\frac{a^2 b^2 y_0}{a^2 x_0^2 + b^2 y_0^2}; \tag{75}$$

which give the coordinates of O in terms of those of C.

Since x, y and x_0, y_0 are symmetrically involved in (74), it follows that the points to which they correspond are thus far at least reciprocal. And as C is called the centre of impulsion, O is called the spontaneous centre. Thus, if O is the centre of impulsion, C is the spontaneous centre through which the spontaneous axis passes; and vice versâ.

Let h and h' be the distances of C and of O respectively from G; then

$$h^2 = x_0^2 + y_0^2,$$

$$h'^2 = x'^2_0 + y'^2_0 = \frac{a^4 b^4 (x_0^2 + y_0^2)}{(a^2 x_0^2 + b^2 y_0^2)^2};$$

$$\therefore \quad hh' = \frac{a^2 b^2 (x_0^2 + y_0^2)}{a^2 x_0^2 + b^2 y_0^2}; \tag{76}$$

and

$$a^2 x_0^2 + b^2 y_0^2 = \frac{a^2 b^2 h}{h'}. \tag{77}$$

354.] Let us interpret these results by means of the central momental ellipsoid. The equation to that ellipsoid is

$$A\xi^2 + B\eta^2 + C\zeta^2 = \mu,$$

where μ is undetermined. Let A, B, C be replaced by ma^2, mb^2, mc^2 respectively; and, for the sake of simplification, let $\mu = ma^2 b^2$; then the equation becomes

$$a^2 \xi^2 + b^2 \eta^2 + c^2 \zeta^2 = a^2 b^2;$$

and thus the trace of this on the plane of (x, y) is the ellipse

$$a^2 \xi^2 + b^2 \eta^2 = a^2 b^2; \tag{78}$$

which we will call the central ellipse. The ξ- and the η-principal semi-axes of this ellipse are evidently b and a respectively; and the moments of inertia about these axes are A and B; which are respectively equal to ma^2 and mb^2; so that

$$A = ma^2 = \frac{\mu}{b^2}; \qquad B = mb^2 = \frac{\mu}{a^2};$$

and consequently the moments of inertia are inversely proportional to the square of the corresponding radii vectores of the ellipse. If K is the moment of inertia about a radius vector r of this ellipse,

$$K = \frac{\mu}{r^2} = \frac{ma^2 b^2}{r^2}. \tag{79}$$

Let (ξ, η) be the point P where this ellipse is intersected by the line G C; see Fig. 43; and let the radius vector $\text{GP} = r$; then

$$\frac{\xi}{x_0} = \frac{\eta}{y_0} = \frac{r}{h};$$

and consequently from (76),

$$hh' = \xi^2 + \eta^2 = r^2; \tag{80}$$

$$\therefore \quad \text{GC} \times \text{GO} = \text{GP}^2; \tag{81}$$

and thus, if the central ellipse is described, the spontaneous centre which is relative to a given centre of impulsion can be determined immediately. Of this theorem, (28), Art. 344, is evidently a particular case.

Hence, if C is a focus of the ellipse, the spontaneous axis is the farther directrix.

Again, draw the diameter G D which is conjugate to G P; its equation is

$$\frac{x\xi}{b^2} + \frac{y\eta}{a^2} = 0; \tag{82}$$

and the spontaneous axis is evidently parallel to this line. Thus, if through O we draw O S parallel to G D, or to the tangent of the central ellipse at P, O S is the spontaneous axis.

Similarly, if through C the line C T is drawn parallel to G D and O S, C T is the spontaneous axis relative to O as a centre of impulsion.

If C is at P, O is at P′; and C T and O S are tangents to the central ellipse at P and P′ respectively. In this case P P′ is the shortest possible distance along the line C G O, between the centre of impulsion and the spontaneous centre. Thus of all minima distances between these centres, A A′ is the least and B B′ is the greatest.

355.] The spontaneous axis and spontaneous centre, which are relative to a given centre of impulsion, give rise to many interesting theorems.

(1) The equation to the spontaneous axis, in reference to a given centre of impulsion (x_0, y_0), being

$$\frac{x_0 x}{b^2} + \frac{y_0 y}{a^2} = 1;$$

it is evident that, if a series of spontaneous axes pass through the same point (x'_0, y'_0), all the corresponding centres of impulsion lie along the straight line

$$\frac{xx'_0}{b^2} + \frac{yy'_0}{a^2} = 1. \tag{83}$$

This line is parallel to G D, which is conjugate to the diameter G O P′ of the central ellipse: and thus all the centres lie along the line C T, which is the spontaneous axis relative to O as a centre of impulsion.

Similarly, of centres of impulsion lying in the line O S, the corresponding spontaneous axes pass through the point C.

Thus, wherever in O S the centre of impulsion is, the place of C is the same; and wherever in C T the centre of impulsion is, the place of O is the same.

(2) Let us suppose the centre of impulsion to move on a given curve; then the spontaneous axis will envelope another curve, of which the equation may be found.

Thus suppose the centre of impulsion to move on the circle

$$x_0^2 + y_0^2 = r^2; \tag{84}$$

then the equation to the spontaneous axis is

$$\frac{x x_0}{b^2} + \frac{y y_0}{a^2} = 1; \tag{85}$$

let these be differentiated, on the supposition that x_0 and y_0 vary; then the envelope of the last is

$$\frac{x^2}{b^4} + \frac{y^2}{a^4} = \frac{1}{r^2}; \tag{86}$$

which is the equation to an ellipse, coaxial and concentric with the central ellipse.

(3) Or, again, the curve may be given, all the tangents to which are to be spontaneous axes; and it may be required to determine the locus of the corresponding centres of impulsion.

Thus, if the spontaneous axes all touch the circle $x^2+y^2=r^2$, the locus of the corresponding centres of impulsion is the ellipse

$$\frac{x^2}{b^4} + \frac{y^2}{a^4} = \frac{1}{r^2}.$$

Indeed these reciprocal properties give rise to a complete system of duality, to a great extent similar to that of polar lines and their reciprocals. This however is not the occasion for a farther development of it.

(4) Or, again, since the coordinates to the centre of impulsion and to the spontaneous centre are related by the equations (75), if the locus of one centre is given, that of the other can be found.

Thus, suppose the centre of impulsion to move along the

ellipse $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, then the locus of the spontaneous centre is also the same ellipse.

If the centre of impulsion moves along a straight line, then the locus of the spontaneous centre is an ellipse. Let the equation to the straight line be put into the form

$$\frac{xx_0}{b^2}+\frac{yy_0}{a^2}=1; \tag{87}$$

where x_0 and y_0 are any constants. Then the locus of the spontaneous centres is

$$\frac{\xi^2}{b^2}+\frac{\eta^2}{a^2}+\frac{\xi x_0}{b^2}+\frac{\eta y_0}{a^2}=0; \tag{88}$$

which is evidently an ellipse similar, and similarly situated, to the central ellipse; whose centre is at $\left(-\frac{x_0}{2}, -\frac{y_0}{2}\right)$, and which passes through the origin. The form of the equation to the straight line which I have taken shews that the line is the spontaneous axis to a centre of impulsion situated at $(-x_0, -y_0)$.

356.] Let thus much suffice for the circumstances of the spontaneous axis of the body in its relation to the centre of impulsion; and let us investigate other incidents of its motion at the first instant.

Let Ω be the angular velocity about the spontaneous axes; then, from (72),

$$\Omega^2=\Omega_1{}^2+\Omega_2{}^2=\frac{Q^2}{m^2}\left\{\frac{x_0{}^2}{b^4}+\frac{y_0{}^2}{a^4}\right\}; \tag{89}$$

$$\therefore\ \Omega=\frac{Q}{mp},$$

if p is the length of the perpendicular from the mass-centre on the spontaneous axis.

The velocity of the mass-centre is evidently $\frac{Q}{m}$.

Also, since the length of the perpendicular from C (x_0, y_0) on the spontaneous axis

$$=\frac{a^2x_0{}^2+b^2y_0{}^2+a^2b^2}{(a^4x_0{}^2+b^4y_0{}^2)^{\frac{1}{2}}}; \tag{90}$$

therefore the velocity at the centre of impulsion at the first instant after the blow

$$=\frac{Q}{m}\left\{\frac{x_0{}^2}{b^2}+\frac{y_0{}^2}{a^2}+1\right\}=\frac{Q}{m}\frac{h+h'}{h'}. \tag{91}$$

357.] If the body impinges against a fixed obstacle, or indeed against any mass at its centre of impulsion C in the plane of the

central principal axes of x and y, the momentum of the blow will be Q: let us inquire what will be the momentum at any other point of impact in the same plane, say at R, which is (x, y), with a view to a farther inquiry of the position of the point when the momentum is a maximum.

Let C be as heretofore (x_0, y_0), and let R be (x, y), Fig. 44; and let the momentum of the blow given at R be P; let O′ be the spontaneous centre relative to R, and let O′U be its spontaneous axis. Join RC, and produce it to meet O′U in U. Resolve Q into two parallel forces, P and P′, acting at U and R, with lines of action parallel to that of Q; so that by the laws of composition of parallel forces,

$$\text{P} + \text{P}' = \text{Q}, \tag{92}$$

$$\text{P} \times \text{UR} = \text{Q} \times \text{UC}, \tag{93}$$

$$\text{P}' \times \text{UR} = \text{Q} \times \text{CR}. \tag{94}$$

As O′U is the spontaneous axis relative to R as centre of impulsion, whatever in O′U is the point at which a blow is given, R remains at rest; so that P′ impressed at U produces no effect on R; and consequently P is the whole effect at R; and P is determined by (93): we have therefore to find UR and UC.

The equations to UO′ and CR are respectively

$$\frac{\xi x}{b^2} + \frac{\eta y}{a^2} + 1 = 0,$$

$$\xi(y-y_0) + \eta(x_0-x) + y_0 x - x_0 y = 0;$$

and if U is (ξ, η),

$$\frac{\text{UC}}{\text{UR}} = \frac{x_0-\xi}{x-\xi} = \frac{a^2 x x_0 + b^2 y y_0 + a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}; \tag{95}$$

$$\therefore \quad \text{P} = \text{Q}\,\frac{a^2 x x_0 + b^2 y y_0 + a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}; \tag{96}$$

which is the momentum of the blow with which the body would strike any obstacle at the point (x, y) in the central principal plane of (x, y).

358.] Certain particular values of P deserve mention. P = 0, if

$$a^2 x x_0 + b^2 y y_0 + a^2 b^2 = 0; \tag{97}$$

that is, when the obstacle is at any point on the spontaneous axis.

And P = Q, when

$$a^2 x^2 + b^2 y^2 - a^2 x_0 x - b^2 y_0 y = 0; \tag{98}$$

that is, when the obstacle is at any point on the ellipse, similar and similarly placed to the central ellipse, of which the line GC

is a diameter, and of which, of course, the centre is at the middle point of GC. The points G and C are on this ellipse; and consequently at both these points P = Q.

The case also in which P has a given value, say n Q, deserves consideration; of course it gives a locus of centres of percussion, which is generally an ellipse; and in certain cases becomes a point; and in certain other cases is imaginary. The subject however does not offer any particular difficulty: and the student can easily work it out for himself.

Also, if the body were originally put into motion by the blow Q at G, so that it has a motion of translation only, then $x_0 = y_0 = 0$, and

$$\text{P} = \text{Q}\frac{a^2b^2}{a^2x^2+b^2y^2+a^2b^2}. \tag{99}$$

Let us also investigate the value of P′, and consider certain particular values of it: from (92) or (94),

$$\text{P}' = \text{Q}\frac{a^2x^2+b^2y^2-a^2xx_0-b^2yy_0}{a^2x^2+b^2y^2+a^2b^2}. \tag{100}$$

Hence P′ = 0, when

$$a^2x^2+b^2y^2-a^2xx_0-b^2yy_0 = 0;$$

that is, when the point of impact is on the ellipse given in (98), in which case P = Q. Hence P′ = 0, if the obstacle is at the mass-centre or at the centre of impulsion.

Also P′ = Q, when

$$a^2xx_0+b^2yy_0+a^2b^2 = 0;$$

that is, when the obstacle is at a point on the spontaneous axis, in which case P = 0.

We might also investigate the locus of the place of the obstacle when P′ = nQ, say; but as the problem presents no particular difficulty, the reader may work it out for himself.

If the centre of impulsion is at the mass-centre, $x_0 = y_0 = 0$;

and
$$\text{P}' = \text{Q}\frac{a^2x^2+b^2y^2}{a^2x^2+b^2y^2+a^2b^2}. \tag{101}$$

359.] But at what points is P a maximum? In this case

$$\left(\frac{d\text{P}}{dx}\right) = \left(\frac{d\text{P}}{dy}\right) = 0.$$

And hence we have

$$\left.\begin{aligned} x_0(a^2x^2+b^2y^2+a^2b^2)-2x(a^2xx_0+b^2yy_0+a^2b^2) &= 0, \\ y_0(a^2x^2+b^2y^2+a^2b^2)-2y(a^2xx_0+b^2yy_0+a^2b^2) &= 0; \end{aligned}\right\} \tag{102}$$

which give
$$\frac{x}{x_0} = \frac{y}{y_0}; \tag{103}$$

whence it follows that the place of the obstacle, when P has a critical value, is on the line GC. If we substitute in either of (102) from (103), we have

$$(a^2 x_0^2 + b^2 y_0^2) x^2 + 2 a^2 b^2 x_0 x - a^2 b^2 x_0^2 = 0, \qquad (104)$$

$$(a^2 x_0^2 + b^2 y_0^2) y^2 + 2 a^2 b^2 y_0 y - a^2 b^2 y_0^2 = 0; \qquad (105)$$

which are quadratic equations in terms of x and y respectively; and thus give two positions of a point of impact for which P has a critical value. Let these be called centres of greatest percussion; and let r be their distance from the mass-centre; then $r^2 = x^2 + y^2$; also $h^2 = x_0^2 + y_0^2$; and

$$\frac{r}{h} = \frac{x}{x_0} = \frac{y}{y_0}; \qquad (106)$$

so that (102) give

$$(a^2 x_0^2 + b^2 y_0^2) r^2 + 2 a^2 b^2 h r - a^2 b^2 h^2 = 0; \qquad (107)$$

therefore, substituting from (77),

$$r^2 + 2h'r - hh' = 0; \qquad (108)$$

$$\therefore \quad r = -h' \pm (h'^2 + hh')^{\frac{1}{2}}; \qquad (109)$$

thus the two centres of greatest percussion are equally distant from the spontaneous centre O. Let V and V′ be these centres; Fig. 45; then

$$\text{OV} = \text{OV}' = (\text{OG} \times \text{OC})^{\frac{1}{2}}; \qquad (110)$$

and the distance is a mean proportional between the distances of the centre of impulsion and of the mass-centre from the spontaneous centre.

Let T and T′ be the corresponding momenta: then making the above substitutions in (96),

$$\text{P} = \text{Q} \frac{rh + hh'}{r^2 + hh'}; \qquad (111)$$

therefore

$$\text{T} = \frac{\text{Q}}{2} \frac{h' + (h'^2 + hh')^{\frac{1}{2}}}{h'}; \qquad (112)$$

$$\text{T}' = \frac{\text{Q}}{2} \frac{h' - (h'^2 + hh')^{\frac{1}{2}}}{h'}; \qquad (113)$$

of which T, which acts at the centre V, is positive, and is greater than Q; T′, which acts at the centre V′, is negative, and thus m gives a blow against an obstacle at V′ in a direction opposite to that in which it strikes the obstacle at V; and thus, as the obstacle at V must be on the upper or positive side of the plane of (x, y), that at V′ must be on the lower or negative side. On applying the

criteria for a maximum or minimum to these values of P, viz., T and T′, it will be found that T is a maximum, and that T′ is a minimum; but as T′ is negative, it is a negative maximum, so that both centres may be called centres of greatest percussion.

If the point of application of the blow Q, by which the body is originally put into motion, is the mass-centre, so that $x_0 = y_0 = 0$, and the body has only motion of translation; then, from (99),

$$\text{P} = \text{Q}\frac{a^2 b^2}{a^2x^2 + b^2y^2 + a^2b^2}; \tag{114}$$

and the maximum value corresponds to $x = 0$, $y = 0$; in which case P = Q; and thus the mass-centre is the centre of greatest percussion.

If $\text{P} = \frac{\text{Q}}{n}$, then

$$a^2x^2 + b^2y^2 = a^2b^2(n-1); \tag{115}$$

which represents an ellipse concentric, coaxial, and similar to the central ellipse; and therefore the intensity of a blow against an obstacle is the same for all points on this ellipse. If $n = 2$, the ellipse is the central ellipse.

360.] Again, if the body is put into motion by a couple whose axis is perpendicular to the axis of z, so that the spontaneous axis passes through the mass-centre, and the body has only rotation about that axis which is in the plane of (x, y), the momentum of a blow P at the point (x, y) in the plane of (x, y) may be determined in the following manner, which is independent of the preceding process:

Let L and M be the components of the moment of the impressed couple about the axes of x and y respectively; then we have

$$\left.\begin{array}{c} \text{X} = \text{Y} = \text{Z} = 0; \\ \Omega_1 = \frac{\text{L}}{ma^2}, \qquad \Omega_2 = \frac{\text{M}}{mb^2}, \qquad \Omega_3 = 0; \end{array}\right\} \tag{116}$$

and the equation to the spontaneous axis is, see (8), Art. 341,

$$\frac{\text{L}y}{a^2} - \frac{\text{M}x}{b^2} = 0. \tag{117}$$

Since the point of application of P is (x, y), the equation to the corresponding spontaneous axis is

$$a^2x\xi + b^2y\eta + a^2b^2 = 0; \tag{118}$$

the perpendicular distance on which from the point (x, y)

$$= \frac{a^2x^2 + b^2y^2 + a^2b^2}{(a^4x^2 + b^4y^2)^{\frac{1}{2}}}. \tag{119}$$

Let us suppose the couple of impulsion, of which the axial components are L and M, to be replaced by a couple whose forces are P at (x, y), and $-$P at the point of intersection of the spontaneous axis with the perpendicular on it from (x, y); then the moment of that couple

$$= \text{P}\,\frac{a^2x^2+b^2y^2+a^2b^2}{(a^4x^2+b^4y^2)^{\frac{1}{2}}}; \tag{120}$$

and the x- and the y-direction-cosines of its axis are

$$\frac{b^2y}{(a^4x^2+b^4y^2)^{\frac{1}{2}}}, \quad \text{and} \quad \frac{-a^2x}{(a^4x^2+b^4y^2)^{\frac{1}{2}}}. \tag{121}$$

And (120) is to be equal to the couple of which L and M are the axial components; hence

$$\text{P}\,(a^2x^2+b^2y^2+a^2b^2) = \text{L}\,b^2y-\text{M}\,a^2x;$$

$$\therefore \quad \text{P} = \frac{\text{L}\,b^2y-\text{M}\,a^2x}{a^2x^2+b^2y^2+a^2b^2}. \tag{122}$$

This quantity might also be deduced from the general expression of P, given in (96). For when the body is put into motion by a couple, that couple is equivalent to a force $= 0$ acting at an infinite distance; so that in the numerator $x_0 = y_0 = \infty$, and consequently a^2b^2 must be omitted; and thus

$$\text{P} = \frac{\text{Q}\,a^2xx_0+\text{Q}\,b^2yy_0}{a^2x^2+b^2y^2+a^2b^2};$$

but $\text{Q}\,y_0 =$ the moment of the couple about the axis of $x = \text{L}$,

and $-\text{Q}\,x_0 =$ - - - - - - - - - - - - - - - $y = \text{M}$;

$$\therefore \quad \text{P} = \frac{\text{L}\,b^2y-\text{M}\,a^2x}{a^2x^2+b^2y^2+a^2b^2}.$$

Now since $-$P acts at a point on the spontaneous axis which corresponds to the centre (x, y), $-$P produces no effect at (x, y); so that P, which is given in (122), is the momentum of the whole blow given by the body on the obstacle.

When P thus determined is a maximum,

$$\left(\frac{d\text{P}}{dx}\right) = \left(\frac{d\text{P}}{dy}\right) = 0;$$

whence we have

$$\left.\begin{aligned} -\text{M}\,(a^2x^2+b^2y^2+a^2b^2) &= 2x\,(\text{L}\,b^2y-\text{M}\,a^2x), \\ \text{L}\,(a^2x^2+b^2y^2+a^2b^2) &= 2y\,(\text{L}\,b^2y-\text{M}\,a^2x); \end{aligned}\right\} \tag{123}$$

$$\therefore \quad \text{L}x+\text{M}y = 0; \tag{124}$$

$$a^2x^2+b^2y^2 = a^2b^2. \tag{125}$$

Thus, the point (x, y) which gives the greatest percussion, is in the central ellipse, at the points in which the plane of the couple of impulsion passing through the mass-centre intersects it. And the greatest value of P

$$= \pm \frac{(b^2 \text{L}^2 + a^2 \text{M}^2)^{\frac{1}{2}}}{2ab}; \tag{126}$$

the two signs corresponding to the two extremities of the diameter of the central ellipse, which coincides with the plane of the couple passing through the mass-centre; at which points the two values of P are equal, but as they have opposite signs, they act in opposite directions.

361.] In continuation of Art. 359 let us investigate the nature of the blow, when the body strikes against a moveable mass at the point (x, y).

Let p be the perpendicular distance from (x, y) to the spontaneous axis; so that

$$p = \frac{a^2 x_0 x + b^2 y_0 y + a^2 b^2}{(a^4 x_0^{\,2} + b^4 y_0^{\,2})^{\frac{1}{2}}}; \tag{127}$$

and let Ω be the angular velocity of the body about the spontaneous axis due to the blow of impulsion; then, from (89),

$$\Omega = \frac{\text{Q}\,(a^4 x_0^{\,2} + b^4 y_0^{\,2})^{\frac{1}{2}}}{m\, a^2 b^2}; \tag{128}$$

and consequently if v is the velocity of the point (x, y) which is due to the blow of impulsion,

$$v = \Omega p = \frac{\text{Q}}{m} \, \frac{a^2 x_0 x + b^2 y_0 y + a^2 b^2}{a^2 b^2}. \tag{129}$$

Hence, if P is the momentum of the blow which the body is capable of at the point (x, y),

$$\text{P} = \text{Q} \frac{a^2 x x_0 + b^2 y y_0 + a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}$$

$$= v \frac{m\, a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}. \tag{130}$$

Let M be the mass which, moving with the velocity v, would produce on the obstacle at (x, y) a blow of this momentum; then

$$\text{M} = \frac{m\, a^2 b^2}{a^2 x^2 + b^2 y^2 + a^2 b^2}. \tag{131}$$

Also, let M$'$ be the mass of a particle which, moving with the velocity of the spontaneous centre (x', y') corresponding to (x, y),

would produce against an obstacle placed there a blow whose momentum is equal to that of the body. Then

$$\text{M}' = \frac{m a^2 b^2}{a^2 x'^2 + b^2 y'^2 + a^2 b^2}$$
$$= \frac{m (a^2 x^2 + b^2 y^2)}{a^2 x^2 + b^2 y^2 + a^2 b^2}, \tag{132}$$

by reason of (75). And thus masses are assigned, which are fractions of m, and which, moving with the velocities of any point and of its corresponding spontaneous centre, would have momenta equal to those of the blows which the body would give to obstacles placed at those points.

362.] The values of these masses thus determined may be conveniently put into another form: let r and r' be the distances of M and M$'$ from the mass-centre; and let δ be the radius vector of the central ellipse which coincides with the line joining M and M$'$: then, as the places of M and M$'$ are reciprocal as a centre of impulsion and a spontaneous centre, $rr' = \delta^2$; and from (77),

$$a^2 x^2 + b^2 y^2 = \frac{a^2 b^2 r}{r'}; \quad \text{so that}$$

$$\left. \begin{aligned} \text{M} &= \frac{m r'}{r + r'}, & \text{M}' &= \frac{m r}{r + r'}, \\ &= \frac{m \delta^2}{r^2 + \delta^2}; & &= \frac{m r^2}{r^2 + \delta^2}. \end{aligned} \right\} \tag{133}$$

In reference to these masses let it be observed, that

(1) $\text{M} + \text{M}' = m$; so that the sum of the two is equal to that of the whole moving body.

(2) $\text{M} r = \text{M}' r'$; so that the mass-centre of M and M$'$ coincides with that of m.

(3) $\text{M}(x^2 + y^2) + \text{M}'(x'^2 + y'^2) = m \delta^2$; so that the radius of gyration of the masses about any axis passing through their mass-centre and perpendicular to the line joining them is equal to the radius vector of the central ellipse which coincides with that line.

In these respects therefore the body may be equivalently replaced by a straight, inflexible, and immaterial bar, having masses M and M$'$ at its two ends, which are determined by equations (133): this bar will not only at its two ends, but at any point in its length, strike an obstacle with a blow of the same momentum as the body.

From (131) it appears, that $M = m$, only when $x = y = 0$; that is, the mass-centre is the only point at which the body will strike an obstacle as if it were a mass equal to its own mass; and in this case $P = Q$.

If the centre of percussion is the point of impact, $P = Q$; but

$$M = \frac{m a^2 b^2}{a^2 x_0{}^2 + b^2 y_0{}^2 + a^2 b^2} = m \frac{h'}{h + h'}; \tag{134}$$

and from (91),

$$v = \frac{Q}{m} \frac{h + h'}{h'};$$

so that the momentum is produced by a mass smaller than m, moving with a greater velocity. Although therefore against a fixed obstacle the momentum of the blow P is the same, whether the obstacle be at the mass-centre or at the centre of percussion; yet against a particle of finite mass, say m', the effects will be different. These we proceed to investigate; and we shall determine both the velocity of m' after impact from the body, as well as the velocity of the impinging point of the body after impact on m'.

363.] Let V and V' be the velocities of the body and of m' after collision at the point (x, y); let $e =$ the elasticity; and let us suppose m' to be at rest when the impact takes place; then, from Art. 263, Vol. III,

$$V = Q \frac{(a^2 x x_0 + b^2 y y_0 + a^2 b^2)}{m a^2 b^2} \frac{(m - e m') a^2 b^2 - e m' (a^2 x^2 + b^2 y^2)}{(m + m') a^2 b^2 + m' (a^2 x^2 + b^2 y^2)}; \tag{135}$$

$$V' = \frac{(1 + e) Q (a^2 x x_0 + b^2 y y_0 + a^2 b^2)}{(m + m') a^2 b^2 + m' (a^2 x^2 + b^2 y^2)}. \tag{136}$$

If we equate to zero the x- and y-differentials of V', the point will be determined at which m' must be struck so that it may move after collision with the greatest velocity: this process gives

$$\frac{x}{x_0} = \frac{y}{y_0}; \tag{137}$$

which shews that the point of greatest percussion is in the line joining the mass-centre and the centre of impulsion. If r is the distance of the required point from the mass-centre, then, as in Article 359,

$$r = -h' \pm \left\{ h'^2 + h h' \left(1 + \frac{m}{m'}\right) \right\}^{\frac{1}{2}}. \tag{138}$$

Thus there are two points at which a body impinging on a

particle m' will cause it to move after collision with a maximum velocity; these points are equidistant from the spontaneous centre which corresponds to the centre of impulsion, and that on the positive side of the spontaneous centre lies farther from it than the mass-centre. These two points are the centres of greatest percussion when $m' = \infty$; that is, when the mass of the particle against which the body impinges is infinitely great, and is thus equivalent to a fixed obstacle.

And corresponding to these distances,

$$\text{V}' = \frac{\text{Q}(1+e)}{2(m+m')}\Big[1 \pm \Big\{1 + \Big(1+\frac{m}{m'}\Big)\frac{h}{h'}\Big\}^{\frac{1}{2}}\Big]; \qquad (139)$$

of which values one is positive and the other is negative: the former shews that the particle m' will move with a velocity whose direction is the same as that of Q; the latter, which corresponds to the point of percussion on the side of the spontaneous axis away from the mass-centre, gives a velocity of m' in the opposite direction.

In a similar way may the point be determined, at which, if the body impinges on m', the velocity of the point of impact after collision will be a maximum; for if we take the x- and y-partial differentials of (135), and equate them to zero, the points will be determined by means of these two equations.

364.] Now at the instant when the body has impinged against a fixed obstacle at the point (x, y), that point of the body is at rest; yet there remains the momentum P′, which is given in (100), whose point of application is U, see Fig. 44: as U however is a point in O′S, which is the spontaneous axis relative to R as a centre of impulsion at which P acts, P′ produces no effect on R; and thus the motion of the body at that instant is due to the force P′ only applied at U; and consequently the spontaneous axis passes through R. U, which must now be considered as a centre of impulsion, is, from Art. 357,

$$\Big(\frac{b^2 y(x_0 y - y_0 x) - a^2 b^2 (x - x_0)}{a^2 x^2 + b^2 y^2 - a^2 x x_0 - b^2 y y_0}, \ \frac{a^2 x(y_0 x - x_0 y) - a^2 b^2 (y - y_0)}{a^2 x^2 + b^2 y^2 - a^2 x x_0 - b^2 y y_0}\Big); \quad (140)$$

the spontaneous axis corresponding to which is

$$\xi\{y(x_0 y - y_0 x) - a^2(x - x_0)\} + \eta\{x(y_0 x - x_0 y) - b^2(y - y_0)\}$$
$$+ a^2 x^2 + b^2 y^2 - a^2 x x_0 - b^2 y y_0 = 0. \quad (141)$$

Now the momentum of the blow of impulsion, namely P′, is

given in (100); and consequently if $u'=$ the velocity of the mass-centre,

$$u' = \frac{\text{P}'}{m} = \frac{\text{Q}}{m} \frac{a^2x^2+b^2y^2-a^2xx_0-b^2yy_0}{a^2x^2+b^2y^2+a^2b^2}. \tag{142}$$

And if $\Omega'=$ the angular velocity about the spontaneous axis through R due to P′, and if $p=$ the perpendicular distance from the centre of gravity on that line,

$$mp\Omega' = \text{P}', \tag{143}$$

$$\frac{(a^2x^2+b^2y^2-a^2xx_0-b^2yy_0)^2}{p^2}$$
$$= \{a^2(x-x_0)-y(x_0y-y_0x)\}^2+\{b^2(y-y_0)+x(x_0y-y_0x)\}^2.$$

If the point (x, y) lies on the line joining the mass-centre and the original centre of impulsion, these expressions become much simplified; because, in that case,

$$x_0y-y_0x=0;$$

thus the equation to the spontaneous axis through R becomes

$$a^2(x-x_0)\xi+b^2(y-y_0)\eta = a^2x^2+b^2y^2-a^2xx_0-b^2yy_0; \tag{144}$$

$$p = \frac{a^2x^2+b^2y^2-a^2xx_0-b^2yy_0}{\{a^4(x-x_0)^2+b^4(y-y_0)^2\}^{\frac{1}{2}}}; \tag{145}$$

$$\Omega' = \frac{\text{Q}}{m} \frac{\{a^4(x-x_0)^2+b^4(y-y_0)^2\}^{\frac{1}{2}}}{a^2x^2+b^2y^2+a^2b^2}; \tag{146}$$

a particular case of this last simplification is that in which the obstacle is placed at a centre of greatest percussion; see Art. 359.

Questions exactly analogous to those which I have alluded to in Art. 359 arise out of the preceding values of u' and Ω', and give points which may be called points of greatest reflexion and of greatest conversion.

365.] Thus as to u'; the problem may be to determine the place of a fixed obstacle, or of a particle of given mass, so that it may be a maximum; or the place of a fixed obstacle, so that it may have a given value; say, be equal to the original value of the velocity of the mass-centre but in an opposite direction; or to find the place of the obstacle, so that u' may be equal to 0.

As to critical values of u' I would observe, that

$$u' = \frac{\text{P}'}{m} = \frac{\text{Q}-\text{P}}{m}, \text{ by reason of (92);}$$

so that whatever values of x and y give critical values for P, also give critical values for P′ and for u': these values have been already investigated in Art. 359, and give what are therein called centres of greatest percussion; these centres then are also centres of greatest reflexion. Also, since there are two such centres, we have also two critical values of u',

$$\left.\begin{aligned} u_1' &= \frac{\text{Q}}{m}\,\frac{h'-(h'^2+hh')^{\frac{1}{2}}}{2h'}, \\ u_2' &= \frac{\text{Q}}{m}\,\frac{h'+(h'^2+hh')^{\frac{1}{2}}}{2h'}. \end{aligned}\right\} \qquad (147)$$

The latter of these is positive, and is evidently greater than $\frac{\text{Q}}{m}$, which is the original value of the velocity of the mass-centre: this is paradoxical: it seems contrary to the first principles of mechanics that a body should strike against a fixed obstacle, and after impact rebound with the velocity of the mass-centre greater than that velocity before impact. But consider this in reference to Fig. 45; V and V′ in it are the centres of greatest percussion, and consequently of greatest reflexion; and u'_2 corresponds to the point V′, so that when the obstacle is placed at that point, and the body impinges against it, the velocity of G after the impact is greater than before. The body moves by the blow Q, which is given at C, from below to above the paper; and rotates about the axis OS; if however it impinges against the obstacle at V′, that angular velocity becomes modified, and OS, which was at rest before the impact, moves in the direction CQ, and the velocity of G is increased. We must not however hence infer that the momentum of the body is increased; for that would be contrary to the principles of mechanics; but some of the momentum, which is due to the angular velocity, by means of the obstacle becomes momentum of translation; and hence it is that the velocity of the mass-centre is after impact greater than it was before. Thus, a ball from a rifled gun, having velocity both of translation of its mass-centre and of rotation about an axis through that centre, may have its velocity of translation increased by meeting with an obstacle, and thus may be carried farther than if it never met with such an obstacle. This is one of the peculiar and surprising facts of *ricochet* practice.

366.] If $u' = -\frac{\text{Q}}{m}$, the velocity of the mass-centre will be

after impact the same as it was before but in an opposite direction; then

$$2a^2x^2+2b^2y^2-a^2xx_0-b^2yy_0+a^2b^2 = 0; \qquad (148)$$

which is the equation to an ellipse similar to the central ellipse, and similarly situated; the points which give this value of u' are called points of perfect reflexion; and the ellipse (148) is called the ellipse of perfect reflexion.

Again, as to Ω'; it is a function of x and y, and the values of those quantities may be found which will give to Ω' a critical value. Also, those may be found which will assign a point on which, when the body impinges, the angular velocity after impact will have a given value; say, be equal to that before impact, and in an opposite direction. Points which give these values to Ω' are called respectively, points of maximum conversion, points of given conversion, and points of perfect conversion.

Thus, if $\Omega' = 0$, $\text{P}' = 0$; and

$$a^2x^2+b^2y^2-a^2xx_0-b^2yy_0 = 0;$$

so that for all points on the ellipse given by (98), $\text{P} = \text{Q}$, $\text{P}' = 0$, $u' = 0$, $\Omega' = 0$; that is, if the obstacle is on that ellipse, the body impinges on it with a momentum equal to that of original impulsion; the mass-centre of the body is brought to rest, and there is also no angular velocity; in fact the body is brought to rest.

If $\Omega' = -\dfrac{\text{Q}}{m}\dfrac{(a^4x_0^2+b^4y_0^2)^{\frac{1}{2}}}{a^2b^2}$, see (89), so that the angular velocity after the impact is equal to, but of contrary direction to, that before impact, then, if we take the particular case given in (146), we have

$$\frac{\{a^4(x-x_0)^2+b^4(y-y_0)^2\}^{\frac{1}{2}}}{a^2x^2+b^2y^2+a^2b^2} = -\frac{(a^4x_0^2+b^4y_0^2)^{\frac{1}{2}}}{a^2b^2}; \qquad (149)$$

which gives an equation of the fourth degree in terms of x and y; all points on the curve expressed by which are points of perfect conversion.

To determine the points for which Ω' has a critical value, the x- and y-partial differentials of (143) or of (146) must be equated to zero: in the general case however they lead to results so much complicated that it is useless to insert them.

367.] In the preceding Articles, see Art. 349, (49) and (50), and Art. 362, (133), it has been shewn, that a body may be equivalently replaced by two particles of definite and determinate masses at the ends of an immaterial rigid and straight bar, so

far as the effects of momentum communicated to the body by a blow, and the effects of impact of the body on a fixed obstacle are concerned. This property is of considerable use in the solution of another problem: A body of given mass moving with a given velocity impinges on a given body at a given point, it is required to determine the motion of the bodies at the instant after impact.

I will assume the line of motion of the moving mass to be in a central principal plane and to be parallel to a principal axis of the body on which it impinges. Let m be the mass of the latter body, $m' =$ the impinging mass, of which let the velocity at the point of impact be v'; let the line of motion of m' be in the central principal plane of (x, y), and be parallel to the axis of y; let C, see Fig. 46, the point of impact, be in the axis of x at a distance $= x$ from the mass-centre G; and let k be the radius of gyration of the body about the axis Gz, which is perpendicular to the line of action of the blow. Let O be the spontaneous centre reciprocal to C; so that $\text{OG} = \frac{k^2}{x}$. Now in Art. 362 we have shewn, that so far as concerns blows given by it, the body m may be replaced equivalently by two masses M and M′, which are therein assigned, of which the former is placed at C, and the latter at O; and that as O is a centre reciprocal to C, the mass M′ placed at O neither affects nor is affected by the blow given at C; so that as far as the momentum of a blow at C is concerned, the effect of the body will be the same as that of the mass M placed there: all this is explained in Art. 362. The problem then which is proposed for solution is this; m' moving with a velocity v' impinges on M at rest: it is required to determine the motion of M and m' after collision. The principles of Art. 263, Vol. III, are sufficient for the purpose, and may be applied as follows:

Let $\text{V} =$ the velocity of M after impact; $\text{V}' =$ the velocity of m' after impact; and let $e =$ the elasticity. Then, since

$$\text{M} = \frac{m\,k^2}{k^2+x^2}, \tag{150}$$

$$\text{V} = \frac{(1+e)\,m'\,(k^2+x^2)\,v'}{(m+m')\,k^2+m'\,x^2}; \tag{151}$$

$$\text{V}' = \frac{m'\,(k^2+x^2)-e\,m\,k^2}{(m+m')\,k^2+m'\,x^2}; \tag{152}$$

thus the momentum of M after impact $= \text{M V}$

$$= \frac{m m' (1+e) k^2 v'}{(m+m') k^2 + m' x^2}; \tag{153}$$

and this is the momentum of the body at the point C, and is that which has hitherto been denoted by Q.

The momentum imparted to the body decreases as x increases; and vanishes when $x = \infty$; and the greatest value is that which corresponds to $x = 0$.

If the bodies are perfectly inelastic, $e = 0$; in which case the momentum imparted to the body

$$= \frac{m m' k^2 v'}{(m+m') k^2 + m' x^2}. \tag{154}$$

Suppose now that m' and v' are variables, with the condition of their product being constant; that is, the momentum of the impinging ball is constant, although its mass and velocity vary; say,

$$m' v' = \mu_0 v_0; \tag{155}$$

and suppose moreover that, whatever is the distance from G at which m' impinges, the momentum imparted to the body is the same; say, $= m \text{V}_0$; then,

$$\text{V}_0 = \frac{k^2 \mu_0 v_0}{m k^2 + m' (k^2 + x^2)};$$

and consequently $m' (k^2 + x^2)$ is constant, $= m_0 b^2$, say;

$$\therefore \quad m' = \frac{m_0 b^2}{k^2 + x^2}; \tag{156}$$

$$v' = \frac{\mu_0 v_0 (k^2 + x^2)}{m_0 b^2}; \tag{157}$$

in which equations m' and v' are expressed as functions of x, and are thus determinable for any distance from the mass-centre of the point of impact.

Thus, if a hammer is to be constructed and used, so that the same quantity of momentum is to be imparted to a body whose mass is m, whatever is the distance of the point of impact from the mass-centre, the momentum of the blow of the hammer being always the same, then the mass of the hammer and the velocity of the blow are given by (156) and (157).

368.] The following are examples in which these principles are further applied.

Ex. 1. Two uniform rods AB and BC, having a joint at B,

are placed in the same straight line, A B is struck at A by a blow whose momentum is Q, in a line perpendicular to A B, it is required to determine the subsequent motion of A B and B C.

Let m and m' be the masses, and $2a$ and $2a'$ the lengths of the rods; and let G and G$'$ be their respective mass-centres. Let V and V$'$ be the velocities of translation of the mass-centres, and Ω and Ω' the angular velocities: let X be the reaction at B, which is at right angles to the rods. Let O and O$'$ be the instantaneous centres, and let G O $= x$, G$'$ O$' = x'$.

$$\therefore \quad x\Omega = \mathrm{V}, \quad x'\Omega' = \mathrm{V}', \quad (a-x)\Omega = (a'+x')\Omega'; \text{ also}$$

$$\left.\begin{aligned} m\mathrm{V} &= \mathrm{Q}+\mathrm{X}, \\ \frac{ma\Omega}{3} &= \mathrm{Q}-\mathrm{X}; \end{aligned}\right\} \qquad \left.\begin{aligned} m'\mathrm{V}' &= \mathrm{X}, \\ \frac{m'a'\Omega'}{3} &= \mathrm{X}; \end{aligned}\right\} \tag{158}$$

$$\therefore \quad x' = \frac{a'}{3}, \qquad x = \frac{am'}{m'+2m}.$$

$$\frac{\Omega}{\Omega'} = \frac{2a'}{3a}\,\frac{m'+2m}{m} = 2, \text{ if } m = m' \text{ and } a = a';$$

$$\frac{\text{Displacement of A}}{\text{Displacement of C}} = \frac{(a+x)\Omega}{(a-x')\Omega'} = \frac{2(m+m')}{m} = 4, \text{ if } m = m'.$$

Ex. 2. Three uniform and equal thin rods A B, B C, C D are arranged as three sides of a square having joints at B and C: the end A is struck in the plane of the rods and at right angles to A B by a blow whose momentum is Q. It is required to compare the angular velocities of A B and C D, and the displacements of A and D.

Let m be the mass, and $2a$ the length of each rod; let G and G$'$ be the mass-centres of A B and C D respectively, G O $= x$, G$'$O$' = x'$; and let T and T$'$ be the reactions at B and C respectively.

Then (1) as to A B,

$$\left.\begin{aligned} m\mathrm{V} &= \mathrm{Q}+\mathrm{T}, \\ \frac{ma\Omega}{3} &= \mathrm{Q}-\mathrm{T}; \end{aligned}\right\} \qquad \left.\begin{aligned} \therefore \quad \mathrm{T} &= \frac{m}{2}\left(\mathrm{V}-\frac{a\Omega}{3}\right), \\ \mathrm{Q} &= \frac{m}{2}\left(\mathrm{V}+\frac{a\Omega}{3}\right); \end{aligned}\right\} \tag{159}$$

$$\Omega x = \mathrm{V}.$$

(2) as to B C,

$$\mathrm{T}-\mathrm{T}' = m(a-x)\Omega = m(a+x')\Omega',$$

the displacements of B and C being equal and in the same direction.

(3) as to CD,

$$\left.\begin{aligned} m\text{V}' &= \text{T}', \\ \frac{m a \Omega'}{3} &= \text{T}'; \end{aligned}\right\} \qquad \therefore \quad \left.\begin{aligned} \Omega' x' = \text{V}' &= \frac{a\Omega'}{3}, \\ x' &= \frac{a}{3}; \end{aligned}\right\} \qquad (160)$$

$$\therefore \quad \frac{\text{T}-\text{T}'}{m} = \frac{\text{V}}{2} - \text{V}' - \frac{a\Omega}{6}$$

$$= \frac{(3x-a)\,\Omega - 2a\,\Omega'}{6} = (a-x)\,\Omega = (a+x')\,\Omega';$$

$$\therefore \quad x = \frac{17a}{21}, \quad \frac{\Omega'}{\Omega} = \frac{1}{7};$$

$$\frac{\text{Displacement of A}}{\text{Displacement of D}} = \frac{(a+x)\,\Omega}{(a-x')\,\Omega'} = 19.$$

If BC were a thin string whose mass might be neglected, then $\text{T}' = \text{T}$:

$$x = \frac{5a}{9}, \qquad x' = \frac{a}{3}, \qquad \frac{\Omega'}{\Omega} = \frac{1}{3};$$

$$\frac{\text{Displacement of A}}{\text{Displacement of D}} = 7.$$

Ex. 3. An uniform rod, moving parallel to itself with a velocity V, impinges on a smooth plane which is perpendicular to the line of motion of the rod, the rod being inclined at an angle to the line of motion. Determine the initial circumstances.

Let m be the mass and $2a$ the length of the rod; then if Ω is the initial angular velocity, V_0 the initial velocity of the mass-centre, R_0 the momentum of the impact, and α the angle between the rod and the normal to the plane, the equations of motion are

$$m\text{V} = \text{R}_0 + m\text{V}_0; \quad \frac{m a^2 \Omega}{3} = a\,\text{R}_0 \sin\alpha; \quad \text{V}_0 = a \sin\alpha\,\Omega,$$ whence Ω, R_0 and V_0 may be found.

369.] Ex. 4. A heavy spherical billiard ball on a rough horizontal table is struck by a cue at a given point with a blow of given intensity in a given direction; it is required to determine the resulting motion of the ball.

Let a = the radius, M = the mass of the ball; Q = the momentum of the blow; α = the angle at which its line of action is inclined to the plane of the table.

Let the horizontal plane which passes through the centre of the ball and is parallel to that of the table be the plane of (x, y); and let the line in it parallel to the vertical plane which con-

tains the line of Q be the axis of x; let $h =$ the horizontal distance from the centre of the ball to the vertical plane which contains the line of blow; and let k be the perpendicular distance on the line of blow from the point where h meets the vertical plane containing that line.

Let $\text{F} \cos\beta$ and $\text{F} \sin\beta$ be the components parallel to the axes of x and y respectively of the friction against the table which is brought into action by the blow; let $\Omega_1, \Omega_2, \Omega_3$ be the resulting angular velocities about axes through the centre of the ball which are parallel to the coordinate axes; and let u_0, v_0 be the resulting expressed velocities of the centre of the ball, parallel to the coordinate axes. Then the equations of translation parallel to the axes of x and y are

$$\left.\begin{aligned} \text{M}u_0 &= \text{Q}\cos\alpha - \text{F}\cos\beta, \\ \text{M}v_0 &= \qquad\qquad -\text{F}\sin\beta; \end{aligned}\right\}$$

and if A is the moment of inertia of the ball about an axis through its centre,

$$\left.\begin{aligned} \text{A}\Omega_1 &= -\text{Q}h\sin\alpha - a\text{F}\sin\beta, \\ \text{A}\Omega_2 &= \quad \text{Q}k + a\text{F}\cos\beta, \\ \text{A}\Omega_3 &= -\text{Q}h\cos\alpha. \end{aligned}\right\}$$

If $\text{R} =$ the pressure of the ball on the table, due to the blow,

$$\text{R} = \text{Q}\sin\alpha;$$

since the line of action of R passes through the centre of the ball, it produces no effect on Ω_1, Ω_2, or Ω_3.

Thus the axial components of the velocity of the point of contact of the ball with the plane are $u_0 - a\Omega_2$, $v_0 + a\Omega_1$; consequently, if $\quad s^2 = (u_0 - a\Omega_2)^2 + (v_0 + a\Omega_1)^2,$

as β is the angle at which the initial path of the point of contact is inclined to the axis of x,

$$\frac{\cos\beta}{u_0 - a\Omega_2} = \frac{\sin\beta}{v_0 + a\Omega_1} = \frac{1}{s}; \tag{161}$$

and since the friction acts as a retarding force along the line of motion of the point of contact, its line of action is thus determined; and the friction is known in terms of the pressure R, so that the four unknown quantities $u_0, v_0, \Omega_1, \Omega_2$ are involved in four independent equations, and may be determined without difficulty; and thus the initial motion of the ball will be determined. Applications of these results will be made hereafter.

370.] Ex. 5. A body m rests on a prop and is struck by a blow whose momentum is Q, the line of motion of the blow being in a central principal plane, and parallel to one central principal axis; and the line of reaction of the prop being in the same principal plane, and parallel to the line of the blow.

Let OGR be a central principal axis of the body whose mass is m, and mass-centre is G. Let C be the prop and R the point of application of the blow whose momentum is Q, and of which QR is the line of action; this line of motion being in the central principal plane xGz: see Fig. 47. It is evident that if R coincides with C, that is, if the blow is given at the prop, the momentum of the pressure borne by the prop = Q; suppose however that the point of impact of the blow is R, where GR $= x$; let P = the momentum borne by C, and let P′ be that applied at O, which is the spontaneous centre reciprocal to C, both these being due to Q; then the pressure P′ does not affect the pressure at C, which is a point reciprocal to O, so that P is the whole pressure on the prop.

Let GC $= h$; and consequently GO $= \frac{k^2}{h}$; then as P and P′ are the components of Q, Q = P − P′;

and
$$\text{P}\left(h+\frac{k^2}{h}\right) = \text{Q}\left(\frac{k^2}{h}+x\right);$$

$$\therefore \quad \text{P} = \text{Q}\frac{k^2+hx}{k^2+h^2}; \tag{162}$$

which assigns the pressure borne by the prop. If $x = h$, P = Q; that is, the blow is applied at the prop, and the pressure borne by the prop is equal to the momentum of the blow. If $x = -\frac{k^2}{h}$, that is, if the blow is struck at O, the spontaneous centre relative to C, P = 0.

P increases as x increases, and P is greater than Q when x is greater than h: it follows therefore that by means of an intervening body m, a blow of given momentum can produce a pressure of any intensity on a given prop. If $x = \infty$, P $= \infty$.

If however the blow is caused by a hammer of mass m', and impinging with a velocity v' on a point whose distance from the mass-centre is x; then, from (154),

$$\text{Q} = \frac{m\,m'\,k^2\,v'}{(m+m')k^2+m'x^2}; \tag{163}$$

and therefore $$\text{P} = \frac{m\,m'\,k^2 v'\,(k^2+h\,x)}{(k^2+h^2)\{(m+m')k^2+m'x^2\}}. \tag{164}$$

If in this expression m' and v' have, for a distance x, the values found for them in (156) and (157), then the momentum of the blow of the hammer is always the same, and the momentum borne by the prop is given by (162), and may consequently be of any magnitude whatever.

P is a maximum in (164) when

$$x = -h' \pm \left(h'^2 + \frac{m+m'}{m}k^2\right)^{\frac{1}{2}}, \tag{165}$$

where $\text{GO} = h' = \frac{k^2}{h}$; which gives two points equidistant from O.

Thus if $m = m'$; and $h = h' = k$,

$$x = (-1 \pm 3^{\frac{1}{2}})\,k.$$

Similarly may the points of impact be determined, so that the momentum of the pressure borne by the obstacle may be of a given value.

I cannot conclude this subject, in which I have borrowed largely from the Memoirs of Poinsot, contained in Vols. II and IV of the second series of Liouville's Journal, without alluding to a remark which he makes of the process by which the circumstances of motion of a rigid system having a fixed axis or a fixed point may be deduced from those of a similar free system. He considers a fixed point to be a particle of a certain definite mass, introduces this mass and its incidents into all the equations of motion, and in the final results makes this mass infinite; and this particle of infinite mass he considers to be a fixed point; on which, of course, as to translation, a finite force has no effect; but for an axis passing through that point the moment of inertia of the body is finite, and consequently the impressed couples will produce their own rotatory effects.

SECTION 2.—*Motion of a free invariable system under the action of finite accelerating forces.*

371.] We now come to the most general case of absolute motion of a body, or of any system or systems of material particles under the action of finite forces. Many processes have been devised for the purpose, and several of them are especially adapted to particular classes of problems. All however are

founded upon the principle of D'Alembert; and their equations of motion are derived from, or are identical with, those six equations in which we have expressed that theorem. I propose to apply these to the solution of problems of motion in preference to other and derived processes; because we shall hereby maintain an uniformity of process and of principle, and because the circumstances of the problems will be resolved into their most simple elements. We shall indeed take the forms which these equations admit of, in virtue of the theorems proved in Section 2 of Chap. III; we shall consider the motion of translation of the mass-centre, by assuming all the forces to act on a particle, whose mass is equal to the whole mass of the moving system, placed therein; and in our inquiry into the rotation of the system, we shall assume the mass-centre to be a fixed point, and the body or material system to rotate about an axis passing through that point. Thus the motion of the system in the first place depends on the two following groups of equations:

$$\left.\begin{aligned} \Sigma . m\left(\mathrm{X}-\frac{d^2 x}{dt^2}\right) &= 0, \\ \Sigma . m\left(\mathrm{Y}-\frac{d^2 y}{dt^2}\right) &= 0, \\ \Sigma . m\left(\mathrm{Z}-\frac{d^2 z}{dt^2}\right) &= 0; \end{aligned}\right\} \qquad (166)$$

$$\left.\begin{aligned} \Sigma . m\left\{y\left(\mathrm{Z}-\frac{d^2 z}{dt^2}\right)-z\left(\mathrm{Y}-\frac{d^2 y}{dt^2}\right)\right\} &= 0, \\ \Sigma . m\left\{z\left(\mathrm{X}-\frac{d^2 x}{dt^2}\right)-x\left(\mathrm{Z}-\frac{d^2 z}{dt^2}\right)\right\} &= 0, \\ \Sigma . m\left\{x\left(\mathrm{Y}-\frac{d^2 y}{dt^2}\right)-y\left(\mathrm{X}-\frac{d^2 x}{dt^2}\right)\right\} &= 0. \end{aligned}\right\} \qquad (167)$$

If $\overline{\mathrm{M}}$ is the mass of the whole moving system, $(\bar{x}, \bar{y}, \bar{z})$ is the place of the mass-centre at the time t, and if (x', y', z') is the place of m relatively to a system of coordinate axes originating at the mass-centre, and parallel to the original system of axes; then, by the theorems of Section 2, Chap. III, these take the forms

$$\left.\begin{aligned} \overline{\mathrm{M}}\frac{d^2 \bar{x}}{dt^2} &= \Sigma . m\mathrm{X}, \\ \overline{\mathrm{M}}\frac{d^2 \bar{y}}{dt^2} &= \Sigma . m\mathrm{Y}, \\ \overline{\mathrm{M}}\frac{d^2 \bar{z}}{dt^2} &= \Sigma . m\mathrm{Z}; \end{aligned}\right\} \qquad (168)$$

$$\left.\begin{aligned}\Sigma . m\left\{y'\left(\text{Z}-\frac{d^2 z'}{dt^2}\right)-z'\left(\text{Y}-\frac{d^2 y'}{dt^2}\right)\right\} &= 0,\\ \Sigma . m\left\{z'\left(\text{X}-\frac{d^2 x}{dt^2}\right)-x'\left(\text{Z}-\frac{d^2 z'}{dt^2}\right)\right\} &= 0,\\ \Sigma . m\left\{x'\left(\text{Y}-\frac{d^2 y'}{dt^2}\right)-y'\left(\text{X}-\frac{d^2 x'}{dt^2}\right)\right\} &= 0.\end{aligned}\right\} \qquad (169)$$

By reason of the former of these last two groups, the motion of translation of the body is reduced to that of a single material particle whose mass is $\overline{\text{M}}$; and to this motion all that has been said in Vol. III is applicable. The second group reduces the motion of rotation to that of a body rotating about an axis passing through a fixed point of it; and consequently to this motion all that has been said in the preceding Chapter is applicable. The problem therefore requires two processes in combination, each of which has been separately discussed; and little else remains than to illustrate the combination by means of particular examples. Indeed I have already anticipated the process in the investigation of the phaenomena of terrestrial precession and nutation in the preceding Chapter; because we have assumed the centre of the earth to be fixed, whereas it has a motion of translation in space.

In investigating the motion of rotation of the body about the point which is assumed to be fixed, we may use the simplifications and substitutions of the last and preceding Chapters. Thus, if ω_x, ω_y, ω_z are the angular velocities at the time t about any three coordinate axes originating at the fixed point, (169) become (54), of Art. 156; which however it is unnecessary to repeat in this place as we shall employ simplified forms of them. We shall investigate the angular velocities of the body at the time t relatively to the three principal axes of the body; and the equations for determining these are

$$\left.\begin{aligned}\text{A}\frac{d\omega_1}{dt}+(\text{C}-\text{B})\,\omega_2\omega_3 &= \text{L},\\ \text{B}\frac{d\omega_2}{dt}+(\text{A}-\text{C})\,\omega_3\omega_1 &= \text{M},\\ \text{C}\frac{d\omega_3}{dt}+(\text{B}-\text{A})\,\omega_1\omega_2 &= \text{N};\end{aligned}\right\} \qquad (170)$$

because hereby (theoretically at least) the angular velocity of the body, and the position of the instantaneous rotation-axis relatively to the principal axes, may be determined at the time

t; and thence we may determine, as in the preceding Chapter, the motion of the body in reference to fixed axes, by means of the three connecting angles θ, ϕ, ψ.

And if the position of the rotation-axis which passes through the mass-centre of the body is invariable relatively to the body, then the rotation is determined by the simple equation,

$$\frac{d\omega}{dt} = \frac{\text{moment of impressed forces}}{\text{moment of inertia}}. \qquad (171)$$

In the solution of mechanical problems, the theorems of vis viva, and of conservation of areas, may frequently be applied, to the saving of considerable trouble; not indeed because they contain any truth besides those involved explicitly or implicitly in the equations of motion, but because they are first integrals of these equations. In a didactic treatise, however, as clearness of conception and accuracy of expression are of paramount importance; and as these will be obtained when the circumstances of a problem are resolved into their simplest elements; so in the following problems, the equations of motion are generally given in their original forms, and for the complete solution of a problem two successive integrations are required. In some cases the equations of areas and the equation of vis viva will be given directly: and the latter will frequently present itself in the derived form which has been proved in Art. 113; viz., the vis viva of the system is equal to the sum of the vis viva of the whole system condensed into its mass-centre, and of the vis viva of the several particles relative to the mass-centre.

372.] The following are problems on the motion of rigid bodies.

Ex. 1. A heavy homogeneous sphere rolls down a rough inclined plane; it is required to determine the motion.

We suppose the sphere to be placed at rest on the plane, and to roll down it so that the point of contact describes a straight line perpendicular to the line of intersection of the inclined and horizontal planes. Let Fig. 48 represent a section of the sphere and plane at the time t, made by a vertical plane passing through C the centre of the sphere. Let A be the point of the sphere which was originally in contact with the plane at the point O; let a be the radius of the sphere; OP $= s$, ACP $= \theta$, M $=$ the mass of the sphere, F $=$ the friction of rolling, R $=$ the pressure of the sphere on the plane, $\alpha =$ the angle of elevation of the plane.

Now C evidently moves along a straight line parallel to the plane; so that for its motion of translation we have

$$\text{M}\,\frac{d^2 s}{dt^2} = \text{M}g\sin\alpha - \text{F};$$

and if C is considered fixed, the sphere evidently rotates about a horizontal axis parallel to the plane; and if k is the radius of gyration of the sphere relative to this axis,

$$\text{M}k^2\,\frac{d^2\theta}{dt^2} = \text{F}a;$$

and since the plane is perfectly rough, so that the sphere does not slide, $ds = a\,d\theta$; also $k^2 = \dfrac{2a^2}{5}$;

$$\therefore\quad \frac{d^2 s}{dt^2} = a\,\frac{d^2\theta}{dt^2} = \frac{5}{7}\,g\sin\alpha;$$

which assigns the motion; also $\text{R} = \text{M}g\cos\alpha$, $\text{F} = \dfrac{2\text{M}g\sin\alpha}{7}$. If the plane were perfectly smooth, the impressed velocity-increment along the plane would be $g\sin\alpha$; so that the roughness of the plane which causes the rolling diminishes the action of gravity along the plane by two-sevenths of its full value.

If the rolling body were a circular cylinder with its axis horizontal, then $k^2 = \dfrac{a^2}{2}$; and

$$\frac{d^2 s}{dt^2} = \frac{2}{3}\,g\sin\alpha;$$

so that the roughness of the plane would diminish the action of gravity along the plane by one-third of its full value.

Ex. 2. A hollow spherical shell is filled with fluid and rolls down a rough inclined plane; determine its motion.

Let M and M′ be the masses of the shell and fluid respectively; and let k and k' be the radii of gyration of them respectively about a diameter; let a and a' be the radii of the exterior and interior surfaces of the shell; then, employing the same notation as in Ex. 1, we have

$$(\text{M}+\text{M}')\,\frac{d^2 s}{dt^2} = (\text{M}+\text{M}')\,g\sin\alpha - \text{F}.$$

As the spherical shell rotates in its descent down the plane, the fluid has only motion of translation; so that the equation of rotation is

$$\text{M}k^2\,\frac{d^2\theta}{dt^2} = \text{F}a;$$

$$\therefore \quad \{(M+M')a^2+Mk^2\}\frac{d^2s}{dt^2}=(M+M')a^2g\sin\alpha.$$

If the interior were solid, and rigidly joined to the shell, the equation of motion would be

$$\{(M+M')a^2+Mk^2+M'k'^2\}\frac{d^2s}{dt^2}=(M+M')a^2g\sin\alpha.$$

Thus if s and s' are the spaces through which the centre moves during the time t in these two cases respectively, then

$$\frac{s}{s'}=\frac{(M+M')a^2+Mk^2+M'k'^2}{(M+M')a^2+Mk^2};$$

so that a greater space is described by the sphere which has the fluid than by that which has the solid in its interior.

If the densities of the solid and the fluid are the same, replacing k and k' by their values,

$$\frac{s}{s'}=\frac{7a^5}{7a^5-2a'^5}.$$

Ex. 3. A heavy right circular cylinder is composed of two substances, whose volumes are equal, and whose densities are ρ and ρ'; these substances are arranged in two different forms; in one case, that whose density is ρ occupies the central part of the wheel, and the other is placed as a ring around it; in the second case, the places of the substances are interchanged; t and t' are the times in which the cylinders roll down a given rough inclined plane from rest; shew that

$$t^2:t'^2::5\rho+7\rho':5\rho'+7\rho.$$

Ex. 4. A homogeneous heavy sphere rolls down within a fixed rough spherical bowl; it is required to determine the motion.

Let the circumstances be as represented in Fig. 49, where b is the radius of the bowl, a is the radius and m is the mass of the rolling sphere; and let us suppose the sphere to have been placed at rest in the bowl. Let OCQ $=\phi$, QPA $=\theta$, BCO $=\alpha$; then as the sphere rolls and does not slide, $a\theta=b(\alpha-\phi)$; let ω be the angular velocity of the sphere at the time t, so that

$$\omega=\frac{d.\text{MPA}}{dt}=\frac{d(\theta+\phi)}{dt}=-\frac{b-a}{a}\frac{d\phi}{dt}.$$

In reference to OM and OC as coordinate axes, let (x,y) be the place of P at the time t; so that

$$y=b-(b-a)\cos\phi, \quad x=(b-a)\sin\phi;$$

then the equation of vis viva is

$$m\left\{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{2a^2}{5}\omega^2\right\}=2mg(\cos\phi-\cos\alpha):$$

whence we have

$$(b-a)\frac{d\phi^2}{dt^2} = \frac{10g}{7}(\cos\phi - \cos\alpha);$$

from which ϕ may be expressed in terms of t by means of elliptic functions.

If R is the pressure of the sphere against the bowl,

$$\text{R} = m(b-a)\frac{d\phi^2}{dt^2} + mg\cos\phi = mg\,\frac{17\cos\phi - 10\cos\alpha}{7};$$

so that the pressure at the lowest point $= mg\dfrac{17-10\cos\alpha}{7}$, and the pressure vanishes when $17\cos\phi = 10\cos\alpha$.

If F is the friction at the point of contact at the time t,

$$\text{F} = m(b-a)\frac{d^2\phi}{dt^2} + mg\sin\phi = \frac{2\,mg\sin\phi}{7};$$

as this $= 0$, when $\phi = 0$, there is no friction at the lowest point of the bowl.

If the ball rolls over only a small arc at the lowest part so that α and ϕ are always small, then replacing $\cos\phi$ and $\cos\alpha$ by $1-\dfrac{\phi^2}{2}$ and $1-\dfrac{\alpha^2}{2}$ respectively, we have

$$\frac{-d\phi}{(\alpha^2-\phi^2)^{\frac{1}{2}}} = \left\{\frac{5g}{7(b-a)}\right\}^{\frac{1}{2}}dt;$$

$$\therefore\quad \phi = \alpha\cos\left\{\frac{5g}{7(b-a)}\right\}^{\frac{1}{2}}t:$$

thus the ball comes to rest at points whose angular distance is α on both sides of O, the lowest point of the bowl; and the periodic time $= \pi\left\{\dfrac{7(b-a)}{5g}\right\}^{\frac{1}{2}}$; consequently the oscillations are performed isochronously with those of a simple pendulum whose length is $\dfrac{7}{5}(b-a)$.

Ex. 5. A heavy uniform beam of length $2a$ and mass m slides down between a smooth vertical wall and a smooth horizontal plane, the beam moving in a vertical plane. Determine the motion.

Let the intersection of the horizontal and vertical planes and of the plane of motion of the beam be the origin, and let (x, y) be the place of the centre of gravity of the beam and θ its inclination to the horizontal plane at the time t; so that $x = a\cos\theta$, $y = a\sin\theta$.

Hence the equation of vis viva is

$$m\left(\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+k^2\frac{d\theta^2}{dt^2}\right)=2mga(\sin\alpha-\sin\theta),$$

if α is the inclination of the beam, when at rest, to the horizontal plane;

$$\therefore\ \frac{4a^2}{3}\frac{d\theta^2}{dt^2}=2ga(\sin\alpha-\sin\theta),$$

whereby θ may be expressed in terms of t by means of elliptic functions.

If X and Y are the pressures against the vertical wall and the horizontal plane respectively,

$$\mathrm{X}=m\frac{d^2x}{dt^2}=-\frac{3mg\cos\theta}{4}(2\sin\alpha-3\sin\theta):$$

$$\mathrm{Y}=mg+m\frac{d^2y}{dt^2}=\frac{mg}{4}\{1-6\sin\alpha\sin\theta+9(\sin\theta)^2\}.$$

Hence, when $\sin\theta=\frac{2}{3}\sin\alpha$, the beam leaves the vertical wall but does not leave the horizontal plane; the pressure on which is a maximum, and is $=\frac{mg(\cos\alpha)^2}{4}$. Thus the subsequent circumstances of motion become changed, the beam has a constant horizontal velocity $=\frac{(2ag)^{\frac{1}{2}}}{3}(\sin\alpha)^{\frac{3}{2}}$, and also an angular velocity $=\left(\frac{g\sin\alpha}{2a}\right)^{\frac{1}{2}}$, about a horizontal axis passing through the centre of gravity; finally the beam becomes horizontal and lies in the plane.

Ex. 6. A heavy homogeneous sphere rolls down the rough face of a wedge; the wedge rests on a smooth horizontal plane, along which it slides by reason of the pressure of the sphere; it is required to determine the motions of the sphere and of the wedge.

The circumstances of motion at the time t are delineated in Fig. 50. $m=$ the mass of the ball, M $=$ the mass of the wedge; $a=$ the radius of the ball, $\alpha=$ the angle of the wedge; Q the apex of the wedge, O the place of Q when $t=0$; O′ the point on the wedge which was in contact with the point A of the sphere, when $t=0$; at which time let us suppose all to be at rest; ACP $=\theta$, the angle through which the sphere has revolved in the time t.

Let O be the origin, and let the horizontal and vertical lines through it be the axes of x and y; OQ $=x'$; and let (x, y) and (h, k) be the places of the centre of gravity of the sphere at the times $t=t$, and $t=0$ respectively.

Then the equation of vis viva is

$$m\left\{\frac{dx^2}{dt^2}+\frac{dy^2}{dt^2}+\frac{2a^2}{5}\frac{d\theta^2}{dt^2}\right\}+\text{M}\frac{dx'^2}{dt^2}=2mg(k-y).$$

Also, as no external force acts parallel to the axis of x, the centre of gravity of the two moving bodies moves in a vertical straight line, so that

$$m(x-h)+\text{M}x'=0.$$

Also from the geometry,

$$x=h+x'-a\theta\cos\alpha, \qquad y=k-a\theta\sin\alpha;$$

$$\therefore\ x=h-\frac{\text{M}}{\text{M}+m}a\theta\cos\alpha.$$

Whence

$$\frac{7\text{M}^2+\{4+10(\sin\alpha)^2\}\text{M}m+\{2+5(\sin\alpha)^2\}m^2}{5(\text{M}+m)^2}\frac{d\theta^2}{dt^2}=\frac{2g\theta\sin\alpha}{a},$$

and thus θ, and thence x and y, may be determined in terms of t.

Also from above

$$(m+\text{M})(x-h)\sin\alpha-\text{M}(y-k)\cos\alpha=0;$$

which shews that the path described by the centre of the sphere is a straight line.

Ex. 7. A heavy beam OP, see Fig. 51, turns about a hinge at O, and its end P rests on a smooth inclined plane or wedge, which slides along a smooth horizontal plane which passes through O: it is required to determine the motion of the beam and of the inclined plane.

Let m and M be the masses of the beam and wedge respectively; $2a=$ the length of the beam, $\alpha=$ the angle of inclination of the plane to the horizon; $\text{POQ}=\theta$, $\text{OQ}=x$. Then, from the geometry, we have,

$$x\sin\alpha=2a\sin(\alpha-\theta).$$

Now the equation of vis viva is

$$m\frac{4a^2}{3}\frac{d\theta^2}{dt^2}+\text{M}\frac{dx^2}{dt^2}=2mag(\sin\theta_0-\sin\theta)$$

where θ_0 is the value of θ, when the system is at rest; and, replacing $\frac{dx}{dt}$ by its value in terms of $\frac{d\theta}{dt}$, we have

$$\left\{\frac{m}{3}+\text{M}\left(\frac{\cos(\alpha-\theta)}{\sin\alpha}\right)^2\right\}\frac{d\theta^2}{dt^2}=\frac{mg}{2a}(\sin\theta_0-\sin\theta),$$

whence θ may be expressed in terms of t by means of elliptic functions.

Ex. 8. A heavy body whose bounding surface is a circular

cylinder, but whose centre of gravity is not in the axis of the surface, rolls on a rough horizontal plane: it is required to determine its motion.

Let M = the mass of the body; and let Fig. 52 represent the circumstances at the time t; in which G is the centre of gravity, C is the point of intersection of the axis of the cylinder by a vertical plane passing through the centre of gravity G; Ox is the horizontal plane; O, the origin, is the point where the cylinder touches the plane when it is in equilibrium, A being the corresponding point of the cylinder. Let CA $= a$, CG $= c$, ACP $= \theta$, so that OP $= a\theta$; and let k be the radius of gyration of the body relative to its rotation-axis through G; let G be (x, y); then from the geometry we have,

$$x = a\theta - c\sin\theta, \qquad y = a - c\cos\theta.$$

The equation of vis viva is

$$m\left\{\frac{dx^2}{dt^2} + \frac{dy^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right\} = 2mcg(\cos\theta - \cos\alpha),$$

if $\theta = \alpha$, when the body is at rest. Hence we have

$$\{a^2 - 2ac\cos\theta + c^2 + k^2\}\frac{d\theta^2}{dt^2} = 2cg(\cos\theta - \cos\alpha); \qquad (172)$$

which gives the angular velocity of the body about a horizontal axis; and whence θ may be expressed in terms of t by means of an elliptic function.

If however the angle through which the body rocks is always small, so that α and θ are always so small that all powers of them above the second may be neglected, and that the second powers may be neglected when they are added to finite quantities; then we may replace (172) by its approximate equation,

$$\{k^2 + (a-c)^2\}\frac{d\theta^2}{dt^2} = cg(\alpha^2 - \theta^2);$$

$$\therefore \quad \theta = \alpha\cos\left\{\frac{cg}{k^2 + (a-c)^2}\right\}^{\frac{1}{2}} t; \qquad (173)$$

so that the body oscillates or rocks through an angle 2α; and the time of an oscillation

$$= \pi\left\{\frac{k^2 + (a-c)^2}{cg}\right\}^{\frac{1}{2}}. \qquad (174)$$

This result applies to a problem which is physically of considerable importance. In making observations with the pendulum, the mode of suspension which is found most convenient for the

determination of the distance between the centres of suspension and oscillation is that of knife edges, resting on horizontal plates of agate or of some other hard material. Although the knife edges are made of steel, and brought to as fine an edge as possible, yet they are not mathematical straight lines, but approximate to cylinders, which we may, without sensible error, suppose to be circular, and of a very small radius; in which case the pendulum is suspended by a horizontal cylinder which rests and rolls on two parallel horizontal bars which are perpendicular to the axis, and of which a diagram is given in Fig. 53. Here the centre of gravity of the rolling body is below the horizontal plane, so that c is greater than a; then, if T is the time of an oscillation,

$$\text{T} = \pi \left\{ \frac{k^2 + (c-a)^2}{cg} \right\}^{\frac{1}{2}};$$

but if the pendulum is suspended by an exact edge, the time of oscillation

$$= \pi \left\{ \frac{k^2 + c^2}{cg} \right\}^{\frac{1}{2}};$$

thus the effect of the want of accuracy in the edge diminishes the time of vibration in the ratio of

$$\{k^2 + (c-a)^2\}^{\frac{1}{2}} \quad \text{to} \quad \{k^2 + c^2\}^{\frac{1}{2}}.$$

Ex. 9. If we suppose the rocking body to be homogeneous, and to be bounded by a cylindrical surface whose section perpendicular to the generating lines is semicircular, as in Fig. 54; then, if

$$\text{CA} = a, \ \text{CG} = c = \frac{4a}{3\pi}; \ k^2 = \frac{a^2}{2} - \left(\frac{4a}{3\pi}\right)^2,$$

and the time of oscillation

$$= \pi \left\{ \frac{9\pi - 16}{8g} a \right\}^{\frac{1}{2}}.$$

If the rocking body is a homogeneous hemisphere, the time of an oscillation

$$= \pi \left(\frac{26a}{15g}\right)^{\frac{1}{2}}.$$

Ex. 10. A thin heavy homogeneous bar is suspended from a horizontal beam by two fine inelastic strings of equal length attached to its extremities so that the bar in its position of rest is horizontal. The bar is displaced (1) in the direction of its length; (2) in a direction perpendicular to its length; (3) by a twist about a vertical axis passing through its centre: the strings in each case remaining taut: required to determine the motions, the displacements being infinitesimal.

Let $2a$ be the length and m the mass of the bar, and c the length of each string of suspension. Let the centre of gravity of the bar in its position of rest be the origin, and let its line be the axis of x. Let (x, y, z) be the place of the centre of gravity at the time t, and let T be the tension of each string at that time.

(1) As the bar moves in only a vertical plane, let ϕ be the angle between each string and the vertical at the time t: then by the geometry $x = c\sin\phi$, $y = 0$, $z = c(1-\cos\phi)$: and the equations of motion are

$$m\frac{d^2x}{dt^2} = -2\text{T}\sin\phi; \quad m\frac{d^2y}{dt^2} = 0; \quad m\frac{d^2z}{dt^2} = -mg+2\text{T}\cos\phi.$$

$$\therefore \quad c\frac{d^2\phi}{dt^2} = -g\sin\phi,$$

$$c\frac{d\phi^2}{dt^2} = 2g(\cos\phi-\cos a),$$

where a is the greatest value of ϕ. As a and ϕ are both small, we may replace them by their approximate values, and thus obtain

$$c\frac{d\phi^2}{dt^2} = g(a^2-\phi^2);$$

whence

$$\phi = a\sin\left(\frac{g}{c}\right)^{\frac{1}{2}}t;$$

and therefore, taking approximate values,

$$x = ca\sin\left(\frac{g}{c}\right)^{\frac{1}{2}}t; \quad y = 0; \quad z = \frac{ca^2}{2}\left\{\sin\left(\frac{g}{c}\right)^{\frac{1}{2}}\frac{t}{2}\right\}^2:$$

so that x and z are periodic functions of t; of which the period is $2\pi\left(\frac{c}{g}\right)^{\frac{1}{2}}$, and of which the amplitudes are respectively ca, and $\frac{ca^2}{2}$.

(2) Let the bar receive a lateral displacement, and let ψ be the angle between each string and the vertical at the time t: then by the geometry $x = 0$, $y = c\sin\psi$, $z = c(1-\cos\psi)$: and the equations of motion are

$$m\frac{d^2x}{dt^2} = 0; \quad m\frac{d^2y}{dt^2} = -2\text{T}\sin\psi; \quad m\frac{d^2z}{dt^2} = -mg+2\text{T}\cos\psi;$$

whence if β is the greatest value of ψ, and β and ψ are both small and $\psi = \beta$, when $t = 0$,

$$\psi = \beta \cos \left(\frac{g}{c}\right)^{\frac{1}{2}} t;$$

and taking approximate values as heretofore,

$$x = 0; \quad y = c\beta \cos\left(\frac{g}{c}\right)^{\frac{1}{2}} t; \quad z = \frac{c\beta^2}{2}\left\{\sin\left(\frac{g}{c}\right)^{\frac{1}{2}} \frac{t}{2}\right\}^2;$$

so that y and z are periodic functions of t, of which the period is $2\pi\left(\frac{c}{g}\right)^{\frac{1}{2}}$, being the same as that in the preceding case: and of which the amplitudes are respectively $c\beta$ and $\frac{c\beta^2}{2}$.

If then the epoch is the same in these cases, and the results are simultaneous and their resultant taken, then

$$\frac{x^2}{c^2\alpha^2} + \frac{y^2}{c^2\beta^2} = 1;$$

$$z = \frac{c}{2}(\alpha^2+\beta^2)\left\{\sin\left(\frac{g}{c}\right)^{\frac{1}{2}} \frac{t}{2}\right\}^2;$$

of which the former is the equation to an ellipse: so that the centre of gravity of the bar moves in a curve, the projection of which on the plane of (x, y) is an ellipse; and the greatest height to which the centre of gravity rises is $\frac{c}{2}(\alpha^2+\beta^2)$, this being its place when $t = (2n+1)\pi\left(\frac{c}{g}\right)^{\frac{1}{2}}$, where n is 0 or any integer.

If the epoch in the two cases is not the same, the projected path is still an ellipse, but neither of its principal axes will be parallel to the line of the bar.

(3) Let the bar receive a twist about a vertical line passing through its centre of gravity, whereby an angular velocity Ω is given to it; and let θ be the angles through which the bar has moved in the time t; and let z be the height of the centre of gravity above its original place of rest at that time. Then by the geometry

$$(a - a\cos\theta)^2 + a^2(\sin\theta)^2 + (c-z)^2 = c^2;$$

$$\therefore \quad (c-z)^2 = c^2 - 2a^2 + 2a^2\cos\theta.$$

Also by the equation of vis viva,

$$m\left(\frac{dz^2}{dt^2} + \frac{a^2}{3}\frac{d\theta^2}{dt^2}\right) = \frac{ma^2\Omega^2}{3} - 2mgz,$$

$$\left\{\frac{a^4(\sin\theta)^2}{(c-z)^2} + \frac{a^2}{3}\right\}\frac{d\theta^2}{dt^2} = \frac{a^2\Omega^2}{3} - 2gz;$$

by means of which equations θ and z may be found in terms of t.

The last equation shews that the bar comes to rest whenever $z = \frac{a^2 \Omega^2}{6g}$.

Let this quantity $= h$; so that the equation of vis viva becomes

$$\left\{\frac{a^4 (\sin\theta)^2}{(c-z)^2} + \frac{a^2}{3}\right\} \frac{d\theta^2}{dt^2} = 2g(h-z).$$

If T is the tension of each string at the time t, we have from the general equations of motion

$$m\frac{d^2 z}{dt^2} = -mg + 2\text{T}\frac{c-z}{c};$$

$$m\frac{a^2}{3}\frac{d^2\theta}{dt^2} = -\frac{2\text{T}a}{c}(2cz - z^2)^{\frac{1}{2}} \cos\frac{\theta}{2}$$

$$= -\frac{2a^2 \sin\theta}{c}\text{T};$$

by either of which equations T may be determined. If the former be multiplied by dz, and the latter by $d\theta$, and the results added, T disappears, and the integral is the equation of vis viva, as given above.

As θ and z cannot generally be determined in terms of t, I will take some particular cases. If $h = c$, so that the bar comes to rest in the horizontal plane of the fixed beam,

$$\frac{d^2\theta}{dt^2} = -3g(c-z)\sin\theta \frac{-3(a\sin\theta)^2 + (c-z)^2(1+6\cos\theta)}{\{3(a\sin\theta)^2 + (c-z)^2\}^2};$$

$$\therefore \quad \text{T} = \frac{mgc(c-z)}{2} \frac{-3(a\sin\theta)^2 + (c-z)^2(1+6\cos\theta)}{\{3(a\sin\theta)^2 + (c-z)^2\}^2}.$$

Therefore $\text{T} = 0$, if $z = c$; so that the tension vanishes when the bar comes to rest. And if $z = 0$, $\text{T} = \frac{7mg}{2}$; that is, the initial tension of the string is seven times its tension in the position of rest.

If $h = c = 2a$, so that the bar comes to rest after turning through an angle of 180°; the initial tension of the string is also seven times that of the tension when at rest.

Again, if the displacement of the bar is small, so that θ and z are always small, $2cz = a^2\theta^2$, and the equation of vis viva becomes

$$\frac{d\theta^2}{dt^2} = \frac{3g}{c}\left(\frac{2ch}{a^2} - \theta^2\right);$$

which gives a periodic function, the extreme values of θ being $\pm\left(\frac{2ch}{a^2}\right)^{\frac{1}{2}}$, and the periodic time being $2\pi\left(\frac{c}{3g}\right)^{\frac{1}{2}}$.

If we take this result in combination with those of the first two cases, and suppose all to be simultaneous, as the principle of the superposition of small motions authorises, the centre of gravity of the bar describes a small ellipse of which the centre is the centre of gravity of the bar when at rest; and the bar oscillates about a vertical axis passing through its centre, its periodic time being to the periodic time of centre of gravity in the ellipse as $3^{\frac{1}{2}}$ to 1.

Ex. 11. A fine string is coiled round a heavy cylindrical wheel; one end of the string is fixed, and the wheel descends, unwinding the string: it is required to determine the motion of the wheel.

Let M = the mass of the wheel; T = the tension of the string at the time t; a = the radius; $OP = x$, see Fig. 55; θ = the angle through which the wheel has revolved from its position of rest; k = the radius of gyration of the wheel. Then the equations of motion are

$$M\frac{d^2x}{dt^2} = Mg - T; \qquad Mk^2\frac{d^2\theta}{dt^2} = aT;$$

also $dx = a\,d\theta$; so that

$$(a^2+k^2)\frac{d^2\theta}{dt^2} = ag; \text{ and since } k^2 = \frac{a^2}{2}, \text{ we have}$$

$$a\frac{d^2\theta}{dt^2} = \frac{2g}{3}; \qquad \therefore \frac{d^2x}{dt^2} = \frac{2g}{3}; \qquad \therefore x = x_0 + \frac{gt^2}{3};$$

so that the space described in a given time is two-thirds of that which would be described by the wheel falling freely.

The equation of vis viva is

$$M\left\{\frac{dx^2}{dt^2} + k^2\frac{d\theta^2}{dt^2}\right\} = 2Mgx,$$

from which the preceding results may evidently be derived.

Ex. 12. To determine the motion of a system of pulleys and weights, each of which hangs by a separate string, as in Fig. 56.

The system consists of a fixed pulley whose centre is C, and of a series of pulleys whose centres at the time t are $C_1, C_2, C_3, \ldots$; we will assume all these pulleys to be equal, a to be the radius, k to be the radius of gyration, and m to be the mass of each;

let M be the mass of the weight which acts round the fixed pulley, and M′ the mass of that which is attached to the last moving pulley. Let T be the tension of the string by which M acts on the fixed pulley, and $T_1, T_2, T_3, \ldots$ be the tensions of the strings which severally pass from the fixed pulley to the first moving pulley, from the first moving pulley to the second moving pulley, and so on; and let $T_1', T_2', T_3', \ldots$ be the tensions of the strings which are fastened severally to $A_1, A_2, A_3, \ldots$; let x be the distance of M, and let $x_1, x_2, x_3, \ldots$ be the distances of $C_1, C_2, C_3, \ldots$ from the horizontal line $A_1 A_2, \ldots$ at the time t; let $d\theta, d\theta_1, d\theta_2, \ldots$ be the angles through which the fixed and the several moving pulleys respectively rotate in the time dt; and, to fix our thoughts, let us suppose M′ to descend, so that $x_1, x_2, x_3, \ldots$ increase as t increases; and let us suppose $\theta, \theta_1, \theta_2, \ldots$ to increase as t increases. Then we have the following series of equations of motion;

$$\left.\begin{aligned}
\mathrm{M}\left(g-\frac{d^2 x}{dt^2}\right) &= \mathrm{T}, \\
m\left(g-\frac{d^2 x_1}{dt^2}\right) &= \mathrm{T}_1+\mathrm{T}_1'-\mathrm{T}_2, \\
m\left(g-\frac{d^2 x_2}{dt^2}\right) &= \mathrm{T}_2+\mathrm{T}_2'-\mathrm{T}_3, \\
&\cdots\cdots \\
m\left(g-\frac{d^2 x_{n-1}}{dt^2}\right) &= \mathrm{T}_{n-1}+\mathrm{T}_{n-1}'-\mathrm{T}_n, \\
(m+\mathrm{M}')\left(g-\frac{d^2 x_n}{dt^2}\right) &= \mathrm{T}_n+\mathrm{T}_n';
\end{aligned}\right\} \qquad (175)$$

$$\left.\begin{aligned}
\frac{mk^2}{a}\frac{d^2\theta}{dt^2} &= \mathrm{T}_1-\mathrm{T}, \\
\frac{mk^2}{a}\frac{d^2\theta_1}{dt^2} &= \mathrm{T}_1'-\mathrm{T}_1, \\
\frac{mk^2}{a}\frac{d^2\theta_2}{dt^2} &= \mathrm{T}_2'-\mathrm{T}_2, \\
&\cdots\cdots \\
\frac{mk^2}{a}\frac{d^2\theta_{n-1}}{dt^2} &= \mathrm{T}_{n-1}'-\mathrm{T}_{n-1}, \\
\frac{mk^2}{a}\frac{d^2\theta_n}{dt^2} &= \mathrm{T}_n'-\mathrm{T}_n.
\end{aligned}\right\} \qquad (176)$$

We have also the two following series of conditions; (177), because the pulleys are rough and the cords roll round them and

do not slip; and (178), by reason of the geometrical relations of the system. (179) follow from (178);

$$\left.\begin{aligned} dx &= -a\,d\theta, \\ dx_1 &= a\,d\theta_1, \\ dx_2 &= a\,d\theta_2, \\ &\dots\dots \\ dx_n &= a\,d\theta_n; \end{aligned}\right\} \quad (177) \qquad \left.\begin{aligned} dx &= -2\,dx_1, \\ dx_1 &= 2\,dx_2, \\ dx_2 &= 2\,dx_3, \\ &\dots\dots \\ dx_{n-1} &= 2\,dx_n; \end{aligned}\right\} \quad (178)$$

$$\left.\begin{aligned} dx &= -2^n\,dx_n, \\ dx_1 &= 2^{n-1}\,dx_n, \\ dx_2 &= 2^{n-2}\,dx_n, \\ &\dots\dots \\ dx_{n-1} &= 2\,dx_n. \end{aligned}\right\} \quad (179)$$

Now $k^2 = \frac{a^2}{2}$; and consequently, taking the horizontal pairs of (175) and (176), and introducing the conditions (177), we have

$$\left.\begin{aligned} \text{M}\left(g - \frac{d^2x}{dt^2}\right) - \frac{m}{2}\frac{d^2x}{dt^2} &= \text{T}_1, \\ m\left(g - \frac{3}{2}\frac{d^2x_1}{dt^2}\right) &= 2\,\text{T}_1 - \text{T}_2, \\ m\left(g - \frac{3}{2}\frac{d^2x_2}{dt^2}\right) &= 2\,\text{T}_2 - \text{T}_3, \\ &\dots\dots\dots\dots \\ m\left(g - \frac{3}{2}\frac{d^2x_{n-1}}{dt^2}\right) &= 2\,\text{T}_{n-1} - \text{T}_n, \\ \text{M}'\left(g - \frac{d^2x_n}{dt^2}\right) + m\left(g - \frac{3}{2}\frac{d^2x_n}{dt^2}\right) &= 2\,\text{T}_n; \end{aligned}\right\} \quad (180)$$

whence, eliminating $\text{T}_1, \text{T}_2, \dots \text{T}_n$, and replacing $x_1, x_2, \dots$ in terms of x_n according to the values given in (179), we have

$$\left\{\text{M}' + \frac{m}{2}(2^{2n+1} - 1) + \text{M}\,2^{2n}\right\}\frac{d^2x_n}{dt^2} = \{\text{M}' + m(2^n - 1) - \text{M}\,2^n\}\,g; \quad (181)$$

and this determines the place of M' at the time t. From this value may be deduced, by means of (179), the places of M and of the centre of every pulley at any time.

If $n = 1$, the system is that of a single fixed and of a single moving pulley; and we have

$$\left\{\text{M}' + \frac{7m}{2} + 4\text{M}\right\}\frac{d^2x_1}{dt^2} = \{\text{M}' + m - 2\text{M}\}\,g.$$

If we equate to zero the right-hand member of (181) we have

the condition of statical equilibrium of the system of weights and pulleys, when the weights of the pulleys are taken account of.

373.] In the course of the preceding problems some circumstances of motion have incidentally arisen which require fuller and more special consideration.

The first is the case of initial circumstances, whether the general circumstances can be determined or not; that is, tensions and stresses may be determined initially, although the values of them at the time t cannot be found in finite integral terms. As these quantities are functions of second time-differential coefficients of coordinates, the process will generally be, to obtain values of these quantities from the geometrical connexions of the system, and to substitute these in the equations of motion, taking the initial values of them. The process will be best understood by means of examples; such as Ex. 10 of the preceding Article. The following are other examples in illustration:

Ex. 1. A heavy homogeneous bar is suspended horizontally by two thin strings of equal length attached to its ends, and fastened to a horizontal beam; one string is cut, find the change which takes place in the tension of the other in consequence.

Let m be the mass and $2a$ the length of the bar; let c be the length of each string, and a the angle which each makes with the vertical in the position of rest; so that if T is the tension in that position $2\text{T} \cos a = mg$. At the time t after one string is cut let ϕ be the angle between the uncut string and the vertical; θ the angle between the bar and the horizontal, and (z, x) the coordinates of the centre of gravity of the bar in reference to its initial place. Then the geometrical conditions are

$$x + a = a \cos\theta + c \sin\phi, \qquad z + c = a \sin\theta + c \cos\phi.$$

Let time-differential coefficients be denoted by accents, as in Art. 126, and initial values by small subscript cyphers. Then differentiating these equations twice, and bearing in mind that initially

$$\theta = 0, \qquad \phi = a, \qquad \theta' = \phi' = 0,$$

we have

$$x_0'' = c \cos a \phi_0'', \qquad z_0'' = a\theta_0'' - c \sin a \phi_0''.$$

Now the equations of motion at the time t are

$$m x'' = -\text{T} \sin\phi, \qquad m z'' = mg - \text{T} \cos\phi,$$

$$\frac{m a}{3} \theta'' = \text{T} a \cos(\theta + \phi).$$

$$\therefore \quad m x_0'' = -\text{T}_0 \sin\alpha, \quad m z_0'' = mg - \text{T}_0 \cos\alpha,$$

$$\frac{m a^2}{3}\theta_0'' = \text{T}_0 a \cos\alpha,$$

whence eliminating x_0'', z_0'', θ_0'', ϕ_0'' we have

$$\text{T}_0 = \frac{mg\cos\alpha}{1+3(\cos\alpha)^2}.$$

If $\alpha = 0$, so that the beam is initially suspended by vertical strings,

$$\text{T}_0 = \frac{mg}{4},$$

that is the tension is diminished by one-half, by the cutting of the string;

T_0 is a maximum, when $3(\cos\alpha)^2 = 1$.

Ex. 2. An elliptical plate is suspended horizontally by three strings fastened to a fixed point above it, one string being attached to a focus, and the others to the ends of the opposite latus rectum: the string attached to the focus being cut, find the initial tension of the two other strings.

Let O be the fixed point, C the centre of the plate, so that OC is vertical; S the focus and L, L′ the extremities of the opposite latus rectum to which the strings are attached; H the other focus so that LL′ is bisected in H.

Let OL = OL′ $= c$; LOH = L′OH $= \beta$, so that OH $= c\cos\beta$. Let HOC $= \alpha$. Now the tensions T along the lines OL and OL′ may be compounded into a single force R along OH, being such that R $= 2$T$\cos\beta$. Then the problem becomes similar to that of the preceding example, and employing like symbols we have

$$x_0'' = c\cos\beta\cos\alpha\,\phi_0'', \quad z_0'' = ae\,\theta_0'' - c\cos\beta\sin\alpha\,\phi_0'';$$

$$m x_0'' = -\text{R}_0 \sin\alpha, \quad m z_0'' = mg - \text{R}_0\cos\alpha;$$

$$\frac{m a^2}{4}\theta_0'' = \text{R}_0 ae\cos\alpha;$$

whence

$$\text{R}_0 = \frac{mg\cos\alpha}{1+4e^2(\cos\alpha)^2},$$

$$\text{T}_0 = \frac{mg\cos\alpha}{2\cos\beta\,\{1+4e^2(\cos\alpha)^2\}}.$$

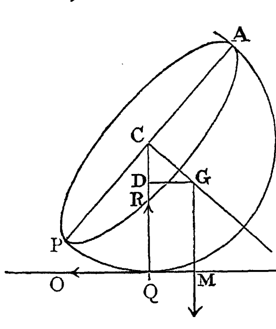
Ex. 3. A heavy homogeneous sphere rests on a rough horizontal plane, and is suddenly divided by a vertical cleavage into two hemispheres; which at once begin to fall. Determine the pressure of each on the plane.

Let one of the falling hemispheres be represented in the annexed figure, having moved through the angle θ. Let AP be the diameter of the cleavage plane which was originally vertical, O being the point at which P rested on the plane. Let m be the mass and a the radius of the hemisphere; let PCQ $= \theta$, and let G be the centre of gravity of the hemisphere, so that

$$\text{CG} = c = \frac{3a}{8}.$$

Also OQ = the arc PQ $= a\theta$. Let (x, y) be the place of G at the time t; so that

$$x = a\theta + c\cos\theta; \quad y = a - c\sin\theta.$$



Differentiating these expressions twice, and bearing in mind that initially $\theta = 0$, $\theta' = 0$,

$$x_0'' = a\theta_0''; \qquad y_0'' = -c\theta_0'';$$

as the equations of motion at the time t are

$$mx'' = \text{F}, \quad my'' = -mg + \text{R},$$
$$mk^2\theta'' = \text{R}c\cos\theta - \text{F}(a - c\sin\theta),$$

where $k^2 = \dfrac{83a^2}{320}$;

$$\therefore \quad mx_0'' = \text{F}_0, \quad my_0'' = -mg + \text{R}_0, \quad mk^2\theta_0'' = \text{R}_0 c - \text{F}_0 a;$$

whence by elimination,

$$\text{R}_0 = mg\frac{a^2 + k^2}{a^2 + k^2 + c^2} = \frac{403}{448}mg;$$

so that the pressure on the plane is initially diminished by the motion.

Ex. 4. If the sphere is divided into n equal sectorial wedges, all having a vertical diameter ACP as a common edge, and being placed on a rough horizontal plane, all are simultaneously set free, the change of pressure on the plane is determined as follows:

Using the same notation as in the preceding example, and taking one sectorial wedge to be the moving body, if M is the mass of the sphere,

$$c = \frac{3na}{16}\sin\frac{\pi}{n}, \quad k^2 = a^2\left\{\frac{2}{5} - \frac{n}{10\pi}\sin\frac{2\pi}{n}\right\} - c^2, \quad m = \frac{\text{M}}{n};$$

and consequently, if R_0 is the whole pressure on the plane,

$$R_0 = Mg\frac{a^2+k^2}{a^2+k^2+c^2} = Mg\left\{1-\frac{c^2}{a^2+k^2+c^2}\right\},$$

whence, substituting for k^2 and c^2, the change of pressure on the plane can be determined.

If $n = \infty$, then

$$c = \frac{3\pi a}{16}, \qquad k^2 = \frac{a^2}{5} - c^2;$$

$$\therefore \quad R_0 = Mg\left\{1-\frac{45\pi^2}{1536}\right\}.$$

374.] The next class of examples which require further and special consideration is the general case of the rocking, or titubation as it has been called, of a heavy body bounded by a cylindrical surface, resting on another rough cylindrical surface, the axes of the two surfaces being parallel and horizontal, when the upper body which rests on the lower surface is slightly displaced from its position of equilibrium.

Fig. 57, which represents a section of the two surfaces by a vertical plane perpendicular to the axes of the cylinders, shews the circumstances at the time t.

G is the centre of gravity of the upper body, whose mass $= m$; and when the upper body is at rest on the lower, A is at the point O, and the line GA, which is the normal of the upper surface at A, is vertical, and is in the same straight line with OC, which is the normal to the lower surface at O. Let the upper body be slightly displaced by rolling, not sliding, on the lower; so that the arcs AP and OP are equal. Let C′ and C be the centres of curvature of the upper and lower surfaces at A and O respectively. The normals to the two surfaces at P are evidently in the same straight line; and since AP and OP are infinitesimal, C′A = C′P, and CO = CP. Let $R =$ the normal pressure of the two surfaces on each other, and let $F =$ the friction of rolling; also let $k =$ the radius of gyration of the moving body relative to an axis through G parallel to the axes of the cylinders.

Let O be the origin, (x, y) the place of G at the time t; CO = CP $= \rho$; C′A = C′P $= \rho'$; C′G $= c$, OCP $= \theta$, AC′P $= \theta'$; consequently $\rho\theta = \rho'\theta'$, and

$$\theta + \theta' = \frac{\rho+\rho'}{\rho'}\,\theta.$$

The equations of motion of m are

$$\left.\begin{aligned} m\frac{d^2x}{dt^2} &= \text{R} \sin\theta - \text{F}\cos\theta, \\ m\frac{d^2y}{dt^2} &= \text{R}\cos\theta + \text{F}\sin\theta - mg, \\ mk^2\frac{d^2(\theta+\theta')}{dt^2} &= -\text{R}c\sin\theta' + \text{F}(\rho'-c)\cos\theta'; \end{aligned}\right\} \quad (182)$$

and the geometrical equations of condition are

$$\left.\begin{aligned} x &= \quad (\rho+\rho')\sin\theta - c\sin(\theta+\theta'), \\ y &= -\rho + (\rho+\rho')\cos\theta - c\cos(\theta+\theta'). \end{aligned}\right\} \quad (183)$$

As the displacement which we are considering is very small, I shall assume θ, θ' and $\frac{d\theta}{dt}$ to be so small that all powers of them above the first may be neglected. Under these suppositions, the preceding give

$$\left.\begin{aligned} \frac{d^2x}{dt^2} &= \left\{\rho+\rho' - c\frac{\rho+\rho'}{\rho'}\right\}\frac{d^2\theta}{dt^2}, \\ \frac{d^2y}{dt^2} &= \left\{-(\rho+\rho')\theta + c\left(\frac{\rho+\rho'}{\rho'}\right)^2\theta\right\}\frac{d^2\theta}{dt^2}; \end{aligned}\right\} \quad (184)$$

and substituting these values in the first two of (182), we have

$$\text{R} = mg + m\frac{(\rho+\rho')\rho}{\rho'}c\theta\frac{d^2\theta}{dt^2}; \quad (185)$$

$$\text{F} = mg\theta - m\frac{\rho+\rho'}{\rho'}(\rho'-c)\frac{d^2\theta}{dt^2}; \quad (186)$$

which determine R and F in terms of $\frac{d^2\theta}{dt^2}$. If we substitute these values in the last of (182), and omit terms involving powers of θ higher than the first, we have

$$\{k^2+(\rho'-c)^2\}\frac{d^2\theta}{dt^2} - \frac{\rho'^2 - c(\rho+\rho')}{\rho+\rho'}g\theta = 0; \quad (187)$$

which is the equation of rotation of the upper body about its rotation-axis through G.

Since the coefficient of $\frac{d^2\theta}{dt^2}$ is positive, the form of the integral of this equation will depend on the sign of the coefficient of θ.

(1) If ρ'^2 is greater than $c(\rho+\rho')$; then the integral of (187) takes the exponential form, and θ will continually increase

as t increases; so that the body moves farther away from its original position of rest, that position being one of unstable equilibrium. The geometrical meaning of this criterion is

$$\frac{1}{\text{AG}} \text{ is less than } \frac{1}{\rho}+\frac{1}{\rho'}.$$

(2) If $\rho'^2 = c(\rho+\rho')$, that is, if

$$\frac{1}{\text{AG}} = \frac{1}{\rho}+\frac{1}{\rho'}, \quad \frac{d^2\theta}{dt^2} = 0,$$

and the body either remains at rest in its new position, or rotates with a constant angular velocity. The original equilibrium in this case is neutral.

(3) If ρ'^2 is less than $c(\rho+\rho')$, (187) is the equation of harmonic motion; in which case the body rocks or titubates; and the time of an oscillation

$$= \pi \frac{\{k^2+(\rho'-c)^2\}^{\frac{1}{2}}(\rho+\rho')^{\frac{1}{2}}}{g^{\frac{1}{2}}\{c(\rho+\rho')-\rho'^2\}^{\frac{1}{2}}}. \tag{188}$$

In this case, the original equilibrium is stable; and we have

$$\frac{1}{\text{AG}} \text{ greater than } \frac{1}{\rho}+\frac{1}{\rho'}.$$

The geometrical criteria for the stability, neutrality, and instability of equilibrium, are the same as those found from statical considerations in Art. 142, Vol. III.

The process and the results of this Article are equally true whatever are the signs and values of ρ, ρ', and c. Thus, if ρ is negative, the lower surface has its concavity upwards, and the problem is that of a body with a convex surface rolling on a concave surface. If ρ' is negative, we have a body with a concave surface rolling on a convex surface. If $\rho = \infty$, the lower surface is plane, and a body with a convex surface rolls upon it. If $\rho' = \infty$, the upper surface is plane, and the body with a plane surface rolls on a convex surface.

375.] Another principle, which requires special consideration, is that of small oscillations.

When a system of material particles, subject to mutual connections or restraints, is slightly disturbed from a position of stable equilibrium, certain forces are brought into action, which tend to restore the system to its original place of rest. We have

had examples of such forces in the preceding Article; and in Examples 4, 8, 9, 10, of Art. 372. Now the equations of connection of the several particles enable us to express the coordinates of the places of these particles at the time t in their disturbed state as functions of new variables θ, ϕ, ψ, ..., which are, all and each, as well as their first t-differentials, infinitesimals when the displacement of the system is infinitesimal, and are equal to zero when the system is in its place of stable equilibrium; and the number of these variables is of course that which is sufficient to determine the places of the particles, subject as they are to their mutual connections. If then we substitute these variables for the old variables in the equations of motion, we thereby obtain new differential equations which correctly represent the circumstances of motion in terms of the new variables. These equations are of the second order, and contain no term independent of the variables θ, ϕ, ψ, ...; because they must be satisfied when the system is in its position of stable equilibrium; that is, when $\theta = \phi = \psi = \ldots = 0$: and since the displacement of the system is, by our assumption, small, these variables and their first differentials are small; if therefore our object is to acquire a general idea of the motion, that is, to obtain the principal terms of the motion, we must in the first place neglect all powers of the variables and of their t-differentials higher than the first. With this object in view the functions of the variables which enter into the equations must be expanded in ascending powers of the variable; and the squares, and all higher powers of them, are to be omitted; thus the differential equations will be linear, of the second order, of constant coefficients, and devoid of second members; and their integrals will be of the following forms:

$$\left.\begin{aligned} \theta &= \text{A}_1 \cos(r_1 t - a_1) + \text{A}_2 \cos(r_2 t - a_2) + \ldots, \\ \phi &= \text{B}_1 \cos(r_1 t - a_1) + \text{B}_2 \cos(r_2 t - a_2) + \ldots, \\ \psi &= \text{C}_1 \cos(r_1 t - a_1) + \text{C}_2 \cos(r_2 t - a_2) + \ldots, \\ &\cdots\cdots\cdots\cdots\cdots ; \end{aligned}\right\} \qquad (189)$$

where r_1, r_2,... are the roots of a certain algebraical equation of an order equal to the number of the variables; A_1, A_2, ..., a_1, a_2, ... are constants depending on the initial or other circumstances of the system; and B_1, B_2, ..., C_1, C_2,... are other constants, which are functions of A_1, A_2, ..., r_1, r_2, ..., and are determinable by means of the given differential equations.

If n is the number of the variables $\theta, \phi, \psi, \ldots$, each of the equations in (189) is the sum of n terms, which are circular functions of t; each term by itself representing a small oscillation of the same nature as that of the simple pendulum; the times of oscillation corresponding to each term being different, and being severally $\frac{\pi}{r_1}, \frac{\pi}{r_2}, \ldots$; and each variable generally containing a term of each period. Thus the motion of the system, slightly disturbed from its position of stable equilibrium, consists of simple oscillations of its several component particles, both the amplitudes and the periodic times being in general different for the several oscillations. As these oscillations coexist, and as each variable is the sum of many, the principle of their combination is commonly called the law of the coexistence of small oscillations.

If the quantities $r_1, r_2, r_3, \ldots$ are commensurable, the system of particles will periodically pass through the same state; for suppose μ to be the greatest common measure of $r_1, r_2, \ldots$; so that

$$r_1 = k_1 \mu, \qquad r_2 = k_2 \mu, \qquad r_3 = k_3 \mu, \ldots;$$

where $k_1, k_2, k_3, \ldots$ are whole numbers which have no common measure; then if T is the time in which the system passes from a given state to the same state again, $k_1 \mu \mathrm{T}, k_2 \mu \mathrm{T}, \ldots$ must all be multiples of 2π; and as $k_1, k_2, \ldots$ have no common factor, the least value of T which will satisfy this condition is

$$\mathrm{T} = \frac{2\pi}{\mu}; \tag{190}$$

this therefore is the time in which the system of particles passes through all its forms from one state to the same state again.

If $\mu = 0$, this time is infinite; that is, if the quantities $r_1, r_2, \ldots$ have no common measure, the system of particles is not periodic; and the state in which the particles may be at a given time is never taken by the particles again.

376.] Since each of these small motions takes effect separately, and independently of other similar motions; and since the whole effect is the sum of these separate and partial effects; the law of co-existence of small oscillations is a particular case of the so-called principle of superposition of small motions. This principle may be explained in the following way.

Suppose that for certain initial values of the variables and their t-differentials, say,

$$\left.\begin{array}{lll} \theta = \theta_1, & \phi = \phi_1, & \psi = \psi_1, \ldots, \\ \frac{d\theta}{dt} = \theta_1', & \frac{d\phi}{dt} = \phi_1', & \frac{d\psi}{dt} = \psi_1', \ldots, \end{array}\right\} \tag{191}$$

the motion is represented by the integrals,

$$\theta = \Theta_1, \quad \phi = \Phi_1, \quad \psi = \Psi_1, \ldots; \tag{192}$$

and suppose that for another system of values, say,

$$\left.\begin{array}{ll} \theta = \theta_2, & \phi = \phi_2, \ldots, \\ \frac{d\theta}{dt} = \theta_2', & \frac{d\phi}{dt} = \phi_2', \ldots, \end{array}\right\} \tag{193}$$

the motion is represented by the integrals,

$$\theta = \Theta_2, \quad \phi = \Phi_2, \quad \psi = \Psi_2, \ldots; \tag{194}$$

and so on for n systems: then for the systems of values which are the sums of all these values, viz.

$$\theta = \theta_1 + \theta_2 + \ldots, \qquad \phi = \phi_1 + \phi_2 + \ldots, \qquad \psi = \psi_1 + \psi_2 + \ldots,$$

$$\frac{d\theta}{dt} = \theta_1' + \theta_2' + \ldots, \qquad \frac{d\phi}{dt} = \phi_1' + \phi_2' + \ldots, \qquad \frac{d\psi}{dt} = \psi_1' + \psi_2' + \ldots,$$

the motion is represented by the sum of the partial integrals; viz.,

$$\left.\begin{array}{l} \theta = \Theta_1 + \Theta_2 + \ldots, \\ \phi = \Phi_1 + \Phi_2 + \ldots, \\ \psi = \Psi_1 + \Psi_2 + \ldots; \end{array}\right\} \tag{195}$$

for these values will satisfy the differential equations of motion by reason of their linearity; and they reduce themselves to the several initial values when $t = 0$; thus they satisfy all the conditions of the problem.

The preceding processes are only applicable when we confine ourselves to small motions and to first approximations. If a more exact determination is required, we must return to the original equations of motion in their complete forms, and substitute in terms of the second degree relatively to the variables those values which we have found in terms of t to a first approximation; and then, neglecting all the terms of a degree higher than the second, we shall have new equations which will differ from the first only by the addition of a new member, which is a known function of t. Values of θ, ϕ, ... will be determined from these to an approximation higher than the former. And if an approximation is required still more exact, we must

introduce the second values of the variables in the original equations, and pursue a process similar to the former.

377.] The following examples are illustrative of the process.

Ex. 1. I will first take the simple case of a conical pendulum, that is, of a heavy particle constrained to move on the inside of a smooth spherical surface; this problem is the same as that which has been considered in Arts. 440 and 441, Vol. III.

Let us refer the position of the moving particle to the point of suspension of the pendulum as the origin, and to two vertical planes passing through that point, and perpendicular to each other, as the planes of (x, z) and (y, z), the axis of z being taken vertically downwards. Let us moreover suppose the rod of the pendulum, whose length $= l$, initially to be in the plane of (x, z), and to be inclined at an angle $= a$ to the z-axis; and the bob to be projected with a velocity $= u$ perpendicularly to the plane of (x, z); let the line of the pendulum at the time t be projected on the planes of (x, z) and (y, z); and let the angles between these projections and the z-axis be respectively θ and ψ: now I shall assume that the oscillations of the pendulum are always small, and I shall consequently consider θ and ψ, the variables which determine its position, to be so small that powers of them higher than the second are to be neglected. The initial values of θ and ψ are respectively a and 0; and of $\frac{d\theta}{dt}$ and $\frac{d\psi}{dt}$ are 0 and $\frac{u}{l}$; then the equations of motion are

$$l\frac{d^2\theta}{dt^2} = -g\theta, \qquad l\frac{d^2\psi}{dt^2} = -g\psi;$$

$$\therefore \quad l\frac{d\theta^2}{dt^2} = g(a^2-\theta^2), \qquad l\left\{\frac{d\psi^2}{dt^2} - \frac{u^2}{l^2}\right\} = -g\psi^2;$$

$$\theta = a\cos\left(\frac{g}{l}\right)^{\frac{1}{2}}t; \qquad \psi = \frac{u}{(gl)^{\frac{1}{2}}}\sin\left(\frac{g}{l}\right)^{\frac{1}{2}}t.$$

Let (x, y) be the projection on the horizontal plane of (x, y) of the place of the bob of the pendulum at the time t; so that

$$\begin{aligned} x &= l\sin\theta & \qquad y &= l\sin\psi \\ &= l\theta & &= l\psi \\ &= la\cos\left(\frac{g}{l}\right)^{\frac{1}{2}}t; & &= u\left(\frac{l}{g}\right)^{\frac{1}{2}}\sin\left(\frac{g}{l}\right)^{\frac{1}{2}}t; \end{aligned}$$

let $y = x\tan\phi$; then

$$\tan\phi = \frac{u}{a(lg)^{\frac{1}{2}}}\tan\left(\frac{g}{l}\right)^{\frac{1}{2}}t;$$

so that ϕ, which determines the plane of oscillation of the pendulum at the time t, does not vary directly with t, and consequently the pendulum does not revolve uniformly. Also

$$\frac{x^2}{l^2 a^2} + \frac{g y^2}{l u^2} = 1, \tag{196}$$

which represents an ellipse; so that the bob of the pendulum describes a path whose projection on the plane of (x, y) is an ellipse. All these results are in accordance with those of Art. 441, Vol. III.

Ex. 2. A system of n heavy rods $\mathrm{O A_1}, \mathrm{A_1 A_2}, \ldots$, of given lengths $2a_1, 2a_2, \ldots, 2a_n$ is formed by means of smooth joints at their extremities $\mathrm{A_1}, \mathrm{A_2}, \ldots$, as in Fig. 58, and is suspended by the extremity O from a fixed point. Determine the small oscillations of the system, when the motions of all are in the same vertical plane, say of (x, y).

Let the angles which the rods respectively make with the vertical Oy be $\theta_1, \theta_2, \ldots$; and let $(x_1, y_1), (x_2, y_2), \ldots$ be the places of their centres of gravity at the time t; let $m_1, m_2, m_3, \ldots$ be the masses of the rods, and $k_1, k_2, \ldots$ their radii of gyration relative to axes passing through their centres of gravity, and perpendicular to the plane of (x, y). Let $\mathrm{X_1}, \mathrm{Y_1}, \mathrm{X_2}, \mathrm{Y_2}, \ldots$ be respectively the horizontal and vertical components of the actions of the joints at $\mathrm{A_1}, \mathrm{A_2}, \ldots$; then denoting time-differentiations by accents, the complete equations of motion are

$$m_1 (a_1^2 + k_1^2)\,\theta_1'' = -m_1 g a_1 \sin\theta_1 + 2a_1 (\mathrm{X_1} \cos\theta_1 - \mathrm{Y_1} \sin\theta_1); \tag{197}$$

$$\left.\begin{aligned} m_2 x_2'' &= -\mathrm{X_1} + \mathrm{X_2}, \qquad m_2 y_2'' = m_2 g - \mathrm{Y_1} + \mathrm{Y_2}, \\ m_2 k_2^2 \theta_2'' &= a_2 \{(\mathrm{X_1} + \mathrm{X_2}) \cos\theta_2 - (\mathrm{Y_1} + \mathrm{Y_2}) \sin\theta_2\}; \end{aligned}\right\} \tag{198}$$

$$\left.\begin{aligned} m_3 x_3'' &= -\mathrm{X_2} + \mathrm{X_3}, \qquad m_3 y_3'' = m_3 g - \mathrm{Y_2} + \mathrm{Y_3}, \\ m_3 k_3^2 \theta_3'' &= a_3 \{(\mathrm{X_2} + \mathrm{X_3}) \cos\theta_3 - (\mathrm{Y_2} + \mathrm{Y_3}) \sin\theta_3\}; \end{aligned}\right\} \tag{199}$$

. . . .

$$\left.\begin{aligned} m_n x_n'' &= -\mathrm{X}_{n-1}, \qquad m_n y_n'' = m_n g - \mathrm{Y}_{n-1}, \\ m_n k_n^2 \theta_n'' &= a_n \{\mathrm{X}_{n-1} \cos\theta_n - \mathrm{Y}_{n-1} \sin\theta_n\}; \end{aligned}\right\} \tag{200}$$

but the variable coordinates are subject to the following equations:

$$\left.\begin{aligned} x_2 &= 2a_1 \sin\theta_1 + a \sin\theta_2, \\ y_2 &= 2a_1 \cos\theta_1 + a_2 \cos\theta; \end{aligned}\right\} \tag{201}$$

$$\left.\begin{aligned} x_3 &= 2a_1 \sin\theta_1 + 2a_2 \sin\theta_2 + a_3 \sin\theta_3, \\ y_3 &= 2a_1 \cos\theta_1 + 2a_2 \cos\theta_2 + a_3 \cos\theta_3; \end{aligned}\right\} \tag{202}$$

. . . .

$$\left.\begin{aligned} x_n &= 2a_1 \sin\theta_1 + \ldots + 2a_{n-1} \sin\theta_{n-1} + a_n \sin\theta_n, \\ y_n &= 2a_1 \cos\theta_1 + \ldots + 2a_{n-1} \cos\theta_{n-1} + a_n \cos\theta_n. \end{aligned}\right\} \tag{203}$$

From these we have, omitting squares and higher powers of $\theta_1, \theta_2, \ldots, \theta_1', \theta_2', \ldots$,

$$\left.\begin{aligned} x_2'' &= \quad 2a_1\theta_1''+a_2\theta_2'', \\ y_1'' &= -2a_1\theta_1\theta_1''-a_2\theta_2\theta_2'', \end{aligned}\right\} \tag{204}$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

and substituting these several values in the first two of each of the preceding groups (198), (199), ..., we shall have equations involving the second t-differentials of the θ's only; from these we can eliminate the X's and the Y's, and thereby obtain a series of equations in terms of the θ's and their second t-differentials only.

Ex. 3. As an oscillating compound pendulum is a problem of considerable interest, I will take a particular case of two equal bars, so that $m_1 = m_2 = m$, say:

$$a_1 = a_2 = a; \qquad k_1^2 = k_2^2 = \frac{a^2}{3};$$

then
$$\frac{4am}{3}\theta_1'' = -mg\sin\theta_1 + 2(\mathrm{X}_1\cos\theta_1 - \mathrm{Y}_1\sin\theta_1); \tag{205}$$

$$\left.\begin{aligned} &mx_2'' = -\mathrm{X}_1; \quad my_2'' = mg-\mathrm{Y}_1, \\ &\frac{ma}{3}\theta_2'' = \mathrm{X}_1\cos\theta_2 - \mathrm{Y}_1\sin\theta_2; \end{aligned}\right\} \tag{206}$$

$$\left.\begin{aligned} x_2 &= 2a\sin\theta_1 + a\sin\theta_2, \\ y_2 &= 2a\cos\theta_1 + a\cos\theta_2; \end{aligned}\right\} \tag{207}$$

whence, omitting small quantities of higher powers,

$$\begin{aligned} x_2'' &= 2a\theta_1''+a\theta_2'', \\ y_2'' &= mg + ma(2\theta_1\theta_1''+\theta_2\theta_2''); \end{aligned}$$

and eliminating $\mathrm{X}_1, \mathrm{Y}_1, x_2'' y_2''$, we have

$$\left.\begin{aligned} \frac{16}{3}\theta_1''+2\theta_2''+\frac{3g}{a}\theta_1 &= 0, \\ 2\theta_1''+\frac{4}{3}\theta_2''+\frac{g}{a}\theta_2 &= 0; \end{aligned}\right\} \tag{208}$$

whence eliminating θ_2, we have

$$\frac{d^4\theta_1}{dt^4}+\frac{3g}{a}\frac{d^2\theta_1}{dt^2}+\frac{27g^2}{28a^2} = 0. \tag{209}$$

Now as the roots of the characteristic of this equation are all impossible, if we denote them by $\pm(-)^{\frac{1}{2}}r_1$, $\pm(-)^{\frac{1}{2}}r_2$, the solution of (209) is

$$\theta_1 = \mathrm{E}_1\cos(r_1t+\beta_1)+\mathrm{F}_1\cos(r_2t+\gamma_1), \tag{210}$$

where E_1, F_1, β_1, γ_1 are four constants introduced in integration, which are to be determined from initial or other circumstances.

If we had eliminated θ_1 from (208), we should have found a linear differential equation of the fourth order of the same form and coefficients as (209) in terms of θ_2, and integrating we should have

$$\theta_2 = E_2 \cos(r_1 t + \beta_2) + F_2 \cos(r_2 t + \gamma_2), \tag{211}$$

where E_2, F_2, β_2, γ_2 are constants introduced in integration, and which depend on initial or other circumstances. Thus the variables which determine the position of the bars consist of two circular functions, of which the periodic times are respectively $\frac{2\pi}{r_1}$ and $\frac{2\pi}{r_2}$; and since r_1 and r_2 have no common measure, what is the position of the beams at any time never recurs; but if r_1 and r_2 had been such as to have a common measure n, a given position of the beams will always recur at intervals $\frac{2\pi}{n}$.

Ex. 4. An uniform heavy rod of length $2a$ is suspended at one end from a fixed point by means of a string of length l, whose weight may be neglected. The rod is slightly displaced from its position of equilibrium; it is required to determine its small oscillations.

Let the point of suspension be taken for the origin; and let the horizontal plane through it be the plane of (x, y), and let the z-axis be taken positively downwards. Let $m =$ the mass of the rod; and at the time t let $T =$ the tension of the string, (x, y, z) be the place of the extremity of the string, (x', y', z') the place of the centre of gravity of the rod; (ξ, η, ζ) the place of any element dm of the rod, the distance of which from the upper extremity $= s$. Then the equations of motion of the centre of gravity of the rod are

$$\left.\begin{array}{cc} m\dfrac{d^2x'}{dt^2} = -T\dfrac{x}{l}, & m\dfrac{d^2y'}{dt^2} = -T\dfrac{y}{l}, \\ \multicolumn{2}{c}{m\dfrac{d^2z'}{dt^2} = mg - T\dfrac{z}{l};} \end{array}\right\} \tag{212}$$

but since the displacement of the string and beam is always small, x, y, x', y' are always small, and approximately, $z = l$, $z' = l + a$; consequently for the preceding equations we have their approximate values

$$\frac{d^2 x'}{dt^2} = -g\frac{x}{l}, \qquad \frac{d^2 y'}{dt^2} = -g\frac{y}{l}, \qquad \text{T} = mg. \tag{213}$$

The equation of moments relative to the x-axis is

$$\Sigma . m \left\{ \eta \frac{d^2 \zeta}{dt^2} - \zeta \frac{d^2 \eta}{dt^2} \right\} = \Sigma . m\, (\eta \text{Z} - \zeta \text{Y});$$

but approximately,

$$\frac{d^2 \zeta}{dt^2} = 0; \qquad \eta = y + (y' - y)\frac{s}{a}; \qquad \zeta = l + s;$$

so that we have

$$-\int_0^{2a} m \frac{l+s}{2a} \left\{ \frac{d^2 y}{dt^2} + \frac{d^2 (y'-y)}{dt^2} \frac{s}{a} \right\} ds = mgy';$$

$$\therefore \quad a\frac{d^2 y}{dt^2} - (4a + 3l)\frac{d^2 y'}{dt^2} = 3gy'. \tag{214}$$

Similarly the equation of moments relative to the y-axis is

$$a\frac{d^2 x}{dt^2} - (4a + 3l)\frac{d^2 x'}{dt^2} = 3gx'. \tag{215}$$

These two equations together with the first two of (213) are sufficient to determine the motion.

Eliminating y' between the second of (213) and (214), we have

$$\frac{d^4 y}{dt^4} + \frac{4a + 3l}{al} g \frac{d^2 y}{dt^2} + \frac{3g^2}{al} y = 0. \tag{216}$$

Now the four roots of the characteristic of this equation are evidently impossible; let them be $\pm (-)^{\frac{1}{2}} r_1$, $\pm (-)^{\frac{1}{2}} r_2$; then the solution of (216) is

$$y = \text{E}_1 \cos (r_1 t - a_1) + \text{F}_1 \cos (r_2 t - a_2);$$

where E_1, F_1, a_1, a_2 are constants depending on the initial or other circumstances of the beam and string. The other variables have the following values:

$$y' = \text{E}_1' \cos (r_1 t - a_1) + \text{F}_1' \cos (r_2 t - a_2),$$
$$x = \text{E}_2 \cos (r_1 t - \beta_1) + \text{F}_2 \cos (r_2 t - \beta_2),$$
$$x' = \text{E}_2' \cos (r_1 t - \beta_1) + \text{F}_2' \cos (r_2 t - \beta_2);$$

and each of these variables involves the same two circular functions, of which the periodic times are respectively $\frac{2\pi}{r_1}$ and $\frac{2\pi}{r_2}$, although the amplitudes of vibration and the commencement of the periodic times are different for each. All the undetermined variables can be found in terms of the initial circumstances of the rod and string.

Ex. 5. A body which has one point in it fixed is in motion

about an instantaneous axis, the angle of inclination of which to a stable principal axis of the body at the fixed point is always small; it is required to determine the motion when the body makes small oscillations about its mean position.

Let the principal axis of the body with which the instantaneous always nearly coincides be the ζ-axis, about which the moment is C; so that ω_1 and ω_2 are always small quantities, the products and powers of which above the second we shall omit in our approximations. Also ω_3 is nearly constant: we shall take n to represent its mean value, and shall replace ω_3 by n in small terms. As the motion of the instantaneous axis is small, L, M, N are also supposed to be small. Under these circumstances Euler's equations are

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{B})\,n\,\omega_2 &= \text{L}, \\ \text{B}\frac{d\omega_2}{dt} + (\text{A}-\text{C})\,n\,\omega_1 &= \text{M}, \\ \text{C}\frac{d\omega_3}{dt} \qquad\qquad &= \text{N}. \end{aligned}\right\} \quad (217)$$

Let us refer the motion to the axes of (x, y, z), fixed in space by means of the direction-cosines given in Art. 2. Let the mean position of the ζ-principal axis be the z-axis, so that the angle between the z- and the ζ-axis is always small; consequently c_3, which is the cosine of the angle contained between these axes, is always nearly equal to 1, and may be replaced by 1 in small terms. Hence also it follows, that c_1, c_2, a_3, b_3 are always small quantities. And since

$$\omega_z = a_3\omega_1 + b_3\omega_2 + c_3\omega_3,$$

we may replace ω_z by ω_3, that is, by n, in small terms. Now replacing ω_x and ω_y by their values given in (83) and (84), Art. 57, we have

$$\begin{aligned} \omega_1 &= a_1\omega_x + a_2\omega_y + a_3\omega_z \\ &= a_1\left\{a_2\frac{da_3}{dt} + b_2\frac{db_3}{dt} + c_2\frac{dc_3}{dt}\right\} \\ &\quad - a_2\left\{a_1\frac{da_3}{dt} + b_1\frac{db_3}{dt} + c_1\frac{dc_3}{dt}\right\} \\ &\quad + a_3 n \\ &= (a_1b_2 - a_2b_1)\frac{db_3}{dt} + (a_1c_2 - a_2c_1)\frac{dc_3}{dt} + a_3 n \\ &= c_3\frac{db_3}{dt} - b_3\frac{dc_3}{dt} + a_3 n; \end{aligned}$$

now $c_3 = 1$ in small terms, and its variation is so small that $\frac{dc_3}{dt}$ must be omitted;

$$\therefore \quad \omega_1 = \frac{db_3}{dt} + a_3 n. \tag{218}$$

Similarly,

$$\omega_2 = -\frac{da_3}{dt} + b_3 n. \tag{219}$$

Let us substitute these values in (217); and we have, omitting the subscript 3,

$$\left.\begin{aligned} \text{A}\frac{d^2 b}{dt^2} + (\text{A}+\text{B}-\text{C})n\frac{da}{dt} + n^2(\text{C}-\text{B})b &= \text{L}, \\ -\text{B}\frac{d^2 a}{dt^2} + (\text{A}+\text{B}-\text{C})n\frac{db}{dt} + n^2(\text{A}-\text{C})a &= \text{M}. \end{aligned}\right\} \tag{220}$$

L and M, which are the moments of the impressed couples whose axes are the ξ- and η-axis respectively, must be expressed in terms of a and b; as these however are small quantities, and as L and M both vanish in the state of equilibrium when $a = b = 0$, they are of the following forms;

$$\left.\begin{aligned} \text{L} &= pa + p'b, \\ \text{M} &= qa + q'b; \end{aligned}\right\} \tag{221}$$

so that finally the differential equations in terms of a and b are of the forms

$$\left.\begin{aligned} \frac{d^2 b}{dt^2} + \alpha\frac{da}{dt} + \beta^2 a + \beta'^2 b &= 0, \\ \frac{d^2 a}{dt^2} - \alpha\frac{db}{dt} + \gamma^2 a + \gamma'^2 b &= 0; \end{aligned}\right\} \tag{222}$$

which are two simultaneous differential equations* of the second order, and are integrable by the processes explained in Vol. II. Hereby a and b will be expressed in terms of t; and as they are the cosines of the angles contained between the fixed z-axis and the moving axes of ξ and η respectively, so will they determine the position of the three principal axes of the body at the time t; and (218) and (219) will give ω_1 and ω_2 in terms of t; and as $\omega_3 = n$, so will also ω be known; and the position of the instantaneous rotation-axis will be given.

378.] The equations (220) may be conveniently applied to the

* These equations are the same in *form* as those given in "An Elementary Treatise on the Dynamics of a System of Rigid Bodies," by E. J. Routh, M.A., Cambridge, 1860; equations A, p. 174. It is indeed to a study of that treatise that I owe the thought of transforming generally the first two of Euler's equations into (222) as their equivalents.

problem of the motion of a heavy body having a point fixed, at which two of the principal moments are equal, say, A = B, and the third principal moment C is greater than A, the centre of gravity being in the C-axis, at a distance h from the fixed point; and the angle between the C-axis and the vertical being always small.

Let the fixed point be the origin, and let the vertical line passing through it be the z-axis: and let h be the distance of the centre of gravity along the ζ-axis from the origin: then

$$\text{L} = mghb; \qquad \text{M} = -mgha; \qquad \text{N} = 0;$$

and (220) become

$$\left.\begin{aligned} \text{A}\frac{d^2 b}{dt^2} + (2\text{A}-\text{C})n\frac{da}{dt} + n^2(\text{C}-\text{A})b &= mghb, \\ \text{A}\frac{d^2 a}{dt^2} - (2\text{A}-\text{C})n\frac{db}{dt} + n^2(\text{C}-\text{A})a &= mgha; \end{aligned}\right\} \tag{223}$$

of which the form is

$$\left.\begin{aligned} \frac{d^2 b}{dt^2} + 2\alpha\frac{da}{dt} + \beta^2 b &= 0, \\ \frac{d^2 a}{dt^2} - 2\alpha\frac{db}{dt} + \beta^2 a &= 0: \end{aligned}\right\} \tag{224}$$

whence eliminating b, we have

$$\frac{d^4 a}{dt^4} + 2(2\alpha^2+\beta^2)\frac{d^2 a}{dt^2} + \beta^4 a = 0. \tag{225}$$

Now the characteristic of this equation is

$$r^4 + 2(2\alpha^2+\beta^2)r^2 + \beta^4 = 0. \tag{226}$$

Solving this, we have

$$\begin{aligned} r &= \pm(-)^{\frac{1}{2}}\{\alpha \pm (\alpha^2+\beta^2)^{\frac{1}{2}}\} \\ &= \pm(-)^{\frac{1}{2}}\left\{\frac{2\text{A}-\text{C}}{2\text{A}}n \pm \frac{(\text{C}^2 n^2 - 4\text{A}mgh)^{\frac{1}{2}}}{2\text{A}}\right\}; \end{aligned} \tag{227}$$

and thus we have three cases:

(1) If $\text{C}^2 n^2$ is greater than $4\text{A}mgh$; that is, if

$$n^2 \text{ is greater than } \frac{4\text{A}mgh}{\text{C}^2}, \tag{228}$$

the four values of r are impossible; let them be

$$\pm(-)^{\frac{1}{2}} r_1, \qquad \pm(-)^{\frac{1}{2}} r_2;$$

then the solution of (225) is

$$a = c_1\sin(r_1 t+\gamma_1) + c_2\sin(r_2 t+\gamma_2); \tag{229}$$

where c_1, c_2, γ_1, γ_2 are four constants, which are to be determined by the initial or other circumstances of the top. In this case the motion is stable, and the top makes oscillations about its mean place. Thus the stability of the motion depends on the inequality (228), which shews that the angular velocity of the top about its own axis must be greater than a certain assigned quantity.

(2) If $\mathrm{C}^2 n^2 = 4\mathrm{A} mgh$; then $r = \pm (-)^{\frac{1}{2}} \alpha$; and

$$a = (c_1 + c_2 t) \sin (\alpha t + \gamma); \tag{230}$$

where c_1, c_2, and γ are constants depending on the initial circumstances. This result shews generally, that a has periodical values; but that, as $c_1 + c_2 t$ increases with the time, its maximum values increase; and thus the motion of the top is still oscillatory, but in an increasing or decreasing orbit, according as c_2 is positive or negative.

(3) If $\mathrm{C}^2 n^2$ is less than $4\mathrm{A} mgh$, then

$$r = \pm \left\{ \pm \frac{(4\mathrm{A} mgh - \mathrm{C}^2 n^2)^{\frac{1}{2}}}{2\mathrm{A}} + (-)^{\frac{1}{2}} \frac{2\mathrm{A} - \mathrm{C}}{2\mathrm{A}} n \right\}$$

$$= \pm \{ \pm \rho + (-)^{\frac{1}{2}} \sigma \};$$

$$\therefore \quad a = c_1 e^{\rho t} \sin (\sigma t + \gamma_1) + c_2 e^{-\rho t} \sin (\sigma t + \gamma_2); \tag{231}$$

where c_1, c_2, γ_1, and γ_2 are constants to be determined by the initial circumstances. This form shews that a increases without limit as t increases without limit; but a is the cosine of an angle, is always small, and cannot exceed 1; consequently this form soon ceases to represent the motion of the top.

If a is eliminated from (224), we shall have an equation in b of the same form and with the same coefficients as (225): hence the solution will be of the same form as that of a, but the constants of integration will be different.

This problem is the same as that already solved in Arts. 313–327, where further elucidation has been given. It is also that of the motion of a top which rotates, with its axis nearly perpendicular, about a fixed peg.

379.] Another subject, which has arisen in the course of the preceding examples, requires a few words of explanation. We have frequently met with a resistance or a force arising from friction; and we have assumed the force to act in a direction contrary to that of the motion: we have spoken too of a friction of *rolling* as distinct from a friction of *sliding*. This distinction,

as well as the dynamical effects of the two kinds, we proceed to explain more fully; and we must begin with certain laws which have been experimentally observed; for although these are *a priori* reasonable, yet they depend upon the physical constitution of matter; and our theoretical knowledge of molecular physics is, as yet, too uncertain, so that any proof derived from that source should supersede proof drawn from observation and experiment.

Friction of sliding has been considered statically in Section 10, Chapter III, of Vol. III, and the three laws therein stated are sufficient for the dynamical effects which we have now to consider. I shall take the case of a heavy body placed on a rough inclined plane, or on a rough curved surface, and sliding on it, so that its velocity is retarded by friction.

From the laws just alluded to it appears, (1) that, so long as the pressure is the same, the friction is independent of the area of the surfaces in contact; (2) that the friction varies as the normal pressure exerted by the heavy body against the surface on which it moves. It also appears that the body does not begin to move unless the inclination of the plane, or if it is on a curved surface, of the tangent plane of the point at which it is placed, is equal to or exceeds a certain angle called "the angle of repose," and that, if $\mu =$ the coefficient of friction, this angle $= \tan^{-1}\mu$. It appears also from law III of the section above cited, that the friction is independent of the velocity of sliding. I propose now to apply these laws to some problems.

Ex. 1. A particle is placed on a rough inclined plane, the angle of inclination of which to the horizon is greater than the angle of repose; it is required to determine the motion.

Let $a =$ the angle of inclination of the plane; $m =$ the mass of the particle; $\text{R} =$ the normal pressure on the plane; $\text{F} =$ the retarding force of friction; $\mu =$ the coefficient of friction. Let $x =$ the distance along the plane through which the particle has moved in the time t; then

$$\text{R} = mg\cos a, \qquad \text{F} = \mu\text{R};$$

$$\therefore \; m\frac{d^2x}{dt^2} = mg\sin a - \text{F} = mg\{\sin a - \mu\cos a\};$$

from which equation the motion may be determined.

Ex. 2. A particle slides down within a rough circular cylinder whose axis is horizontal: determine the equation of motion.

Let $\theta =$ the angular distance from the lowest point at the time t; then the equation of motion is evidently

$$-a\frac{d^2\theta}{dt^2} = g\{\sin\theta - \mu\cos\theta\};$$

$$\therefore \quad -a\frac{d\theta^2}{dt^2} = 2g\{\cos\alpha - \cos\theta + \mu(\sin\alpha - \sin\theta)\};$$

where α is the value of θ, when the particle is at rest.

380.] In these cases the friction has been that of sliding only; and although the inclination of the plane has, in the latter example, been less than the angle of repose, yet, by reason of the previously acquired momentum, the particle has still continued to move. In cases however of bodies moving in contact with rough surfaces there may be friction of rolling as well as friction of sliding. If a cube is placed on an inclined plane whose inclination to the horizon is less than 45°, the cube will not fall over, and will slide down if the angle of repose is less than that of the inclination of the plane. If however a heavy sphere is placed on a rough inclined plane it will always roll; it will moreover slide as well as roll if the angle of inclination of the plane is greater than its angle of repose. Now, if the sphere rolls only, the process taken in Ex. 1, Art. 372, determines the motion; whereas, if it slides as well as rolls, other terms are required in the equations.

We may form a tolerably precise notion of the friction of rolling by imagining a heavy cylinder or wheel rolling on a horizontal plane. By reason of the compressibility of the matter in contact, the cylinder and the plane mutually penetrate each other; and hence arise reactions, acting on the cylinder in a direction contrary to that in which it is moving, and which act as obstacles to its rolling. The friction of rolling is measured by the horizontal force which it is necessary to apply to the axis of the wheel to maintain an uniform velocity of translation of the cylinder. Experiments were made in this subject by Coulomb; and he discovered the following law: the force of rolling friction for a heavy cylinder and a given plane varies directly as the pressure, and inversely as the radius of the cylinder. Thus, if $R =$ the pressure of the cylinder on the plane, $r =$ the radius of the cylinder, $F =$ the rolling friction,

$$F = \nu\frac{R}{r}, \tag{232}$$

where ν is a constant, called the coefficient of rolling friction, which depends on the nature of the surfaces in contact.

Rolling friction as a retarding force is much less than sliding friction, and may be neglected when the latter acts.

In mechanical problems the difficulty frequently is to determine whether a body will slide and roll, or only roll: now let $F =$ the friction, and $R =$ the normal pressure; then if the ratio of F to R, which is equal to μ, is equal to, or greater than, $\tan\alpha$, the body only rolls; in which case a geometrical condition will exist, which may take the form of a relation between the space of translation described by the centre of gravity and that due to the rotation about the instantaneous axis passing through the centre of gravity: the use of this relation is evident in the examples of Art. 372. If however the ratio of F to R is less than $\tan\alpha$, the body will slide as well as roll, and the geometrical relation just alluded to ceases to hold good. The method of solution then to be applied will be evident from the following examples:

381.] Ex. 1. Let us take the following simple case. An angular velocity Ω is given to a heavy sphere about a horizontal axis, which is free from all constraint, and the sphere is placed on a rough horizontal plane, and all restraint is removed. It is required to determine the subsequent motion of the sphere.

So soon as the sphere is brought into contact with the plane, inasmuch as the centre of the sphere is not in motion at that instant, a slipping arises between the sphere and the plane, its line being at right angles to the horizontal axis of rotation of the sphere; by this slipping a friction is caused and energy is withdrawn from the sphere, but the centre of the sphere moves with a horizontal velocity along a line parallel to the line of the sliding friction, so that the initial kinetic energy of the sphere becomes broken up into (1) the kinetic energy of translation of the mass of the sphere collected at its centre; (2) the kinetic energy of rotation of the sphere about the horizontal rotation-axis; (3) the kinetic energy which has been withdrawn by the friction; this is evident also from the following equation of kinetic energy. This slipping continues until the horizontal velocity of the mass-centre and the rotation about the horizontal axis are so related, that if v denotes the former and ω the angular velocity of the latter, a being the radius of the sphere,

$v = a\omega$, and the friction of sliding then ceases; but as the plane is rough a friction of rolling comes into action, in which however if no kinetic energy is withdrawn, the sphere continues to roll with uniform velocities of translation and of rotation. Hence these two parts of the motion require distinct consideration; the former being that in which sliding friction operates, and the latter that in which there is only rolling friction.

Having regard to the laws of sliding friction which are given in Art. 118, Vol. III, and especially to I and III, the equations in the former part of the motion are as follows: R being the pressure on the plane, and F being friction;

$$R = mg; \quad F = \mu R = \mu mg.$$

Let x be the horizontal distance through which the centre of the sphere has moved in the time t; let Ω be the initial angular velocity of the sphere; and ω the angular velocity at the time t; so that if $d\theta$ is the angle through which the sphere revolves in dt, $d\theta = \omega\, dt$. The effect of F is to increase x and to diminish θ;

$$\therefore \quad m\frac{d^2x}{dt^2} = F = \mu mg;$$

$$m\frac{2a^2}{5}\frac{d^2\theta}{dt^2} = -aF = -\mu amg.$$

Thus the force acting on the sphere is of uniform acceleration. The equation of kinetic energy is

$$m\frac{dx^2}{dt^2} + \frac{2ma^2}{5}\frac{d\theta^2}{dt^2} + 2F(a\theta - x) = \frac{2ma^2}{5}\Omega^2; \qquad (233)$$

the right-hand being the initial kinetic energy, and the left-hand member being the energy at the time t, the first two terms being kinetic, and the last being that which is withdrawn by sliding friction and transmuted into some other form. So long as there is sliding friction $a\,d\theta$ is greater than dx, and so according to the above initial conditions $a\theta$ is greater than x.

Also from the equations we have

$$\frac{dx}{dt} = \mu g t; \qquad x = \frac{\mu g t^2}{2};$$

$$\frac{d\theta}{dt} = -\frac{5\mu g t}{2a} + \Omega; \qquad \theta = -\frac{5\mu g t^2}{4a} + \Omega t;$$

which give the distance through which the centre of the sphere moves in a given time t, and also the angle through which it has revolved.

Now as soon as x has so far increased and the angular velocity has so far decreased, that $dx = a d\theta$, sliding friction ceases, and rolling friction comes into operation, and the relation $dx = a d\theta$ holds true throughout the subsequent motion. If V_1 is the velocity of the mass-centre, and Ω_1 is the angular velocity at that instant, and e_1 is the excess of the distance through which the initial point of contact in the sphere has travelled over that through which the point of contact has moved, then from the preceding equations,

$$\text{V}_1^2 = a^2 \Omega_1^2 = \frac{2a^2}{7}\Omega^2 - \frac{10}{7}\mu g e_1.$$

Thus the velocities of translation of the mass-centre, and of rotation about a horizontal axis through it are assigned at the instant when sliding friction ceases. Also when $dx = a d\theta$,

$$t = \frac{2a\Omega}{7\mu g},$$

which assigns the time when sliding friction ceases.

If F_1 is the rolling friction, the equations of motion become

$$m\frac{d^2 x}{dt} = -\text{F}_1,$$

$$m\frac{2a^2}{5}\frac{d^2\theta}{dt} = a\text{F}_1;$$

$$\therefore \quad m\frac{dx^2}{dt} + m\frac{2a^2}{5}\frac{d\theta^2}{dt^2} = m\text{V}_1^2 + m\frac{2a^2}{5}\Omega_1^2,$$

and as $dx = a d\theta$, $\frac{dx}{dt}$ and $\frac{d\theta}{dt}$ are constant, and the sphere rolls with a constant angular velocity.

Ex. 2. A heavy sphere moves down a rough inclined plane, whose angle of inclination to the horizon is greater than that of repose; it is required to determine the motion.

Let $\alpha =$ the angle of inclination; $\mu =$ the coefficient of sliding friction; $\text{F} =$ the sliding friction; $\text{R} =$ the normal pressure; the equations of motion are as follows:

$$\text{R} = mg\cos\alpha; \quad \text{F} = \mu\text{R} = \mu mg\cos\alpha;$$

$$m\frac{d^2 s}{dt^2} = mg\sin\alpha - \text{F} = mg(\sin\alpha - \mu\cos\alpha),$$

$$m\frac{2a^2}{5}\frac{d^2\theta}{dt^2} = a\text{F} = a\mu mg\cos\alpha;$$

these being the equations so long as sliding friction continues,

and giving results similar to those of the preceding example. Thus the sliding ceases and the sphere only rolls when $ds = a\,d\theta$; in which case

$$\mu = \frac{2}{7}\tan a;$$

consequently the sphere will slide and roll, or will roll only, according as $\tan a$ is greater or not greater than $\frac{7\mu}{2}$.

Ex. 3. If the body moving down the plane is a circular cylinder of radius $= a$, with its axis horizontal; then

$$3\mu = \tan a;$$

and the body will slide and roll, or roll only, according as a is greater or not greater than $\tan^{-1} 3\mu$.

Ex. 4. A heavy body whose bounding surface is a circular cylinder, but whose centre of gravity is not in the axis of the surface, makes small titubations on a rough horizontal plane. Determine the limits of the angle of titubation so that it may not slide on the plane.

Take the figure and symbols of Ex. 8, Art. 372; and let $\mu =$ the coefficient of sliding friction between the rocking body and the plane. Then, as the angle through which the body rocks is small, powers of θ higher than the second may be neglected. Hence we have

$$\frac{d^2 x}{dt^2} = (a-c)\frac{d^2\theta}{dt^2};$$

$$\therefore \quad -k^2 \mathrm{F} = (a-c)^2 \mathrm{F} - \mathrm{R}\,c\,(a-c)\,\theta;$$

$$\therefore \quad \frac{\mathrm{F}}{\mathrm{R}} = \mu = \frac{c\,(a-c)\,\theta}{(a-c)^2 + k^2};$$

and consequently, if the body rolls and does not slide, θ must not be greater than $\frac{(a-c)^2+k^2}{c\,(a-c)}\,\mu$.

382.] The following examples of motion of rigid bodies, free and constrained, involve various modes of application of preceding principles, and are inserted in illustration, as well as because many of them are in themselves of considerable interest.

A heavy homogeneous solid ellipsoid is struck by a blow whose momentum is Q, in a line parallel to one of its principal axes, and subsequently moves freely under the action of its weight; it is required to determine the motion.

Let $\mathrm{M} =$ the mass of the ellipsoid; and let the equation to

the bounding surface relative to the centre as origin, and its three principal axes as coordinate axes, be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1;$$

and let us suppose the line of blow to be parallel to the z-axis, and to intersect the plane of (x, y) in the point (x_0, y_0).

The centre of gravity, which is the centre of the ellipsoid, will move as if it were a particle of mass $= \mathrm{M}$; and consequently its path is a parabola; and if V is the initial velocity,

$$\mathrm{V}=\frac{\mathrm{Q}}{\mathrm{M}};$$

and its initial line is that of the blow. Thus all the elements of its path are known.

The ellipsoid also rotates under the action of Q, as if the centre were a fixed point. Consequently, if A and B are the principal moments of inertia of the solid ellipsoid about the x- and y-axes respectively, the initial instantaneous axis through the centre is, see (73), Art. 353,

$$\frac{x x_0}{\mathrm{B}}+\frac{y y_0}{\mathrm{A}}=0; \qquad (234)$$

and if Ω is the initial angular velocity, see (72), Art. 353,

$$\Omega=\mathrm{Q}\left\{\frac{y_0^{\,2}}{\mathrm{A}^2}+\frac{x_0^{\,2}}{\mathrm{B}^2}\right\}^{\frac{1}{2}}; \qquad (235)$$

the combined effect of these two motions is indeed a rotation about an initial spontaneous axis, whose equation relative to the moving ellipsoid is

$$\frac{x x_0}{\mathrm{B}}+\frac{y y_0}{\mathrm{A}}+\frac{1}{\mathrm{M}}=0. \qquad (236)$$

Since the initial rotation-axis, given by (234), is not a principal axis of the body it is not a permanent axis; consequently it continually moves both in the body and in space; that determined above being only its initial position. Its motion will be determined by Euler's three equations, simplified by the condition, that momenta are impressed only at the origin; so that we have

$$\left.\begin{aligned}\mathrm{A}\frac{d\omega_1}{dt}+(\mathrm{C}-\mathrm{B})\,\omega_2\omega_3=0,\\ \mathrm{B}\frac{d\omega_2}{dt}+(\mathrm{A}-\mathrm{C})\,\omega_3\omega_1=0,\\ \mathrm{C}\frac{d\omega_3}{dt}+(\mathrm{B}-\mathrm{A})\,\omega_1\omega_2=0;\end{aligned}\right\} \qquad (237)$$

the initial values of $\omega_1, \omega_2, \omega_3$ being respectively $\frac{\mathrm{Q}y_0}{\mathrm{A}}$, $-\frac{\mathrm{Q}x_0}{\mathrm{B}}$,

and 0. These equations however have been so fully discussed in the preceding Chapter that it is unnecessary to say more on the subject.

If either x_0 or $y_0 = 0$, the initial instantaneous axis is a principal axis, and therefore is a permanent axis; and the ellipsoid during its motion in space uniformly revolves about this axis. Thus, if $x_0 = 0$, the line of the blow Q is in the plane of (y, z) and is parallel to the z-axis, and the x-axis is the instantaneous rotation-axis, which is also the permanent rotation-axis; and the permanent angular velocity of the body about it

$$= \frac{\text{Q} y_0}{\text{A}} = \frac{5\,\text{Q} y_0}{\text{M}(b^2+c^2)}.$$

383.] A right cone is placed with its slant side on a perfectly rough inclined plane, and rolls on it by the action of its weight; it is required to determine its motion.

Let $2\alpha =$ the vertical angle of the cone; $\text{M} =$ its mass; $a =$ its height; and let $\beta =$ the inclination of the plane to the horizon.

The forces acting on the cone are, its weight, the rolling friction of the cone on the inclined plane, and the normal reaction of the plane. As the plane is perfectly rough, and the cone rolls on its convex surface without sliding, the place of the vertex of the cone is always the same, and the motion is that of a rotating body which has a fixed point in its axis. We shall therefore investigate it by means of Euler's three equations. Now the line of contact of the cone with the plane is evidently always the instantaneous rotation-axis; and as the force of rolling friction, as well as the normal reaction of the plane, acts through this line, they produce relatively to it no angular velocity: it will be convenient to derive from Euler's equations the equation of rotation relative to this line.

Let C be the principal moment of inertia of the cone relative to its own axis; and let A be that relative to an axis perpendicular to the axis of the cone and passing through the vertex; so that Euler's equations are

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{A})\,\omega_2\omega_3 &= \text{L}, \\ \text{A}\frac{d\omega_2}{dt} - (\text{C}-\text{A})\,\omega_3\omega_1 &= \text{M}, \\ \text{C}\frac{d\omega_3}{dt} \qquad\qquad &= \text{N}. \end{aligned}\right\} \qquad (238)$$

Let us suppose the x-axis, to which A corresponds, to be initially in the inclined plane, and θ to be the angle through which this axis, and consequently the cone, rotates in the time t. Let ω be the instantaneous angular velocity at the time t; then, relative to the three principal axes of the body, the direction-cosines of the instantaneous axis are evidently $\sin\alpha\cos\theta$, $\sin\alpha\sin\theta$, and $\cos\alpha$; so that

$$\omega_1 = \omega\sin\alpha\cos\theta, \qquad \omega_2 = \omega\sin\alpha\sin\theta, \qquad \omega_3 = \omega\cos\alpha; \tag{239}$$

and if G is the moment of the impressed forces relative to the instantaneous rotation-axis,

$$\text{G} = \text{L}\sin\alpha\cos\theta + \text{M}\sin\alpha\sin\theta + \text{N}\cos\alpha. \tag{240}$$

Also

$$\omega = \frac{d\theta}{dt}. \tag{241}$$

Now, substituting these values in (240), we have

$$\{\text{A}(\sin\alpha)^2 + \text{C}(\cos\alpha)^2\}\frac{d^2\theta}{dt^2} = \text{G}.$$

Let ϕ be the angle at the time t contained between the line of contact of the cone with the plane, and a straight line on the plane perpendicular to a horizontal line. Then, as the cone rolls on the plane, evidently

$$-a\,d\phi = a\tan\alpha\,d\theta; \tag{242}$$

and as the weight, which acts at the centre of gravity of the cone, is the only force which impresses angular velocity on the body relative to the instantaneous rotation-axis, and tends to increase θ,

$$\text{G} = \text{M}g\,\frac{3a}{4}\sin\alpha\sin\beta\sin\phi. \tag{243}$$

Also $\text{A}(\sin\alpha)^2 + \text{C}(\cos\alpha)^2 = \frac{3\text{M}}{20}\{6 + (\tan\alpha)^2\}a^2(\sin\alpha)^2$;

so that

$$a\frac{d^2\phi}{dt^2} = -5g\,\frac{(\cos\alpha)^2\sin\beta}{(\sin\alpha)^3\{6+(\tan\alpha)^2\}}\sin\phi; \tag{244}$$

which equation determines the motion.

If $\phi = \phi_0$ when the cone is at rest, then, integrating (244), we have

$$a\left(\frac{d\phi}{dt}\right)^2 = 10g\,\frac{(\cos\alpha)^2\sin\beta}{(\sin\alpha)^3\{6+(\tan\alpha)^2\}}\{\cos\phi - \cos\phi_0\}. \tag{245}$$

If the cone makes small oscillations on the plane, the time of an oscillation

$$= \pi\left\{\frac{a(\sin\alpha)^3\{6+(\tan\alpha)^2\}}{5g(\cos\alpha)^2\sin\beta}\right\}^{\frac{1}{2}}.$$

If the cone is fixed by its vertex to a point in a rough perpen-

dicular wall, and rolls on the wall, then the above equations determine the motion, when $\beta = 90°$.

384.] Determine the motion of a top whose apex moves on a smooth horizontal plane.

The motion of a top has already come twice into consideration; viz., in Arts. 318–327, where we have generally investigated the motion of a heavy rigid body with a point fixed, and having two equal principal moments of inertia relative to that point; and also again in Art. 378, where the small motions of it about a mean position have been investigated in illustration of the general law of small oscillations. In both these cases the apex of the top has been assumed to be a fixed point; and this condition is approximately satisfied when the top moves on a perfectly rough plane. If however the plane is smooth, the apex moves in the plane; and we propose now to investigate its motion, and the motion of the top.

I shall assume the centre of gravity to be in the geometrical axis of the top, and at a distance equal to h from the apex or peg of the top. And I shall also use the same symbols in the same significance as in Art. 318, except that the origin of the several axes to which the rotation is referred will be taken at the centre of gravity.

As the plane is perfectly smooth, the only forces acting on the body are the vertical reaction of the plane $= \text{R}$ (say), and the weight of the top; so that if (ξ, η, ζ) is the place of the centre of gravity relative to a system of axes fixed in space, of which that of ζ is vertical, and those of ξ and η are in the given plane, then

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} = \frac{d^2\eta}{dt^2} &= 0, \\ m\frac{d^2\zeta}{dt^2} &= \text{R} - mg; \end{aligned}\right\} \tag{246}$$

from the first two of which it is plain that in horizontal motion the centre of gravity either remains at rest or moves uniformly in a rectilineal path, the elements of which depend on the initial impulsion.

Also, since $\zeta = h\cos\theta$,

$$\text{R} = m\left\{\frac{d^2.h\cos\theta}{dt^2} + g\right\}. \tag{247}$$

Next let us consider the motion of the top relative to the centre of gravity. The only force which produces a moment

about that point is the pressure of the plane at the apex, of which the value is given in (247); and we have, as in Art. 318,

$$\left.\begin{aligned} \text{L} &= \text{R}h\sin\theta\cos\phi, \\ \text{M} &= -\text{R}h\sin\theta\sin\phi, \\ \text{N} &= 0. \end{aligned}\right\} \tag{248}$$

Now $\text{A} = \text{B}$; and consequently the third of Euler's equations, by reason of the third of (248), gives

$$\omega_3 = \text{a constant} = n \text{ (say)}; \tag{249}$$

and thus the first two of Euler's equations are

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} + (\text{C}-\text{A})n\omega_2 &= \text{R}h\sin\theta\cos\phi, \\ \text{A}\frac{d\omega_2}{dt} - (\text{C}-\text{A})n\omega_1 &= -\text{R}h\sin\theta\sin\phi. \end{aligned}\right\} \tag{250}$$

Let us suppose the whole initial angular velocity of the body to be about the axis of the top, and to be n; so that initially $\omega_1 = \omega_2 = 0$, and consequently $\frac{d\theta}{dt} = \frac{d\psi}{dt} = 0$; and let us suppose the initial values of θ to be θ_0, and of ϕ and ψ to be zero. From (250) we have, as in Art. 318,

$$\begin{aligned} \text{A}\left\{\omega_1\frac{d\omega_1}{dt} + \omega_2\frac{d\omega_2}{dt}\right\} &= \text{R}h\sin\theta\,\{\omega_1\cos\phi - \omega_2\sin\phi\} \\ &= \text{R}h\sin\theta\frac{d\theta}{dt} \\ &= mh\sin\theta\frac{d\theta}{dt}\left\{g - h\cos\theta\left(\frac{d\theta}{dt}\right)^2 - h\sin\theta\frac{d^2\theta}{dt^2}\right\}; \end{aligned}$$

$$\therefore\quad \text{A}(\omega_1{}^2 + \omega_2{}^2) = 2mhg(\cos\theta_0 - \cos\theta) - mh^2(\sin\theta)^2\left(\frac{d\theta}{dt}\right)^2;$$

therefore

$$\{\text{A} + mh^2(\sin\theta)^2\}\left(\frac{d\theta}{dt}\right)^2 + \text{A}(\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2 = 2mhg(\cos\theta_0 - \cos\theta). \tag{251}$$

Also, multiplying the first of (250) by $\sin\phi$, and the second by $\cos\phi$, adding, and integrating, as in Art. 318, we have

$$\text{A}(\sin\theta)^2\frac{d\psi}{dt} = \text{C}n(\cos\theta_0 - \cos\theta); \tag{252}$$

and eliminating $\frac{d\psi}{dt}$ by means of (251) and (252), we have

$$dt = \pm\frac{\{\text{A}^2 + mh^2\text{A}(\sin\theta)^2\}^{\frac{1}{2}}\sin\theta\,d\theta}{\{\cos\theta_0 - \cos\theta\}^{\frac{1}{2}}\{2mhg\text{A}(\sin\theta)^2 - \text{C}^2n^2(\cos\theta_0 - \cos\theta)\}^{\frac{1}{2}}}. \tag{253}$$

In these equations the following results are implied. Since the left-hand member of (251) is necessarily positive, the right-hand

member is also positive; so that θ is never less than θ_0: thus the angle at which the axis of the top is inclined to the vertical is never less than its initial value, and θ increases until it reaches a value, say θ_1, at which the second radical in the denominator of (253) vanishes; then $\frac{d\theta}{dt} = 0$, and the inclination of the axis to the vertical is a maximum; θ_1 is always less than π, because the expression which determines it is positive when $\theta = \theta_0$, and is negative when $\theta = \pi$: also the time in which the value of θ passes from θ_0 to θ_1 is finite, as we have proved in Art. 319. Thus the axis of the top makes isochronal oscillations in a vertical plane, as that plane revolves about the vertical axis of z.

That vertical plane however does not revolve uniformly; in other words, its precessional velocity, which $= \frac{d\psi}{dt}$, is not constant; (252) shews this.

According as n is positive or negative, so is the precession direct or retrograde; that is, the line of intersection of the equatorial plane of the top with the horizontal plane revolves in the same direction as the top rotates.

And the variations of the precessional velocity are periodic, having the same period as those of the inclination of the axis to the vertical; now the precessional velocity vanishes when $\theta = \theta_0$, and becomes a maximum when $\theta = \theta_1$; and continues to make these periodical oscillations. This is explained at greater length in Art. 320. Thus, if the centre of gravity of the top does not move, the apex of the top describes on the horizontal plane the curve delineated in Fig. 35, where the radius of the interior and exterior circles are respectively $h \sin \theta_0$, and $h \sin \theta_1$; and where the arcs of the path respectively touch the exterior circle when $\frac{d\psi}{dt}$ is a maximum; and meet the interior at right angles when $\frac{d\psi}{dt} = 0$.

It is evident, by the principle of vis viva, that the angular velocity of the top is a maximum when $\theta = \theta_1$, and is a minimum when $\theta = \theta_0$.

If n is very large, so that θ_1 is very little greater than θ_0, the values of θ are confined within very small limits. In this case we can, as in Art. 322, integrate (253) approximately, and

obtain results which give an accurate representation of the motion of the top.

385.] On the motion of a heavy homogeneous spherical ball, (an ivory billiard ball,) on a rough horizontal table.

I shall take M to be the mass of the sphere, a to be its radius, and A to be the moment of inertia about a diameter; so that $5\text{A} = 2a^2\text{M}$.

I shall suppose the ball to be put into motion initially by means of a blow, of which the intensity, line of action, and point of application are known; all these circumstances being given in billiards by the stroke of the cue. Thus, the initial velocity of translation of the centre of gravity, and the angular velocity relative to the initial rotation-axis, will be known. During the subsequent motion, the ball will both roll and slide, so that retarding forces of rolling and sliding friction will be in action on it. That of sliding friction acts at the point of contact of the ball with the table, and in the line along which the point of contact slides on the table. That of rolling friction acts in the line along which at the time the centre of gravity is moving. This latter friction however is, in ordinary cases, only a very small fraction of the former, and may consequently be neglected. As by the roughness of the table the sliding motion of the ball is diminished much more rapidly than the rolling motion, so the sliding motion soon ceases, and the ball only rolls. The following equations will determine with accuracy the time and the place at which this cessation of sliding takes place.

Let the plane in which the centre of the ball moves be that of (x, y); this plane is consequently parallel to that of the table, and is horizontal; let (x, y) be the place of the centre of the ball at the time t; so that at that time $(x, y, -a)$ is the place of the point of contact. Let F be the force of sliding friction which acts at the point of contact, and let β be the angle at which its line of action is inclined to the axis of x. Let R $(= \text{M}g)$ be the pressure of the ball on the plane, and let $\mu =$ the coefficient of sliding friction; so that $\text{F} = \mu\text{R} = \mu\text{M}g$.

The equations of motion of the centre of gravity are

$$\left.\begin{aligned} \text{M}\frac{d^2x}{dt^2} &= -\text{F}\cos\beta, \qquad \text{M}\frac{d^2y}{dt^2} = -\text{F}\sin\beta, \\ \text{M}\frac{d^2z}{dt^2} &= -\text{M}g + \text{R} = 0. \end{aligned}\right\} \qquad (254)$$

Let us consider the rotation in reference to a system of axes originating at the centre of gravity, and parallel to the fixed axes of (x, y, z); then we have

$$\left.\begin{aligned} \text{A}\frac{d\omega_1}{dt} &= -a\text{F}\sin\beta = a\text{M}\frac{d^2y}{dt^2}, \\ \text{A}\frac{d\omega_2}{dt} &= a\text{F}\cos\beta = -a\text{M}\frac{d^2x}{dt^2}; \end{aligned}\right\} \quad (255)$$

$$\text{A}\frac{d\omega_3}{dt} = 0; \quad (256)$$

$$\therefore \left.\begin{aligned} \text{A}(\omega_1 - \Omega_1) &= a\text{M}\left(\frac{dy}{dt} - v_0\right), \\ \text{A}(\omega_2 - \Omega_2) &= -a\text{M}\left(\frac{dx}{dt} - u_0\right), \\ \omega_3 - \Omega_3 &= 0; \end{aligned}\right\} \quad (257)$$

where u_0, v_0 are the axial components of the initial velocities of the centre of gravity, and Ω_1, Ω_2, Ω_3 are the initial angular velocities about three axes originating at the centre of gravity and parallel to the fixed axes.

These equations connect the instantaneous angular velocity with the velocity of the centre of gravity of the ball. Thus ω_3 is constant, and ω_1 and ω_2 only vary when $\frac{dx}{dt}$ and $\frac{dy}{dt}$ vary; that is, when F acts. And therefore if the centre of gravity moves uniformly in a straight line, the angular velocity of the ball and the direction of the rotation-axis do not vary, and there is no sliding friction; therefore, from (257),

$$\omega_1 - \Omega_1 = \frac{5}{2a}\left(\frac{dy}{dt} - v_0\right), \qquad \omega_2 - \Omega_2 = -\frac{5}{2a}\left(\frac{dx}{dt} - u_0\right). \quad (258)$$

386.] By means of these we can determine the path of the centre of gravity of the ball, so long as the ball continues to slide. The line of action of F, as we have said, is that of the motion of the point of contact. Now the projections on the x- and y-axes respectively of space described by this point in the time dt are $dx - a\omega_2 dt$, and $dy + a\omega_1 dt$; and substituting for ω_1 and ω_2 from (258) we have

$$\left.\begin{aligned} dx - a\omega_2 dt &= \frac{7\,dx}{2} - \left(a\Omega_2 + \frac{5\,u_0}{2}\right)dt, \\ dy + a\omega_1 dt &= \frac{7\,dy}{2} + \left(a\Omega_1 - \frac{5\,v_0}{2}\right)dt; \end{aligned}\right\} \quad (259)$$

but these are proportional to $\cos\beta$ and to $\sin\beta$, which enter into (254). To simplify these expressions however let

$$u_1 = \frac{2}{7}\left(\frac{5u_0}{2} + a\Omega_2\right), \qquad v_1 = \frac{2}{7}\left(\frac{5v_0}{2} - a\Omega_1\right), \tag{260}$$

so that

$$u_0 - u_1 = \frac{2}{7}(u_0 - a\Omega_2), \qquad v_0 - v_1 = \frac{2}{7}(v_0 + a\Omega_1);$$

then

$$dx - a\omega_2 dt = \frac{7}{2}(dx - u_1 dt), \quad dy + a\omega_1 dt = \frac{7}{2}(dy - v_1 dt); \tag{261}$$

and consequently from (254),

$$\frac{\frac{d^2x}{dt^2}}{\frac{dx}{dt} - u_1} = \frac{\frac{d^2y}{dt^2}}{\frac{dy}{dt} - v_1}; \tag{262}$$

and integrating,

$$\frac{\frac{dx}{dt} - u_1}{u_0 - u_1} = \frac{\frac{dy}{dt} - v_1}{v_0 - v_1};$$

hence also we have

$$\frac{\frac{d^2x}{dt^2}}{u_0 - u_1} = \frac{\frac{d^2y}{dt^2}}{v_0 - v_1}; \tag{263}$$

and consequently,

$$\frac{\cos\beta}{u_0 - u_1} = \frac{\sin\beta}{v_0 - v_1}; \tag{264}$$

and thus not only is F, the force of friction which retards the ball, constant in magnitude, being equal to $\mu M g$, but the line of action of it has a constant direction. And therefore the centre of gravity of the ball describes a parabolic path, like a heavy projectile, the axis of the parabola being parallel to the line of action of the constant force of friction.

And β is the angle at which the line of action of F is inclined to the axis of x; and

$$\tan\beta = \frac{v_0 - v_1}{u_0 - u_1} = \frac{v_0 + a\Omega_1}{u_0 - a\Omega_2}. \tag{265}$$

Now u_1 and v_1 are the axial components of the velocity of the centre of gravity of the ball at the instant when it ceases to slide, for then

$$\frac{dx}{dt} = a\omega_2 = u_1, \qquad \frac{dy}{dt} = -a\omega_1 = v_1;$$

the last terms of these equations following by reason of (261).

This is also evident from (262); for when the ball only rolls, F $= 0$; and consequently

$$\frac{d^2 x}{dt^2} = \frac{d^2 y}{dt^2} = 0, \quad \text{and} \quad \frac{dx}{dt} = u_1, \quad \frac{dy}{dt} = v_1.$$

It is observed, that u_1 and v_1 do not depend on the friction, but on the initial circumstances of motion.

If

$$(u_0-u_1)^2+(v_0-v_1)^2 = s^2 = \frac{4}{49}\{(u_0-a\,\Omega_2)^2+(v_0+a\,\Omega_1)^2\}, \qquad (266)$$

(254) become

$$\frac{d^2 x}{dt^2} = -\mu g\frac{u_0-u_1}{s}, \qquad \frac{d^2 y}{dt^2} = -\mu g\frac{v_0-v_1}{s};$$

and therefore, if the initial place of the centre of gravity of the ball is taken as the origin,

$$\left.\begin{aligned} \frac{dx}{dt} &= u_0 - \frac{\mu g(u_0-u_1)}{s}t, \\ \frac{dy}{dt} &= v_0 - \frac{\mu g(v_0-v_1)}{s}t; \end{aligned}\right\} \qquad (267)$$

$$\left.\begin{aligned} x &= u_0 t - \frac{\mu g(u_0-u_1)}{2s}t^2, \\ y &= v_0 t - \frac{\mu g(v_0-v_1)}{2s}t^2; \end{aligned}\right\} \qquad (268)$$

which give the position of the centre of gravity in terms of t.

Eliminating t, we have

$$\{x(v_0-v_1)-y(u_0-u_1)\}^2+\frac{2s}{\mu g}(u_1 v_0-v_1 u_0)(u_0 y-v_0 x) = 0; \qquad (269)$$

which is the equation to the parabolic path of the centre of gravity of the sphere.

387.] Now this equation becomes $y = x$, that is, represents a straight line, when u_1, v_1 are respectively proportional to u_0, v_0; in which case we have, from (259),

$$u_0\,\Omega_1 + v_0\,\Omega_2 = 0; \qquad (270)$$

which shews that the initial rotation-axis is in a vertical plane at right angles to the line of initial velocity of the centre of gravity.

The equation (269) expresses the path of the centre of the ball, so long as the ball slides as well as rolls; at the instant however when the sliding ceases, and the ball only rolls,

$$\frac{dx}{dt} = u_1 = a\,\omega_2, \qquad \frac{dy}{dt} = v_1 = -a\,\omega_1; \qquad (271)$$

and from either of (267) we have

$$t = \frac{s}{\mu g} = \frac{2}{7\mu g}\{(u_0 - a\Omega_2)^2 + (v_0 + a\Omega_1)^2\}^{\frac{1}{2}}; \qquad (27$$

at which time, from (268), we have

$$x = \frac{s}{\mu g}\frac{u_0+u_1}{2}, \qquad y = \frac{s}{\mu g}\frac{v_0+v_1}{2}. \qquad (27$$

After this instant, if there were no friction of rolling, the ba would continue to move uniformly in a straight line, with a vel city of which u_1 and v_1 are the axial components. But as friction of rolling acts to retard it, the ball continues its rect lineal course with a decreasing velocity, until it finally comes rest; and the equation to the line along which the centre gravity moves, and which is also the path of the point of conta with the table, is

$$2\mu g\{u_1 y - v_1 x\} + s(u_0 v_1 - v_0 u_1) = 0; \qquad (27$$

this line is evidently a tangent to the parabola at the point give by (273).

388.] Thus we have arrived at the exact course which billiard ball takes on a table, in the most general circumstanc of a stroke of a cue. The motion is at first a mixed one sliding and of rolling; and the centre of the ball moves in parabola so long as the ball slides on the table, which caus a sliding friction at the point of contact; this sliding howev eventually ceases, and before the ball comes to rest; and t centre of the ball then takes a rectilineal path, which is a tange to the parabola at the point where the sliding has ceased.

We may now give to the equations determined in the precedi Articles those interpretations which arise from initial circu stances produced by the stroke of a cue.

Let Q be the momentum with which the cue strikes the bal and let a be the angle at which its line of action is inclined the plane of the table. Let h be the horizontal distance fro the centre of the ball to the vertical plane containing the ax of the cue, which is the line of blow; and let k be the perpe dicular distance on this line from a horizontal line through t centre perpendicular to the vertical plane, containing the axis the cue; let us moreover take the plane of (x, z) to be parall to the axis of the cue. Let F be the friction which is broug into action between the ball and the table by the blow of t

cue, and let β be the angle between the line along which F acts and the axis of x; let $\mu =$ the coefficient of friction; then, as in Art. 370, the equations of motion of the centre of gravity are,

$$\left.\begin{aligned} \mathrm{M} u_0 &= \quad \mathrm{Q}\cos a - \mathrm{F}\cos\beta, \\ \mathrm{M} v_0 &= \qquad\qquad -\mathrm{F}\sin\beta, \\ 0 &= -\mathrm{Q}\sin a + \mathrm{R} - \mathrm{M}g; \end{aligned}\right\} \tag{275}$$

and the equations of rotation about the centre of gravity are

$$\left.\begin{aligned} \mathrm{A}\,\Omega_1 &= -\mathrm{Q}h\sin a - a\mathrm{F}\sin\beta, \\ \mathrm{A}\,\Omega_2 &= \quad \mathrm{Q}k + a\mathrm{F}\cos\beta, \\ \mathrm{A}\,\Omega_3 &= -\mathrm{Q}h\cos a. \end{aligned}\right\} \tag{276}$$

Also
$$\mathrm{F} = \mu\mathrm{R}, \qquad \mathrm{A} = \frac{2a^2\mathrm{M}}{5};$$

$$\frac{\cos\beta}{u_0 - a\,\Omega_2} = \frac{\sin\beta}{v_0 + a\,\Omega_1}. \tag{277}$$

From these equations we have

$$\tan\beta = \frac{5h\sin a}{5k - 2a\cos a}; \tag{278}$$

which assigns the direction in which the point of contact of the ball with the plane begins to move. This also is the direction of the constant retarding force of friction, and of the axis of the parabola in which the ball moves until sliding friction ceases. Hence also we can determine $u_0, v_0, \Omega_1, \Omega_2$; also we have

$$u_1 = \frac{5\mathrm{Q}(a\cos a + k)}{7a\mathrm{M}}, \qquad v_1 = \frac{5\mathrm{Q}h\sin a}{7a\mathrm{M}}; \tag{279}$$

which values are independent of the friction, as we have before observed in Art. 386.

If the path of the ball is rectilineal, and in the direction of the stroke of the cue, $\tan\beta = 0$, and $v_1 = 0$; hence, either $h = 0$, or $\sin a = 0$; in the former case, the centre of the ball is in the vertical plane containing the axis of the cue; in the latter, the axis of the cue is horizontal. Under either of these circumstances therefore, and under these only, is the path of the ball rectilineal; in all other cases the path is parabolic.

If the path of the centre of the ball is not rectilinear, the value of v_1, given in (279), shews that the deviation lies on that side of the centre in its initial position, on which the stroke is given, for v_1 and h are of the same sign.

389.] These several circumstances are represented in Fig. 59. O, the centre of the ball at the instant of the blow, is the origin;

and the horizontal plane containing O is that of (x, y). The plane of the table is parallel to it, and at a distance below it equal to a, the radius of the ball. P is the point of contact of the ball with the table. QRT is the axis of the cue, the vertical plane parallel to which, and containing O, is the plane of (x, z). The circle is the section of the ball by the plane of (x, z). QRT pierces the plane of (y, z) at R, and the plane of the table at T, and α is the angle which it makes with the table. $\text{OH} = h$ is the perpendicular distance from O to the vertical plane RET, containing the axis of the cue; and $\text{HK} = k$ is the perpendicular from H to the axis of the cue; so that h, k, and α determine the position of the line of blow. Let PP_1 be the curve which is the parabolic path of the point of contact of the ball with the table; P_1 being the point given by the coordinates (273), at which sliding friction ceases, and after which the path becomes rectilineal; let $\text{P}_1\,\text{S}_1$ be that rectilineal path; let $\text{RE} = l$; so that

$$k = (l-a)\cos\alpha; \tag{280}$$

thus $$u_1 = \frac{5\,\text{Q}\,l\cos\alpha}{7\,a\,\text{M}}, \qquad v_1 = \frac{5\,\text{Q}\,h\sin\alpha}{7\,a\,\text{M}};$$

therefore $$\frac{v_1}{u_1} = \frac{h}{l\cot\alpha} = \frac{\text{MT}}{\text{MP}}$$
$$= \tan\text{MPT}, \tag{281}$$

but $\frac{v_1}{u_1}$ is the tangent of the angle between the axis of x and the rectilineal path taken by the ball when friction ceases. Consequently the rectilineal path $\text{P}_1\,\text{S}_1$ is parallel to PT. Hence the final direction of the ball when friction ceases is easily determined; it is parallel to the line drawn from the point of contact of the ball with the table to the point at which the axis of the cue pierces the table.

Hence it follows, that if the axis of the cue is inclined to the plane of (x, y), at an angle so large that the point T falls on the negative side of the line PL, the ball in its final state moves in a direction opposite to that of the stroke.

Explanation of these formulae, in further application of them to the game of billiards, will be found in "Théorie Mathématique des effets du jeu de Billard," par G. Coriolis; Paris, 1835.

CHAPTER VIII.

RELATIVE MOTION OF A MATERIAL SYSTEM.

SECTION 1.—*Investigation of the general equations.*

390.] IN the preceding parts of this work the motion of a material system has been investigated relatively to a fixed origin, and to a fixed system of coordinate axes. The coordinate axes indeed, to which we have found it convenient to refer the motion primarily, have not always been fixed; for in the last Chapter two systems were used, one of which was fixed in the body and moved with it; the other system however was absolutely fixed, and to it ultimately the motion of the material system was referred; and its incidents were deduced from the position of the parts of it relatively to that system. Now I propose to consider a more general case; and to investigate the motion of a material system relatively to a moving origin and a moving system of rectangular axes, all the incidents of motion of these latter, as well as those of the material system, being given, and the material system also moving relatively to it. This general case is that of *relative motion;* that is, of the motion of a material system relatively to moving coordinate axes, to which allusion has already been made in Vol. III, Arts. 317–319, in the case of a single particle; under circumstances however in which a single material particle moves in a plane, and the motion of the rectangular axes is also in that plane. As the formulae which express the circumstances of relative motion are long and complicated, it will be convenient to consider primarily the motion of one particle; and thus be free of signs of summation; but the results will be capable of the most general application, because, by D'Alembert's principle, they can, mutatis mutandis, be extended to systems of moving particles. Our method will be purely analytical. Doubtless hereby we are in danger of overlooking the full meaning of the symbols; we may perhaps lose sight in them of the kinematical and mechanical truths which they represent; and thus our apprehension of things in their actual state may be indistinct. Consequently I shall interpret the equations, and shall shew that they are the expression of results arrived at from the consideration

of relative motion and its incidents in their first principles, and from (as it is commonly said) general reasoning. The kinematics of relative motion first require investigation.

391.] Let m be the mass of the particle whose motion is to be considered; and let (x, y, z) be its place at the time t relatively to an origin and to a system of axes fixed absolutely in space. At the same time let (x_0, y_0, z_0) be the place of another origin, which moves; and relatively to it, and to a moving system of axes originating at it, let (ξ, η, ζ) be the place of m. Let the system of direction-cosines connecting these two systems of axes at the time t be that of Art. 2; then

$$\left.\begin{aligned} x &= x_0 + a_1\xi + b_1\eta + c_1\zeta, \\ y &= y_0 + a_2\xi + b_2\eta + c_2\zeta, \\ z &= z_0 + a_3\xi + b_3\eta + c_3\zeta; \end{aligned}\right\} \qquad (1)$$

and, as all the quantities in the right-hand members are functions of t,

$$\left.\begin{aligned} \frac{dx}{dt} &= \frac{dx_0}{dt} + \xi\frac{da_1}{dt} + \eta\frac{db_1}{dt} + \zeta\frac{dc_1}{dt} + a_1\frac{d\xi}{dt} + b_1\frac{d\eta}{dt} + c_1\frac{d\zeta}{dt}, \\ \frac{dy}{dt} &= \frac{dy_0}{dt} + \xi\frac{da_2}{dt} + \eta\frac{db_2}{dt} + \zeta\frac{dc_2}{dt} + a_2\frac{d\xi}{dt} + b_2\frac{d\eta}{dt} + c_2\frac{d\zeta}{dt}, \\ \frac{dz}{dt} &= \frac{dz_0}{dt} + \xi\frac{da_3}{dt} + \eta\frac{db_3}{dt} + \zeta\frac{dc_3}{dt} + a_3\frac{d\xi}{dt} + b_3\frac{d\eta}{dt} + c_3\frac{d\zeta}{dt}. \end{aligned}\right\} \qquad (2)$$

Let us examine the several terms of these equations. The left-hand members are the axial components of the velocity of m relatively to the three fixed axes of (x, y, z); that is, of the absolute velocity of m. The first four terms of the right-hand members give the value of these components when ξ, η, ζ do not vary; that is, when the place of m is fixed relatively to the moving origin and the moving axes; and thus they are the axial components of the velocity of m considered as a point fixed relatively to the moving coordinate system of reference; the first terms of the several equations being due to the motion of translation of the origin; and the following three to the rotation of that system about an axis passing through that origin; and the last three terms in each equation are the projections on the fixed axes of the axial components of the velocity of m relatively to the moving axes and the moving origin. Equations (2) then yield the following theorem:

The absolute velocity of the particle m is the resultant of its

velocity relatively to the system of moving axes and of the velocity of it, considered as a fixed point of that system; or,

The velocity of a particle relative to a system of moving axes is the excess of its absolute velocity over the velocity of the system of axes.

If v_ξ, v_η, v_ζ are the components of the absolute velocity of m along the moving axes: that is, along the ξ-, η-, ζ-axes respectively; then

$$\begin{aligned} v_\xi &= a_1 \frac{dx}{dt} + a_2 \frac{dy}{dt} + a_3 \frac{dz}{dt} \\ &= a_1 \frac{dx_0}{dt} + a_2 \frac{dy_0}{dt} + a_3 \frac{dz_0}{dt} \\ &\quad + \eta \left\{ a_1 \frac{db_1}{dt} + a_2 \frac{db_2}{dt} + a_3 \frac{db_3}{dt} \right\} \\ &\quad + \zeta \left\{ a_1 \frac{dc_1}{dt} + a_2 \frac{dc_2}{dt} + a_3 \frac{dc_3}{dt} \right\} + \frac{d\xi}{dt}; \end{aligned} \tag{3}$$

$$\begin{aligned} v_\eta &= b_1 \frac{dx_0}{dt} + b_2 \frac{dy_0}{dt} + b_3 \frac{dz_0}{dt} \\ &\quad + \zeta \left\{ b_1 \frac{dc_1}{dt} + b_2 \frac{dc_2}{dt} + b_3 \frac{dc_3}{dt} \right\} \\ &\quad + \xi \left\{ b_1 \frac{da_1}{dt} + b_2 \frac{da_2}{dt} + b_3 \frac{da_3}{dt} \right\} + \frac{d\eta}{dt}; \end{aligned} \tag{4}$$

$$\begin{aligned} v_\zeta &= c_1 \frac{dx_0}{dt} + c_2 \frac{dy_0}{dt} + c_3 \frac{dz_0}{dt} \\ &\quad + \xi \left\{ c_1 \frac{da_1}{dt} + c_2 \frac{da_2}{dt} + c_3 \frac{da_3}{dt} \right\} \\ &\quad + \eta \left\{ c_1 \frac{db_1}{dt} + c_2 \frac{db_2}{dt} + c_3 \frac{db_3}{dt} \right\} + \frac{d\zeta}{dt}; \end{aligned} \tag{5}$$

in which equivalents for v_ξ, v_η, v_ζ all the terms except the last of each arise from the motion of the coordinate system of reference, and the last arise from the motion of m relatively to that system.

If we suppose the motion of the moving system of reference to be made up of a motion of translation by reason of which the new origin is moved in the time dt over a space of which the projections on the fixed axes are dx_0, dy_0, dz_0 respectively, and of a rotation about an axis passing through the origin, of which ω_ξ, ω_η, ω_ζ are the axial components; then, as m at (ξ, η, ζ) is a point fixed relatively to the moving system, substituting from (88), Art. 58, we have

$$\left.\begin{aligned} v_\xi &= a_1 \frac{dx_0}{dt} + a_2 \frac{dy_0}{dt} + a_3 \frac{dz_0}{dt} + \zeta\omega_\eta - \eta\omega_\zeta + \frac{d\xi}{dt}, \\ v_\eta &= b_1 \frac{dx_0}{dt} + b_2 \frac{dy_0}{dt} + b_3 \frac{dz_0}{dt} + \xi\omega_\zeta - \zeta\omega_\xi + \frac{d\eta}{dt}, \\ v_\zeta &= c_1 \frac{dx_0}{dt} + c_2 \frac{dy_0}{dt} + c_3 \frac{dz_0}{dt} + \eta\omega_\xi - \xi\omega_\eta + \frac{d\zeta}{dt}. \end{aligned}\right\} \qquad (6)$$

And thus along the axes of the moving system the axial components of the absolute velocity of m are the sums of the axial components of (1) the absolute velocity of the origin, (2) the velocity of m at (ξ, η, ζ) due to the angular velocities of the coordinate axes, (3) the velocity of m relative to the moving axes.

If the origin of the moving system is fixed, and the axes only move, these become

$$\left.\begin{aligned} v_\xi &= \zeta\omega_\eta - \eta\omega_\zeta + \frac{d\xi}{dt}, \\ v_\eta &= \xi\omega_\zeta - \zeta\omega_\xi + \frac{d\eta}{dt}, \\ v_\zeta &= \eta\omega_\xi - \xi\omega_\eta + \frac{d\zeta}{dt}. \end{aligned}\right\} \qquad (7)$$

392.] Suppose now that the absolute velocity of m varies, then, taking the t-differentials of (2), we have

$$\begin{aligned} \frac{d^2x}{dt^2} = \frac{d^2x_0}{dt^2} + \xi\frac{d^2a_1}{dt^2} + \eta\frac{d^2b_1}{dt^2} + \zeta\frac{d^2c_1}{dt^2} \\ + 2\left\{\frac{d\xi}{dt}\frac{da_1}{dt} + \frac{d\eta}{dt}\frac{db_1}{dt} + \frac{d\zeta}{dt}\frac{dc_1}{dt}\right\} \\ + a_1\frac{d^2\xi}{dt^2} + b_1\frac{d^2\eta}{dt^2} + c_1\frac{d^2\zeta}{dt^2}; \qquad (8) \end{aligned}$$

$$\begin{aligned} \frac{d^2y}{dt^2} = \frac{d^2y_0}{dt^2} + \xi\frac{d^2a_2}{dt^2} + \eta\frac{d^2b_2}{dt^2} + \zeta\frac{d^2c_2}{dt^2} \\ + 2\left\{\frac{d\xi}{dt}\frac{da_2}{dt} + \frac{d\eta}{dt}\frac{db_2}{dt} + \frac{d\zeta}{dt}\frac{dc_2}{dt}\right\} \\ + a_2\frac{d^2\xi}{dt^2} + b_2\frac{d^2\eta}{dt^2} + c_2\frac{d^2\zeta}{dt^2}; \qquad (9) \end{aligned}$$

$$\begin{aligned} \frac{d^2z}{dt^2} = \frac{d^2z_0}{dt^2} + \xi\frac{d^2a_3}{dt^2} + \eta\frac{d^2b_3}{dt^2} + \zeta\frac{d^2c_3}{dt^2} \\ + 2\left\{\frac{d\xi}{dt}\frac{da_3}{dt} + \frac{d\eta}{dt}\frac{db_3}{dt} + \frac{d\zeta}{dt}\frac{dc_3}{dt}\right\} \\ + a_3\frac{d^2\xi}{dt^2} + b_3\frac{d^2\eta}{dt^2} + c_3\frac{d^2\zeta}{dt^2}; \qquad (10) \end{aligned}$$

which are severally the equivalents of the expressed velocity-increments along the fixed coordinate axes in terms of the elements of the moving system of axes.

Let v'_ξ, v'_η, v'_ζ, be the axial components of the expressed velocity-increment of m along the moving axes of ξ, η, ζ, then

$$\begin{aligned}
v'_\xi = {} & a_1\frac{d^2x}{dt^2}+a_2\frac{d^2y}{dt^2}+a_3\frac{d^2z}{dt^2} \\
= {} & a_1\left\{\frac{d^2x_0}{dt^2}+\xi\frac{d^2a_1}{dt^2}+\eta\frac{d^2b_1}{dt^2}+\zeta\frac{d^2c_1}{dt^2}\right\} \\
& +a_2\left\{\frac{d^2y_0}{dt^2}+\xi\frac{d^2a_2}{dt^2}+\eta\frac{d^2b_2}{dt^2}+\zeta\frac{d^2c_2}{dt^2}\right\} \\
& +a_3\left\{\frac{d^2z_0}{dt^2}+\xi\frac{d^2a_3}{dt^2}+\eta\frac{d^2b_3}{dt^2}+\zeta\frac{d^2c_3}{dt^2}\right\} \\
& +2\left\{a_1\frac{db_1}{dt}+a_2\frac{db_2}{dt}+a_3\frac{db_3}{dt}\right\}\frac{d\eta}{dt} \\
& +2\left\{a_1\frac{dc_1}{dt}+a_2\frac{dc_2}{dt}+a_3\frac{dc_3}{dt}\right\}\frac{d\zeta}{dt}+\frac{d^2\xi}{dt^2}; \qquad (11)
\end{aligned}$$

$$\begin{aligned}
v'_\eta = {} & b_1\frac{d^2x}{dt^2}+b_2\frac{d^2y}{dt^2}+b_3\frac{d^2z}{dt^2} \\
= {} & b_1\left\{\frac{d^2x_0}{dt^2}+\xi\frac{d^2a_1}{dt^2}+\eta\frac{d^2b_1}{dt^2}+\zeta\frac{d^2c_1}{dt^2}\right\} \\
& +b_2\left\{\frac{d^2y_0}{dt^2}+\xi\frac{d^2a_2}{dt^2}+\eta\frac{d^2b_2}{dt^2}+\zeta\frac{d^2c_2}{dt^2}\right\} \\
& +b_3\left\{\frac{d^2z_0}{dt^2}+\xi\frac{d^2a_3}{dt^2}+\eta\frac{d^2b_3}{dt^2}+\zeta\frac{d^2c_3}{dt^2}\right\} \\
& +2\left\{b_1\frac{dc_1}{dt}+b_2\frac{dc_2}{dt}+b_3\frac{dc_3}{dt}\right\}\frac{d\zeta}{dt} \\
& +2\left\{b_1\frac{da_1}{dt}+b_2\frac{da_2}{dt}+b_3\frac{da_3}{dt}\right\}\frac{d\xi}{dt}+\frac{d^2\eta}{dt^2}; \qquad (12)
\end{aligned}$$

$$\begin{aligned}
v'_\zeta = {} & c_1\frac{d^2x}{dt^2}+c_2\frac{d^2y}{dt^2}+c_3\frac{d^2z}{dt^2} \\
= {} & c_1\left\{\frac{d^2x_0}{dt^2}+\xi\frac{d^2a_1}{dt^2}+\eta\frac{d^2b_1}{dt^2}+\zeta\frac{d^2c_1}{dt^2}\right\} \\
& +c_2\left\{\frac{d^2y_0}{dt^2}+\xi\frac{d^2a_2}{dt^2}+\eta\frac{d^2b_2}{dt^2}+\zeta\frac{d^2c_2}{dt^2}\right\} \\
& +c_3\left\{\frac{d^2z_0}{dt^2}+\xi\frac{d^2a_3}{dt^2}+\eta\frac{d^2b_3}{dt^2}+\zeta\frac{d^2c_3}{dt^2}\right\} \\
& +2\left\{c_1\frac{da_1}{dt}+c_2\frac{da_2}{dt}+c_3\frac{da_3}{dt}\right\}\frac{d\xi}{dt} \\
& +2\left\{c_1\frac{db_1}{dt}+c_2\frac{db_2}{dt}+c_3\frac{db_3}{dt}\right\}\frac{d\eta}{dt}+\frac{d^2\zeta}{dt^2}. \qquad (13)
\end{aligned}$$

393.] These quantities will be more conveniently expressed, as follows, in terms of ω_ξ, ω_η, ω_ζ, by means of the equivalents which are given in Arts. 55–59. By (11) we have

$$
\begin{aligned}
v'_\xi = a_1\frac{d^2x_0}{dt^2} + a_2\frac{d^2y_0}{dt^2} + a_3\frac{d^2z_0}{dt^2}
& + \xi\left\{a_1\frac{d^2a_1}{dt^2} + a_2\frac{d^2a_2}{dt^2} + a_3\frac{d^2a_3}{dt^2}\right\} \\
& + \eta\left\{a_1\frac{d^2b_1}{dt^2} + a_2\frac{d^2b_2}{dt^2} + a_3\frac{d^2b_3}{dt^2}\right\} \\
& + \zeta\left\{a_1\frac{d^2c_1}{dt^2} + a_2\frac{d^2c_2}{dt^2} + a_3\frac{d^2c_3}{dt^2}\right\} \\
& + 2\left\{\omega_\eta\frac{d\zeta}{dt} - \omega_\zeta\frac{d\eta}{dt}\right\} + \frac{d^2\xi}{dt^2}.
\end{aligned}
$$

But $$a_1\frac{d^2a_1}{dt^2} + a_2\frac{d^2a_2}{dt^2} + a_3\frac{d^2a_3}{dt^2} = \frac{d}{dt}\left\{a_1\frac{da_1}{dt} + a_2\frac{da_2}{dt} + a_3\frac{da_3}{dt}\right\} - \left\{\left(\frac{da_1}{dt}\right)^2 + \left(\frac{da_2}{dt}\right)^2 + \left(\frac{da_3}{dt}\right)^2\right\}$$

$$= -\{\omega_\eta^2 + \omega_\zeta^2\};$$

similarly

$$a_1\frac{d^2b_1}{dt^2} + a_2\frac{d^2b_2}{dt^2} + a_3\frac{d^2b_3}{dt^2} = -\frac{d\omega_\zeta}{dt} + \omega_\xi\omega_\eta;$$

$$a_1\frac{d^2c_1}{dt^2} + a_2\frac{d^2c_2}{dt^2} + a_3\frac{d^2c_3}{dt^2} = \frac{d\omega_\eta}{dt} - \omega_\xi\omega_\zeta;$$

so that

$$
\begin{aligned}
v'_\xi = a_1\frac{d^2x_0}{dt^2} + a_2\frac{d^2y_0}{dt^2} + a_3\frac{d^2z_0}{dt^2}
& + \zeta\frac{d\omega_\eta}{dt} - \eta\frac{d\omega_\zeta}{dt} + \omega_\eta(\eta\omega_\xi - \xi\omega_\eta) - \omega_\zeta(\xi\omega_\zeta - \zeta\omega_\xi) \\
& + 2\left(\omega_\eta\frac{d\zeta}{dt} - \omega_\zeta\frac{d\eta}{dt}\right) + \frac{d^2\xi}{dt^2}; \qquad (14)
\end{aligned}
$$

similarly

$$
\begin{aligned}
v'_\eta = b_1\frac{d^2x_0}{dt^2} + b_2\frac{d^2y_0}{dt^2} + b_3\frac{d^2z_0}{dt^2}
& + \xi\frac{d\omega_\zeta}{dt} - \zeta\frac{d\omega_\xi}{dt} + \omega_\zeta(\zeta\omega_\eta - \eta\omega_\zeta) - \omega_\xi(\eta\omega_\xi - \xi\omega_\eta) \\
& + 2\left(\omega_\zeta\frac{d\xi}{dt} - \omega_\xi\frac{d\zeta}{dt}\right) + \frac{d^2\eta}{dt^2}; \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
v'_\zeta = c_1\frac{d^2x_0}{dt^2} + c_2\frac{d^2y_0}{dt^2} + c_3\frac{d^2z_0}{dt^2}
& + \eta\frac{d\omega_\xi}{dt} - \xi\frac{d\omega_\eta}{dt} + \omega_\xi(\xi\omega_\zeta - \zeta\omega_\xi) - \omega_\eta(\zeta\omega_\eta - \eta\omega_\zeta) \\
& + 2\left(\omega_\xi\frac{d\eta}{dt} - \omega_\eta\frac{d\xi}{dt}\right) + \frac{d^2\zeta}{dt^2}. \qquad (16)
\end{aligned}
$$

Hence the equations of motion can be formed: for let X, Y, Z be the axial components of the impressed velocity-increments on m parallel to the fixed coordinate axes of x, y, z; and let X′, Y′, Z′ be the components of the same impressed velocity-increments parallel to the moving axes of ξ, η, ζ; so that

$$\left.\begin{aligned} X' &= a_1 X + a_2 Y + a_3 Z, \\ Y' &= b_1 X + b_2 Y + b_3 Z, \\ Z' &= c_1 X + c_2 Y + c_3 Z; \end{aligned}\right\} \qquad (17)$$

then the equations which express the relative motion of m are,

$$X' - v'_{\xi} = 0, \qquad Y' - v'_{\eta} = 0, \qquad Z' - v'_{\zeta} = 0. \qquad (18)$$

394.] Before, however, we apply them to the solution of particular problems let us examine them, and analyse them with the view of detecting the origin and meaning of their several terms. In the values of $v'_{\xi}, v'_{\eta}, v'_{\zeta}$, given in (14), (15), (16), it is evident that the first three terms in each express the components along the moving axes of the acceleration of the moving origin, and the same is of course true of the corresponding terms in (11), (12), (13). It is also evident that the next six terms in each arise from the angular velocity and the angular velocity-increment of the system of moving axes about its instantaneous rotation-axis, the origin and the place of m in reference to the moving axes being considered fixed at that instant; the next two terms depend on both the angular velocity of the moving system of reference about its instantaneous axis and the relative velocity of m; and the last terms in each are severally the axial components of the relative acceleration of m. If therefore the coordinate system of (ξ, η, ζ) to which the place of m at the time t is referred, were fixed and immoveable, the several right-hand members would be reduced to their last terms; and the equations of motion would be for m the same as those which have been found for the motion of a particle in Vol. III; and for a material system, the same as those which have been found in Chapter III of this Volume. Observation of this fact leads us to consider the first eleven terms in each right-hand member in (14), (15), (16) as impressed velocity-increments due to certain forces; which forces are indeed fictitious, but are assumed and are useful for the purpose of giving to the equations a form identical with that of the equations of absolute motion. This is the view of the subject which has been taken by Clairaut, Coriolis, and

Bertrand*. And I proceed to investigate the nature of the forces to which we may suppose these several velocity-increments to be due.

395.] Taking the values for v'_ξ, v'_η, v'_ζ which are given in (14), (15), and (16), the first nine terms in each are due to the motion of the moving system of reference, and do not depend on the motion of m relative to that system; that is, they are independent of the t-differentials of ξ, η, ζ. Now we will suppose these terms to arise from a fictitious force, which we will call the force of transference, because it is caused by the transference or the shifting of the system of reference from one position to another. Coriolis has called it "force d'entrainement," and it is such that if X_t, Y_t, Z_t are the axial components of the velocity-increments due to this force, then

$$\text{X}_t = a_1 \frac{d^2 x_0}{dt^2} + a_2 \frac{d^2 y_0}{dt^2} + a_3 \frac{d^2 z_0}{dt^2} + \zeta \frac{d\omega_\eta}{dt} - \eta \frac{d\omega_\zeta}{dt} + \omega_\eta (\eta\,\omega_\xi - \xi\,\omega_\eta) - \omega_\zeta (\xi\,\omega_\zeta - \zeta\,\omega_\xi), \quad (19)$$

with similar values for Y_t and Z_t.

Now let d_ω denote a change in a variable due to a rotation $d\omega$ about an axis. Then by (72), Art. 53,

$$\zeta\omega_\eta - \eta\,\omega_\zeta = \frac{d_\omega \xi}{dt}, \quad \xi\,\omega_\zeta - \zeta\,\omega_\xi = \frac{d_\omega \eta}{dt}, \quad \eta\,\omega_\xi - \xi\,\omega_\eta = \frac{d_\omega \zeta}{dt}; \quad (20)$$

whence substituting in (19) we have

$$\text{X}_t = a_1 \frac{d^2 x_0}{dt^2} + a_2 \frac{d^2 y_0}{dt^2} + a_3 \frac{d^2 z_0}{dt^2} + \frac{d^2{}_\omega \xi}{dt^2}; \quad (21)$$

similarly we have

$$\text{Y}_t = b_1 \frac{d^2 x_0}{dt^2} + b_2 \frac{d^2 y_0}{dt^2} + b_3 \frac{d^2 z_0}{dt^2} + \frac{d^2{}_\omega \eta}{dt^2}; \quad (22)$$

$$\text{Z}_t = c_1 \frac{d^2 x_0}{dt^2} + c_2 \frac{d^2 y_0}{dt^2} + c_3 \frac{d^2 z_0}{dt^2} + \frac{d^2{}_\omega \zeta}{dt^2}; \quad (23)$$

whence it appears that the velocity-increment due to the force of transference is the sum of those which arise from the motion of the moving origin and from the rotation of the moving system of reference about its instantaneous axis. This fact is also evident from general reasoning.

* See Mémoires de l'Académie des Sciences de Paris, 1742, p. 1. Journal de l'Ecole Polytechnique; XXI and XXIV Cahiers. Traité de la Mécanique des corps solides et du calcul de l'effet des Machines par G. Coriolis; 2nd edit., Paris, 1844, p. 46.

396.] The last three terms in (14), (15), and (16) respectively, arise from the motion of m relative to the moving system of reference, and depend on the relative velocity and acceleration of m at the time t; inasmuch as they involve $\frac{d\xi}{dt}$, $\frac{d\eta}{dt}$, $\frac{d\zeta}{dt}$, and also their t-differentials. The last terms are respectively the components of the relative acceleration of m; but the two preceding terms in each expression depend on the relative velocity of m and also on the angular velocity of the system of reference about its instantaneous axis, and may be conveniently considered as the effects of a certain fictitious force, the nature of which is as follows:

The axial components of the velocity-increment due to this force are

$$2\left(\omega_\eta \frac{d\zeta}{dt} - \omega_\zeta \frac{d\eta}{dt}\right), \quad 2\left(\omega_\zeta \frac{d\xi}{dt} - \omega_\xi \frac{d\zeta}{dt}\right), \quad 2\left(\omega_\xi \frac{d\eta}{dt} - \omega_\eta \frac{d\xi}{dt}\right). \quad (24)$$

Let us denote the force by F; then squaring and adding these expressions, we have

$$\text{F}^2 = 4\left\{\left(\omega_\eta \frac{d\zeta}{dt} - \omega_\zeta \frac{d\eta}{dt}\right)^2 + \left(\omega_\zeta \frac{d\xi}{dt} - \omega_\xi \frac{d\zeta}{dt}\right)^2 + \left(\omega_\xi \frac{d\eta}{dt} - \omega_\eta \frac{d\xi}{dt}\right)^2\right\}; \quad (25)$$

so that if ω is the angular velocity of the moving system of reference about its instantaneous axis, and v the velocity of m relative to the moving system, and θ is the angle between the instantaneous rotation-axis and the line of v,

$$\text{F} = 2\,\omega v \sin\theta. \quad (26)$$

Thus F is twice the product of the angular velocity of the moving system about its rotation-axis and the projection of the relative velocity of m on a plane perpendicular to this rotation-axis.

It is, in the first place, to be observed that $\text{F} = 0$, whenever the line of the relative path of m is parallel to the rotation-axis, because in that case $\sin\theta = 0$; and that F is a maximum whenever, caeteris paribus, the line of motion of m is perpendicular to the rotation-axis of the system, because then $\sin\theta$ is a maximum.

As to its line of action; let α, β, γ be its direction angles, so that

$$\frac{\omega_\eta d\zeta - \omega_\zeta d\eta}{\cos\alpha} = \frac{\omega_\zeta d\xi - \omega_\xi d\zeta}{\cos\beta} = \frac{\omega_\xi d\eta - \omega_\eta d\xi}{\cos\gamma}; \quad (27)$$

$$\left.\begin{aligned} \therefore \quad \omega_\xi \cos a + \omega_\eta \cos \beta + \omega_\zeta \cos \gamma = 0, \\ \frac{d\xi}{dt} \cos a + \frac{d\eta}{dt} \cos \beta + \frac{d\zeta}{dt} \cos \gamma = 0: \end{aligned}\right\} \qquad (28)$$

which shew that the line of action of F is perpendicular to the instantaneous axis of rotation of the moving system of reference and also to the line of the relative motion of m. Thus F acts along the line of intersection of the normal plane of the path of m at the time t with the plane perpendicular to the rotation-axis of the moving system, and is consequently a normal force.

F has been called by Coriolis the compound centrifugal force; The origin of the force is as follows. The acceleration due to it depends on the angular velocity of the moving system about its rotation-axis. Through the place of m at the time t let a plane be drawn perpendicular to the rotation-axis of the system, then $v \sin \theta$ is the projection of v on that plane, and the relative distance, say $d\sigma$, described by m in this plane in the time dt is $v \sin \theta \, dt$. Also if ω is the angular velocity at that time, the angle described by $d\sigma$ about an axis passing through its extremity and parallel to the rotation-axis is $\omega \, dt$; so that the space described by the extremity of $d\sigma$ is $v \sin \theta \, dt \times \omega \, dt$: as this is the space due to the force F acting for the time dt and invariable for that time

$$\omega v \sin \theta (dt)^2 = \frac{F}{2} (dt)^2;$$

$$\therefore \quad F = 2 \omega v \sin \theta.$$

This explanation of the origin of F shews why the name of compound centrifugal force has been given to it.

397.] Hence the equations of relative motion which are given in (18) become

$$\left.\begin{aligned} X' - X_t - F \cos a - \frac{d^2\xi}{dt^2} = 0, \\ Y' - Y_t - F \cos \beta - \frac{d^2\eta}{dt^2} = 0, \\ Z' - Z_t - F \cos \gamma - \frac{d^2\zeta}{dt^2} = 0. \end{aligned}\right\} \qquad (29)$$

In respect of these I would observe, that as we have supposed the motion of m and of the system of reference to be in a positive direction relative to the fixed coordinate axes according to our usual image, the effect of the moving system on the motion of m is a diminution of its relative coordinates, and consequently the

fictitious forces tend, as t increases, to diminish the relative coordinates, and consequently act in directions opposite to those of X', Y', Z', as is shewn in the preceding equations.

These three equations may be combined into a single equation by means of the principle of virtual velocities: for if $\delta\sigma$ is a geometrical displacement of the place of m, of which $\delta\xi$, $\delta\eta$, $\delta\zeta$ are the axial projections, the equation of virtual velocities is

$$\Big(\mathrm{X}'-\mathrm{X}_t-\mathrm{F}\cos\alpha-\frac{d^2\xi}{dt^2}\Big)\delta\xi+\Big(\mathrm{Y}'-\mathrm{Y}_t-\mathrm{F}\cos\beta-\frac{d^2\eta}{dt^2}\Big)\delta\eta$$
$$+\Big(\mathrm{Z}'-\mathrm{Z}_t-\mathrm{F}\cos\gamma-\frac{d^2\zeta}{dt^2}\Big)\delta\zeta=0. \quad (30)$$

398.] Before we apply these equations in their general form, let us consider certain reduced forms which they assume in particular cases, for we shall thereby be better able to estimate the meaning and importance of their several terms.

If the motion of the system of coordinate axes is that of translation only, then $\omega=0$, and the second fictitious force vanishes; in this case, the axes of the moving system will be, or may be taken to be, always parallel to those of the fixed system, so that there will be no t-variations of the nine direction-cosines; and all the direction-cosines will vanish except a_1, b_2, and c_3, each of which becomes equal to unity; and the equations (29) become

$$\left.\begin{aligned}\mathrm{X}'-\frac{d^2x_0}{dt^2}-\frac{d^2\xi}{dt^2}&=0,\\ \mathrm{Y}'-\frac{d^2y_0}{dt^2}-\frac{d^2\eta}{dt^2}&=0,\\ \mathrm{Z}'-\frac{d^2z_0}{dt^2}-\frac{d^2\zeta}{dt^2}&=0;\end{aligned}\right\} \quad (31)$$

which are the equations already found in Art. 332, Vol. III. These are sufficient for the determination of the relative motion of translation of a planet considered to be condensed into a particle at its mass-centre, and have been applied to that problem in Arts. 365 and 367, Vol. III.

If moreover the moving origin travels along a straight line with uniform velocity, then also $\frac{d^2x_0}{dt^2}=\frac{d^2y_0}{dt^2}=\frac{d^2z_0}{dt^2}=0$; and the equations (31) have the same form as those which express the ordinary absolute motion. It is evident however from the equations which express the relative velocities, that these will not

be the same as the absolute velocities; and consequently some quantities will be introduced in the integration-process which depend on the elements of the line of motion and on the velocity of the moving origin.

399.] One particular form of the preceding general results requires especial mention; viz., that in which the origin moves in the plane of (x, y), and the moving axes of (ξ, η) are always in that plane, so that the axes of z and ζ are always parallel. The equations which express this motion have been derived from first principles in Arts. 318 and 319, Vol. III: but it is desirable to shew how they follow from the preceding equations. Let θ be the angle between the ξ-axis and the x-axis at the time t: then $z_0 = \zeta = 0$, and

$$\left.\begin{aligned} a_1 &= \cos\theta, \\ a_2 &= \sin\theta, \\ a_3 &= 0; \end{aligned}\right\} \qquad \left.\begin{aligned} b_1 &= -\sin\theta, \\ b_2 &= \cos\theta, \\ b_3 &= 0; \end{aligned}\right\} \qquad \left.\begin{aligned} c_1 &= 0, \\ c_2 &= 0, \\ c_3 &= 1. \end{aligned}\right\}$$

$$\omega_\xi = \omega_\eta = 0; \qquad \omega_\zeta = \frac{d\theta}{dt}.$$

$$\begin{aligned} \text{X}_t &= \cos\theta\frac{d^2 x_0}{dt^2} + \sin\theta\frac{d^2 y_0}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2 - \eta\frac{d^2\theta}{dt^2}, \\ \text{Y}_t &= -\sin\theta\frac{d^2 x_0}{dt^2} + \cos\theta\frac{d^2 y_0}{dt^2} - \eta\left(\frac{d\theta}{dt}\right)^2 + \xi\frac{d^2\theta}{dt^2}, \\ \text{Z}_t &= 0; \end{aligned}$$

$$\text{F}\cos\alpha = 2\frac{d\eta}{dt}\frac{d\theta}{dt}, \qquad \text{F}\cos\beta = -2\frac{d\xi}{dt}\frac{d\theta}{dt}, \qquad \text{F}\cos\gamma = 0;$$

so that equations (29) become

$$\left.\begin{aligned} \text{X}' - \cos\theta\frac{d^2 x_0}{dt^2} - \sin\theta\frac{d^2 y_0}{dt^2} + \xi\left(\frac{d\theta}{dt}\right)^2 + \eta\frac{d^2\theta}{dt^2} + 2\frac{d\eta}{dt}\frac{d\theta}{dt} - \frac{d^2\xi}{dt^2} &= 0, \\ \text{Y}' + \sin\theta\frac{d^2 x_0}{dt^2} - \cos\theta\frac{d^2 y_0}{dt^2} + \eta\left(\frac{d\theta}{dt}\right)^2 - \xi\frac{d^2\theta}{dt^2} - 2\frac{d\xi}{dt}\frac{d\theta}{dt} - \frac{d^2\eta}{dt^2} &= 0; \end{aligned}\right\} \quad (32)$$

and the third equation disappears.

These equations may be conveniently expressed in the following form; viz.,

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + \frac{d^2 x_0}{dt^2}\cos\theta + \frac{d^2 y_0}{dt^2}\sin\theta - \xi\left(\frac{d\theta}{dt}\right)^2 - \frac{1}{\eta}\frac{d}{dt}\left(\eta^2\frac{d\theta}{dt}\right) &= \text{X}', \\ \frac{d^2\eta}{dt^2} - \frac{d^2 x_0}{dt^2}\sin\theta + \frac{d^2 y_0}{dt^2}\cos\theta - \eta\left(\frac{d\theta}{dt}\right)^2 + \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right) &= \text{Y}'; \end{aligned}\right\} \quad (33)$$

and are thus the general equations of relative motion of a particle moving in the plane of (ξ, η); by means of them ξ and η

may be expressed in terms of t and known quantities. If t is eliminated from these last two equations, the resulting equation in terms of ξ and η will be that of the relative path in which the particle moves. The equation to the absolute path will be found, when ξ and η are expressed in terms of x and y.

If the origin of the moving axes does not move, and the axes revolve with an uniform angular velocity ω, then (33) become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} &= \mathrm{X}', \\ \frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} &= \mathrm{Y}'. \end{aligned}\right\} \qquad (34)$$

These equations however refer to a very special case of the general motion.

SECTION 2.—*The relative motion of a material particle.*

400.] Although our object is the discovery of equations which represent the relative motion of a material system, and the investigation of the preceding equations which apply to a single material particle has been subordinate to that object; yet it is desirable to shew their applicability to the solution of certain problems, in which only a particle moves, more generally than has been done in Vol. III, Arts. 437, 438; because we shall hereby obtain a clearer insight into the several parts of them, and shall also solve some problems of considerable interest.

Let us first take the case of a particle moving within a tube, which also itself moves, and carries with it the system of axes to which it is referred; I shall consider the tube to be perfectly smooth, so that it offers no resistance to the moving particle in the direction of its motion; but presents a reaction acting along the normal, if the tube is a plane curve; and along the principal normal, if it is a curve of double curvature; and I shall also consider the small bore of the tube and the size of the particle to be such that the particle may exactly fill the tube.

In the most simple form of the problem the tube is a plane curve lying in the plane of (x, y), and rotating with a constant angular velocity ω, about the axis of ζ or z, the origin of the moving axes being fixed at the fixed origin; and no external force acts on the particle. Let R be the normal reacting pres-

sure of the tube against the particle; let m be the mass of the particle, and $ds =$ the length-element of the tube at the point (ξ, η), which is the place of m at the time t; then the equations (34) are

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2}-\omega^2\xi-2\omega\frac{d\eta}{dt} &= \pm\frac{\text{R}}{m}\frac{d\eta}{ds},\\ \frac{d^2\eta}{dt^2}-\omega^2\eta+2\omega\frac{d\xi}{dt} &= \mp\frac{\text{R}}{m}\frac{d\xi}{ds}.\end{aligned}\right\} \tag{35}$$

From these equations the following general theorems are deduced;

$$\frac{d\xi d^2\xi+d\eta d^2\eta}{dt^2}-\omega^2\{\xi d\xi+\eta d\eta\} = 0; \tag{36}$$

let $\xi^2+\eta^2 = r^2$, so that r is the distance of the particle from the fixed origin at the time t; then, assuming the relative velocities of the particle to be v and v_0, when the distances of its place from the origin are respectively r and r_0, this equation gives by integration

$$v^2-v_0^2 = \omega^2(r^2-r_0^2); \tag{37}$$

and thus assigns the relative velocity of the particle.

Also from (35) we have

$$\frac{d\eta d^2\xi-d\xi d^2\eta}{dt^2}-\omega^2(\xi d\eta-\eta d\xi)-2\omega\frac{ds^2}{dt} = \pm\frac{\text{R}}{m}ds; \tag{38}$$

now if ρ is the radius of curvature of the curve of the tube at the point (ξ, η), and p is the perpendicular on the tangent from the origin, this becomes

$$\pm\frac{\text{R}}{m} = \frac{v^2}{\rho}+\omega^2 p-2\omega v; \tag{39}$$

which assigns the pressure on the tube.

401.] The following are simple problems in illustration of these equations.

Ex. 1. Find the motion of a particle placed within a rectilinear tube which revolves with a constant angular velocity about an axis which intersects it at right angles.

Let the line of the tube be the moving ξ-axis; so that always $\eta = 0$; let $\omega =$ the constant angular velocity of the tube. Then (35) become

$$\frac{d^2\xi}{dt^2}-\omega^2\xi = 0, \qquad 2\omega\frac{d\xi}{dt} = \pm\frac{\text{R}}{m}.$$

Then if the particle is at relative rest at a distance a from the origin, when $t = 0$;

$$\xi = \frac{a}{2}\{e^{\omega t}+e^{-\omega t}\};$$

$$\text{R} = m a\omega^2\{e^{\omega t}-e^{-\omega t}\};$$

and if θ is the angle described by the tube during the time t, $\theta = \omega t$; and the equation to the relative path of the particle is

$$\xi = \frac{a}{2}\{e^{\theta} + e^{-\theta}\}.$$

If r is the absolute radius vector, $r = \xi$, and

$$r = \frac{a}{2}\{e^{\theta} + e^{-\theta}\};$$

and this is the absolute polar equation of the path.

Ex. 2. A particle is placed within a rectilinear tube which revolves about an axis intersecting it at right angles, with an angular velocity such that the tangent of the angle described in a given time is proportional to the time: it is required to determine the motion of the particle.

Let, as in the preceding problem, the line of the tube be the ξ-axis, so that always $\eta = 0$; let θ be the angle through which the tube has moved in the time t; then, by the conditions of the problem, if $\theta = 0$, when $t = 0$, and if k is a constant,

$$\tan\theta = kt;$$

and (33) become

$$\frac{d^2\xi}{dt^2} - \xi\left(\frac{d\theta}{dt}\right)^2 = 0, \qquad \frac{1}{\xi}\frac{d}{dt}\left(\xi^2\frac{d\theta}{dt}\right) = \pm\frac{\text{R}}{m}.$$

Eliminating t by means of the first two of these three equations, and taking θ to be an equicrescent variable, we have

$$\cos\theta\frac{d^2\xi}{d\theta^2} - 2\sin\theta\frac{d\xi}{d\theta} - \xi\cos\theta = 0.$$

When $t = 0$, let us suppose the tube to lie along the x-axis, and the particle to be projected with a velocity u along the tube from a point whose distance from the origin $= a$; then from above, by integration, we have

$$\cos\theta\,\frac{d\xi}{d\theta} - \xi\sin\theta = \frac{u}{k};$$

$$\therefore\quad \xi\cos\theta = a + \frac{u}{k}\theta;$$

which is the relative equation of the path of the particle. And in reference to the fixed axes of x and y,

$$x = a + \frac{u}{k}\theta, \qquad y = \left(a + \frac{u}{k}\theta\right)\tan\theta;$$

which give the absolute path of the particle.

If we substitute from above in the third of the preceding equations, we have $\quad \pm\text{R} = 2muk(\cos\theta)^3.$

Ex. 3. A particle is placed within a thin tube which is of the form of an equiangular spiral; the tube revolves with an uniform angular velocity about an axis passing through its pole, and perpendicular to the plane of the tube: it is required to determine the relative path of the particle.

Let a be the constant angle at which the tangent is inclined to the radius vector at every point of the curve. Then

$$p = r\sin a, \qquad dr = ds\cos a,$$

are the equations to the curve. And (37) gives

$$\left(\frac{dr}{dt}\right)^2 (\sec a)^2 = \omega^2 r^2 - \omega^2 r_0^{\,2} + v_0^{\,2}.$$

Let

$$\omega^2 r_0^{\,2} - v_0^{\,2} = \omega^2 k^2;$$

then if $t = 0$, when $r = k$, this equation gives by integration

$$r = \frac{k}{2}\{e^{\omega t\cos a} + e^{-\omega t\cos a}\};$$

which assigns the relative motion of the particle in the tube.

402.] For an example, in which the origin of coordinates itself moves, let us consider the motion of a particle within a circular tube, which revolves about an axis through its centre perpendicular to its plane with an uniform angular velocity ω; and the centre of the circular tube also describes a circle in the plane of the tube with an uniform angular velocity Ω.

Let us suppose the particle and tube to be situated at the time t, as they are placed in Fig. 61; wherein AQB is the circle in which Q, the centre of the circular tube, moves, and P is the place of the particle. Let us also suppose the centre of the tube to have been at A, on the axis of x, when $t = 0$, and the particle at that time to have been at relative rest at C, C being at C′ at the time t. Let OA $= a$, AC $= c$; then

$$x_0^{\,2} + y_0^{\,2} = a^2, \qquad \xi^2 + \eta^2 = c^2, \qquad \theta = (\Omega + \omega)t;$$

and we have also

$$x_0 = a\cos\Omega t, \qquad y_0 = a\sin\Omega t;$$

then from (33), we have

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - a\Omega^2\cos\omega t - (\Omega+\omega)^2\xi - 2(\Omega+\omega)\frac{d\eta}{dt} &= \pm\frac{R}{m}\frac{d\eta}{ds}, \\ \frac{d^2\eta}{dt^2} + a\Omega^2\sin\omega t - (\Omega+\omega)^2\eta + 2(\Omega+\omega)\frac{d\xi}{dt} &= \mp\frac{R}{m}\frac{d\xi}{ds}; \end{aligned}\right\} \quad (40)$$

which are the required equations of motion, and do not generally admit of integration.

If $\omega = 13\Omega$, the path is approximately that of the centre of the moon projected on the plane of the ecliptic.

403.] Another simple problem, to which the preceding principles are applicable, is that in which a heavy particle moves in a smooth tube, which rotates about a vertical axis with an uniform angular velocity.

Let us first suppose the tube to be of single curvature, and the axis about which it rotates to lie in its plane; let this axis be taken for the axis of z; and let the positive direction of it be measured from the origin in a direction opposite to that of gravity. Let this axis also be taken as the ζ-axis to which the curve is referred, and let a perpendicular to this line through the origin be the ξ-axis; so that the equations to the curve of the tube are

$$\zeta = f(\xi), \qquad \eta = 0.$$

Let ω be the constant angular velocity with which the tube revolves, the plane of the tube being in the plane of (z, x) when $t = 0$. Then from (29), the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi &= \pm\frac{\mathrm{R}}{m}\frac{d\zeta}{ds}, \\ \frac{d^2\zeta}{dt^2} \qquad &= \mp\frac{\mathrm{R}}{m}\frac{d\xi}{ds} - g; \end{aligned}\right\} \qquad (41)$$

whence we have

$$v^2 - v_0^2 = \omega^2(\xi^2 - \xi_0^2) - 2g(\zeta - \zeta_0),$$

$$\frac{v^2}{\rho} - \omega^2\xi\frac{d\zeta}{ds} = \pm\frac{\mathrm{R}}{m} + g\frac{d\xi}{ds}.$$

These are the tangential and normal components of (41).

If the velocity is given in terms of the coordinates of the place of m at the time t, the equation of the curve of the tube may be found; and if the equation of the curve of the tube is given, the velocity of m and the other circumstances of motion of it at any time may be found.

Simple applications of these equations having been given in Art. 437, Vol. III, we will take a more complicated case, and apply them to the determination of the position of the two heavy balls in Watt's centrifugal governor of the steam engine.

The arrangement of this contrivance will be understood from

Fig. 62, where O is the fixed point on the vertical axis at which the rods carrying the heavy balls cross each other. We shall take the plane of the rods and the balls to be that of (ξ, ζ), and shall take the vertical line drawn downwards from O to be the positive direction of ζ: we shall assume the weight of the rods to be so small in comparison of that of each ball, that the former may be neglected without sensible error. Let $\text{OP} = \text{OP}' = a$; and let their inclinations to the vertical be θ and θ_0 at the times t and 0 respectively; and let us suppose θ to be greater than θ_0 and $\frac{d\theta}{dt}$ to be zero, when $t = 0$; let ω be the angular velocity with which the plane containing the balls and rods rotates about $\text{O}z$; and let T be the tension of the rods. Then the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi &= -\text{T}\sin\theta, \\ \frac{d^2\zeta}{dt^2} \qquad &= -\text{T}\cos\theta + g; \end{aligned}\right\}$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2 = \omega^2(\xi^2 - \xi_0^2) + 2g(\zeta - \zeta_0);$$

$$a^2\left(\frac{d\theta}{dt}\right)^2 = a^2\omega^2\{(\cos\theta_0)^2 - (\cos\theta)^2\} + 2ga\{\cos\theta - \cos\theta_0\},$$

$$a\left(\frac{d\theta}{dt}\right)^2 = (\cos\theta_0 - \cos\theta)\{a\omega^2(\cos\theta_0 + \cos\theta) - 2g\},$$

$$\therefore \quad dt = \frac{a^{\frac{1}{2}}\,d\theta}{\{\cos\theta_0 - \cos\theta\}^{\frac{1}{2}}\{a\omega^2(\cos\theta_0 + \cos\theta) - 2g\}^{\frac{1}{2}}};$$

in which equation the variables are separated; and if the integration be effected, θ will be given in terms of t.

$$\frac{d\theta}{dt} = 0, \text{ when } (1)\ \theta = \theta_0; \quad (2)\ \cos\theta = \frac{2g}{a\omega^2} - \cos\theta_0;$$

so that θ varies between the angles given by these two limits.

Let ω_0 be the angular velocity with which the plane of the balls revolves, when the angle at which they are inclined to the vertical axis is θ_0, and does not vary. In this case, as they have no change of ξ or ζ, the preceding equations of motion give

$$g = \omega_0^2 a\cos\theta_0;$$

so that

$$dt = \frac{d\theta}{(\cos\theta_0 - \cos\theta)^{\frac{1}{2}}\{(\omega^2 - 2\omega_0^2)\cos\theta_0 + \omega^2\cos\theta\}^{\frac{1}{2}}};$$

and if $\text{OB} = \text{BQ} = b$; so that $\text{OQ} = 2b\cos\theta$,

$$d.\text{OQ} = -2b\sin\theta\,d\theta;$$

which assigns the vertical displacement of Q due to the change of angle of inclination of OP to the vertical.

404.] Next let us suppose the curve of the tube in which the particle moves to be of double curvature, and to rotate about the z- or ζ-axis with a constant angular velocity ω; we will also suppose the particle to be under the action of forces, the components of the relative impressed velocity-increments being X, Y, Z; then the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2}-\omega^2\xi-2\omega\frac{d\eta}{dt} &= \mathrm{X}+\frac{\mathrm{R}}{m}\cos a, \\ \frac{d^2\eta}{dt^2}-\omega^2\eta+2\omega\frac{d\xi}{dt} &= \mathrm{Y}+\frac{\mathrm{R}}{m}\cos\beta, \\ \frac{d^2\zeta}{dt^2} &= \mathrm{Z}+\frac{\mathrm{R}}{m}\cos\gamma; \end{aligned}\right\} \tag{42}$$

where R is the pressure of the particle on the tube, and a, β, γ are the direction-angles of the principal normal to the curve of the tube at the place of m. From these equations we have

$$v^2-v_0^2-\omega^2(r^2-r_0^2) = 2\int_0^1(\mathrm{X}\,d\xi+\mathrm{Y}\,d\eta+\mathrm{Z}\,d\zeta), \tag{43}$$

where $r^2 = \xi^2+\eta^2$; and the right-hand member expresses twice the work done relatively by the acting forces; and the term involving R disappears, because

$$d\xi\cos a+d\eta\cos\beta+d\zeta\cos\gamma = 0;$$

since the tangent and the principal normal are at right angles to each other.

If the tube had been fixed, so that $\omega = 0$, and V had been the velocity of m in the same position as that in which v is the velocity when the tube revolves,

$$\mathrm{V}^2-v_0^2 = 2\int_0^1(\mathrm{X}\,d\xi+\mathrm{Y}\,d\eta+\mathrm{Z}\,d\zeta);$$

$$\therefore \quad v^2-\mathrm{V}^2 = \omega^2(r^2-r_0^2); \tag{44}$$

so that the velocity in the moving tube is greater than it would be if the tube were fixed, by the amount due to the centrifugal force arising from the rotation of the system of reference.

If $\mathrm{X} = \mathrm{Y} = 0$, $\mathrm{Z} = -g$, so that the particle is heavy and no force besides gravity acts, then

$$v^2 = v_0^2+\omega^2(r^2-r_0^2)-2g(z-z_0);$$

which gives the velocity of the particle in any position. If the

velocity is the same at all points of the tube so that $v = v_0 =$ a constant, then the preceding equation is that to a paraboloid of revolution; and hence the tube in the form of a generator of this paraboloid satisfies the condition that a particle remains at rest in it, whatever its place may be; the latus rectum of this parabola is $\frac{2g}{\omega^2}$.

405.] Also let us consider the case in which a particle m moves in a tube capable of rotation about a given axis, say that of z or ζ; but in which the angular velocity is not constant. In this case, if ω is the angular velocity of the moving system at the time t, the equations of relative motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi - \frac{1}{\eta}\frac{d.\omega\eta^2}{dt} &= \text{X} + \frac{\text{R}}{m}\cos\alpha, \\ \frac{d^2\eta}{dt^2} - \omega^2\eta + \frac{1}{\xi}\frac{d.\omega\xi^2}{dt} &= \text{Y} + \frac{\text{R}}{m}\cos\beta, \\ \frac{d^2\zeta}{dt^2} &= \text{Z} + \frac{\text{R}}{m}\cos\gamma; \end{aligned}\right\} \qquad (45)$$

where α, β, γ are the direction-angles of the principal normal at the time t, and R is the pressure of the tube on the particle.

In illustration of these equations, I will take the following examples.

Determine the motion of a heavy particle m within a smooth tube, so thin that the mass of it may be neglected in the equations of motion, which is bent into the form of a helix and rotates with a constant velocity ω about the axis of the helix which is vertical, and with which the helix is rigidly connected by means of thin wires.

Let the horizontal plane in which m is at relative rest be that of (x, y), the point where it intersects the axis of the helix being the origin, the axis of z being measured vertically downwards, and the axis of x passing through the initial place of m. Let ϕ be the angle through which the vertical plane, containing the axis of the helix and the place of m, has revolved in the time t. Then the equations to the helix are

$$x = a\cos\phi, \quad y = a\sin\phi, \quad z = a(\phi - \omega t)\tan\alpha;$$

$$\xi = a\cos(\phi - \omega t), \quad \eta = a\sin(\phi - \omega t), \quad \zeta = a(\phi - \omega t)\tan\alpha;$$

and the equations of motion are

$$\frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} = \frac{\text{R}}{m}\cos(\phi - \omega t),$$

$$\frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} = \frac{\text{R}}{m}\sin(\phi - \omega t),$$

$$\frac{d^2\zeta}{dt^2} = g;$$

multiplying these respectively by $d\xi$, $d\eta$, $d\zeta$, adding and integrating, bearing in mind that $\xi^2 + \eta^2 = a^2$, and that the action-line of R lies along the principal normal and is consequently perpendicular to the tangent, we have

$$\frac{ds^2}{dt^2} - a^2\omega^2(\sec\alpha)^2 = 2gz;$$

which is the equation of vis viva.

Also multiplying the first equation by η and the second by ξ and subtracting, we have, after integration,

$$\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt} = a^2\omega;$$

$$\therefore \quad \frac{d\phi}{dt} = 2\omega.$$

So that the angular velocity of m is constant, being twice that of the tube.

406.] Let us take an example in which the angular velocity of the moving tube is not constant, such as is the case in the following problem.

Determine the motion of a heavy particle m within a smooth tube, so thin that M, the mass of it, may be neglected in the equations of motion, which is bent into a helix rotating about a vertical axis, parallel to the axis of the helix and touching it.

Let the horizontal plane in which m is, when all is at rest, be that of (x, y), the point where it intersects the rotation-axis being the origin, and the line passing through the point where m is at rest being the axis of x, and the axis of z being vertical downwards. Let θ be the angle through which the tube has revolved, and $\theta + \phi$ the angle through which the vertical plane containing the place of m and the rotation-axis has revolved, in the time t. Then the coordinates of m at the time t are

$$x = 2a\cos\phi\cos(\theta+\phi), \quad y = 2a\cos\phi\sin(\theta+\phi), \quad z = 2a\phi\tan\alpha;$$

$$\xi = a(1+\cos 2\phi), \quad \eta = a\sin 2\phi, \quad \zeta = 2a\phi\tan\alpha;$$

and the equations of motion are

$$\frac{d^2\xi}{dt^2} - \xi\frac{d\theta^2}{dt^2} - \eta\frac{d^2\theta}{dt^2} - 2\frac{d\eta}{dt}\frac{d\theta}{dt} = \frac{\text{R}}{m}\cos 2\phi,$$
$$\frac{d^2\eta}{dt^2} - \eta\frac{d\theta^2}{dt^2} + \xi\frac{d^2\theta}{dt^2} + 2\frac{d\xi}{dt}\frac{d\theta}{dt} = \frac{\text{R}}{m}\sin 2\phi,$$
$$\frac{d^2\zeta}{dt^2} = g;$$
$$\text{M}k^2\frac{d^2\theta}{dt^2} = 2a\text{R}(\cos\phi)^2.$$

But since M, the mass of the tube, is to be neglected, from the last equation, R $= 0$; and consequently the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \xi\frac{d\theta^2}{dt^2} - \eta\frac{d^2\theta}{dt^2} - 2\frac{d\eta}{dt}\frac{d\theta}{dt} &= 0, \\ \frac{d^2\eta}{dt^2} - \eta\frac{d\theta^2}{dt^2} + \xi\frac{d^2\theta}{dt^2} + 2\frac{d\xi}{dt}\frac{d\theta}{dt} &= 0, \\ \frac{d^2\zeta}{dt^2} &= g. \end{aligned}\right\} \quad (46)$$

$$\therefore\quad \eta\frac{d^2\xi}{dt^2} - \xi\frac{d^2\eta}{dt^2} - (\eta^2+\xi^2)\frac{d^2\theta}{dt^2} - 2\left(\xi\frac{d\xi}{dt} + \eta\frac{d\eta}{dt}\right)\frac{d\theta}{dt} = 0,$$
$$\eta\frac{d\xi}{dt} - \xi\frac{d\eta}{dt} - (\xi^2+\eta^2)\frac{d\theta}{dt} = 0,$$

since $\frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\theta}{dt} = 0$, when $t = 0$; whence replacing ξ and η in terms of ϕ, we have

$$\frac{d\phi}{dt} + \frac{d\theta}{dt} = 0: \qquad \therefore\quad \phi + \theta = 0.$$

This equation shews that the angular velocities of m and of the tube about the vertical rotation-axis are equal but in contrary directions, so that m moves in the fixed vertical plane of (z, x). Also

$$\frac{d\xi d^2\xi + d\eta\, d^2\eta + d\zeta d^2\zeta}{dt^2} - (\xi d\xi + \eta d\eta)\frac{d\theta^2}{dt^2} - (\eta d\xi - \xi d\eta)\frac{d^2\theta}{dt^2} = g\,d\zeta:$$

whence

$$\{(\sec a)^2 - (\cos\phi)^2\}\frac{d^2\phi}{dt^2} + 2\sin\phi\cos\phi\frac{d\phi^2}{dt^2} = \frac{g\tan a}{4a},$$

and integrating,

$$(\sec a)^2\frac{d\phi^2}{dt^2} - (\cos\phi)^2\frac{d\phi^2}{dt^2} = \frac{g\tan a}{2a}\phi.$$
$$\therefore\quad \frac{d\phi^2}{dt^2} = \frac{d\theta^2}{dt^2} = \frac{g\phi\tan a}{2a\{(\sec a)^2 - (\cos\phi)^2\}},$$

which is the equation of relative vis viva; and from which ϕ and θ might be determined in terms of t, were integration possible.

407.] Lastly, let us consider the motion of a heavy particle moving in contact with a surface which rotates with an uniform angular velocity about its ζ-axis, which is vertical and is the fixed z-axis.

Let the equation to the surface be

$$\mathrm{F}(\xi, \eta, \zeta) = 0;$$

of which let the partial derived functions be U, V, W; also let

$$\mathrm{U}^2 + \mathrm{V}^2 + \mathrm{W}^2 = \mathrm{Q}^2;$$

then we may suppose the particle to move in a thin space contained between two parallel surfaces infinitesimally near to each other; in which case the equations of relative motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} &= \frac{\mathrm{R}}{m}\frac{\mathrm{U}}{\mathrm{Q}}, \\ \frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} &= \frac{\mathrm{R}}{m}\frac{\mathrm{V}}{\mathrm{Q}}, \\ \frac{d^2\zeta}{dt^2} \qquad\qquad &= \frac{\mathrm{R}}{m}\frac{\mathrm{W}}{\mathrm{Q}} - g; \end{aligned}\right\} \qquad (47)$$

if we multiply these respectively by $d\xi$, $d\eta$, and $d\zeta$, and add and integrate, we have

$$v^2 - \omega^2(\xi^2 + \eta^2) = -2g\zeta + \mathrm{C};$$

where C is a constant depending on the initial values of the several quantities.

If v is constant, or if the particle remains at rest wherever it is put, so that $v = 0$, this is the equation to a paraboloid of revolution.

If v varies as the distance from the fixed origin, the surface is a quadric surface of revolution.

The following example is in illustration of these equations:

On a rough horizontal whirling table revolving about a vertical axis with a constant velocity ω a heavy particle m is placed, in relative rest: determine its subsequent motion.

Let F be the friction between the particle and the table; then the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - \omega^2\xi - 2\omega\frac{d\eta}{dt} &= -\frac{\mathrm{F}}{m}\frac{d\xi}{ds}, \\ \frac{d^2\eta}{dt^2} - \omega^2\eta + 2\omega\frac{d\xi}{dt} &= -\frac{\mathrm{F}}{m}\frac{d\eta}{ds}; \end{aligned}\right\} \qquad (48)$$

whence by the equation of vis viva,

$$m v^2 - m \omega^2 (r^2 - a^2) = -\text{F} s,$$

if the particle is placed at rest at a distance a from the rotation-axis, and s is the length of the path described by the particle.

Again, taking the normal component

$$\frac{d\xi d^2\eta - d\eta d^2\xi}{ds\, dt^2} - \omega^2 \frac{\eta d\xi - \xi d\eta}{ds} - 2\omega \frac{ds}{dt} = 0$$

$$\frac{v^2}{\rho} - \omega^2 r^2 \frac{d\phi}{ds} - 2\omega \frac{ds}{dt} = 0,$$

where ϕ is the angle between r and the axis of ξ. If t is eliminated between this and the preceding equation, we have an equation in terms of geometrical quantities only which is that to the path of the particle on the table.

408.] The principles and equations of the preceding articles are applicable to the solution of a problem of considerable interest; viz. to the motion of a particle, either free or constrained, near to the earth's surface, relative to a system of axes originating on the earth's surface and moving with it.

We may without error assume the centre of gravity of the earth to be fixed, if we impress forces on the moving particle which are equal to the excess of those which act on it over those which act on the earth at its centre of gravity: but as the sun, which is the main force acting on the earth, impresses velocity-increments nearly equal on both the earth and the particle, we may suppose this excess, either positive or negative, to be so small that it may be neglected without sensible error. We may also suppose the position of the rotation-axis of the earth to be fixed and the angular velocity to be constant.

The two systems of axes are imagined to have that arrangement which is drawn in Fig. 63. O is the centre of the earth; the axis of z is measured from O towards C the north pole; the axes of x and y are taken in the plane of the equator. Let ω be the angular velocity of the earth, with which indeed the earth rotates from the y-axis to the x-axis: it will be convenient however to take it at present in the contrary direction, and to change the sign in the final equations, ere we apply them to the particular problem.

Let P be the place of observation, and let us suppose it to be in the northern hemisphere of the earth. Let P be the origin of the moving system of rectangular axes to which the motion of

m is referred: let the axis of ζ be the vertical line at P measured away from the earth towards the zenith of P; this may be assumed, without sensible error, to pass through the earth's centre. Let axes of ξ and η be in the horizontal plane at P, and be respectively N and S, and E and W; the positive direction of ξ being taken towards the south, and that of η towards the west. Let the latitude of P, viz. POQ, $= \lambda$. Let the plane of the meridian of P, when $t = 0$, be that of (x, z); and let the earth's radius be r. Then MON $= \omega t$: and

$$\left.\begin{aligned} \text{OM} &= x_0 = r\cos\lambda\cos\omega t, \\ \text{MN} &= y_0 = r\cos\lambda\sin\omega t, \\ \text{NP} &= z_0 = r\sin\lambda; \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore\quad \frac{d^2 x_0}{dt^2} &= -\omega^2 r\cos\lambda\cos\omega t, \\ \frac{d^2 y_0}{dt^2} &= -\omega^2 r\cos\lambda\sin\omega t, \\ \frac{d^2 z_0}{dt^2} &= 0. \end{aligned}\right\} \qquad (49)$$

Also resolving ω along the axes of ξ, η, ζ, we have

$$\omega_\xi = -\omega\cos\lambda, \qquad \omega_\eta = 0, \qquad \omega_\zeta = \omega\sin\lambda; \qquad (50)$$

of which the first is the component about the line running due S and N in the horizontal plane, and is the only component in that plane: and the last is the component about the vertical at P.

Now if the place of m at the time t is (x, y, z) relatively to the fixed axes, and is (ξ, η, ζ) relatively to the moving axes which originate at P; then

$$\left.\begin{aligned} x &= r\cos\lambda\cos\omega t + \xi\sin\lambda\cos\omega t - \eta\sin\omega t + \zeta\cos\lambda\cos\omega t, \\ y &= r\cos\lambda\sin\omega t + \xi\sin\lambda\sin\omega t + \eta\cos\omega t + \zeta\cos\lambda\sin\omega t, \\ z &= r\sin\lambda - \xi\cos\lambda + \zeta\sin\lambda. \end{aligned}\right\} \qquad (51)$$

On comparing these with (2), Art. 2, we have

$$\left.\begin{aligned} a_1 &= \sin\lambda\cos\omega t, & b_1 &= -\sin\omega t, & c_1 &= \cos\lambda\cos\omega t, \\ a_2 &= \sin\lambda\sin\omega t, & b_2 &= \cos\omega t, & c_2 &= \cos\lambda\sin\omega t, \\ a_3 &= -\cos\lambda; & b_3 &= 0; & c_3 &= \sin\lambda; \end{aligned}\right\} \qquad (52)$$

and differentiating,

$$\left.\begin{aligned} \frac{d^2 a_1}{dt^2} &= -\omega^2\sin\lambda\cos\omega t, & \frac{d^2 b_1}{dt^2} &= \omega^2\sin\omega t, & \frac{d^2 c_1}{dt^2} &= -\omega^2\cos\lambda\cos\omega t, \\ \frac{d^2 a_2}{dt^2} &= -\omega^2\sin\lambda\sin\omega t, & \frac{d^2 b_2}{dt^2} &= -\omega^2\cos\omega t, & \frac{d^2 c_2}{dt^2} &= -\omega^2\cos\lambda\sin\omega t, \\ \frac{d^2 a_3}{dt^2} &= 0; & \frac{d^2 b_3}{dt^2} &= 0; & \frac{d^2 c_3}{dt^2} &= 0; \end{aligned}\right\}$$

and substituting these quantities in (21), (22), and (23), Art. 395, the force of transference has the following components, viz.,

$$\left.\begin{aligned} X_t &= -\omega^2 r\sin\lambda\cos\lambda - \xi\omega^2(\sin\lambda)^2 - \zeta\omega^2\sin\lambda\cos\lambda, \\ Y_t &= -\eta\,\omega^2, \\ Z_t &= -\omega^2 r(\cos\lambda)^2 - \xi\omega^2\sin\lambda\cos\lambda - \zeta\omega^2(\cos\lambda)^2; \end{aligned}\right\} \quad (53)$$

and substituting from (50) in (24), the components of the compound centrifugal force are

$$\left.\begin{aligned} F\cos\alpha &= -2\,\omega\sin\lambda\frac{d\eta}{dt}, \\ F\cos\beta &= 2\Big(\omega\sin\lambda\frac{d\xi}{dt} + \omega\cos\lambda\frac{d\zeta}{dt}\Big), \\ F\cos\gamma &= -2\,\omega\cos\lambda\frac{d\eta}{dt}. \end{aligned}\right\} \quad (54)$$

When these several quantities are substituted in the equations of motion given in (29), these last equations become

$$\left.\begin{aligned} &\frac{d^2\xi}{dt^2} - \omega^2 r\sin\lambda\cos\lambda - \xi\omega^2(\sin\lambda)^2 - \zeta\omega^2\sin\lambda\cos\lambda - 2\,\omega\sin\lambda\frac{d\eta}{dt} = X', \\ &\frac{d^2\eta}{dt^2} - \eta\omega^2 + 2\,\omega\Big(\sin\lambda\frac{d\xi}{dt} + \cos\lambda\frac{d\zeta}{dt}\Big) = Y', \\ &\frac{d^2\zeta}{dt^2} - \omega^2 r(\cos\lambda)^2 - \xi\omega^2\sin\lambda\cos\lambda - \zeta\omega^2(\cos\lambda)^2 - 2\,\omega\cos\lambda\frac{d\eta}{dt} = Z'. \end{aligned}\right\} \quad (55)$$

These equations may be deduced directly from (51) without the intervention of the general process, which has been investigated in the preceding Articles. For we may take the second t-differentials of x, y, and z, and equate the sum of their several components along the axes of ξ, η, ζ to the impressed velocity-increments acting along those axes. In particular problems this is the most convenient method.

409.] In reference to (53) it is to be observed that they are evidently the axial components along the moving axes of the acceleration due to the so-called centrifugal force which the particle m would have by virtue of the earth's rotation, if it were at relative rest on the earth in the place which it has at the time t. For suppose ρ to be the perpendicular distance from the place of m at the time t on the earth's rotation-axis, then $\rho\omega^2$ is the expressed normal acceleration due to the earth's rotation. Let this, as acting on m at (ξ, η, ζ), be resolved in and perpendicular to the plane passing through P and the earth's axis; then these components are respectively $\omega^2(r\cos\lambda + \xi\sin\lambda + \zeta\cos\lambda)$ and

$\omega^2\eta$; and of the former the ξ- and ζ-axial components are respectively

$$\omega^2(r\cos\lambda+\xi\sin\lambda+\zeta\cos\lambda)\sin\lambda,$$

and
$$\omega^2(r\cos\lambda+\xi\sin\lambda+\zeta\cos\lambda)\cos\lambda;$$

and these are X_t, Y_t, and Z_t as given in (53).

It is evident from first principles that these are the effects due to the force of transference. Similarly (54) are evidently the effects of the compound centrifugal force.

410.] To adapt these equations to the actual circumstances of the earth, the sign of ω must be changed, because the earth revolves from west to east, which is a direction opposite to that assumed in the preceding Articles. To determine its value, we will take a second for the unit of time; then, since a mean sidereal day contains 86164.09 seconds,

$$\omega = \frac{2\pi}{86164.09} = \frac{1}{13713} = .00007292,$$

which is a small fraction; and consequently ω^2, which enters into the preceding equations, is a very small quantity. Also, in the problems to which we shall apply the equations, ξ, η, ζ will be always very small fractions of the earth's radius; and thus we may for a first approximate solution of a problem, without sensible error, neglect those terms in the left-hand members of the equations which involve products of these coordinates and of ω^2; and the equations become

$$\left.\begin{aligned}
\frac{d^2\xi}{dt^2}-\omega^2 r\sin\lambda\cos\lambda+2\omega\sin\lambda\frac{d\eta}{dt} &= X',\\
\frac{d^2\eta}{dt^2}-2\omega\left(\sin\lambda\frac{d\xi}{dt}+\cos\lambda\frac{d\zeta}{dt}\right) &= Y',\\
\frac{d^2\zeta}{dt^2}-\omega^2 r(\cos\lambda)^2+2\omega\cos\lambda\frac{d\eta}{dt} &= Z';
\end{aligned}\right\} \qquad (56)$$

where X', Y', Z' are the components along the moving axes of all the absolute velocity-increments impressed on m.

411.] Now I propose to apply these equations in the first place to the motion of a particle projected with a given velocity and in a given direction from P, the place of observation, which is also the origin of the moving system of axes. Although the power of our weapons of projection has been very greatly increased of late, yet still, for all points of the path, ξ, η, ζ are but small fractions of the earth's radius; consequently $\omega^2\xi$, $\omega^2\eta$, $\omega^2\zeta$

are small quantities which we may omit, and (56) are applicable to practical problems in gunnery practice.

In the right-hand members, for the same reasons, I shall assume the earth's attraction to be the same at all points of the particle's path, and to be what it is at P, the place of observation. Although gravity varies at different points of the earth's surface, according to a law which is accordant with Clairaut's theorem, yet I shall take it to be the same at all latitudes; and no sensible error will, within the compass of our approximations, thereby be introduced into the results. I shall also consider the projectile to move in vacuo, and shall consequently neglect the resistance of the air. Thus the particle moves under the action of the earth's attraction, and the centrifugal force due to the rotation of the earth, the resultant of these being that force which is commonly known as gravity at a particular place; hence

$$\left.\begin{aligned} X' &= -\omega^2 r \sin\lambda\cos\lambda, \\ Y' &= 0, \\ Z' &= -\omega^2 r (\cos\lambda)^2 - g; \end{aligned}\right\} \tag{57}$$

and the equations of motion (56) become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + 2\omega\sin\lambda\frac{d\eta}{dt} &= 0, \\ \frac{d^2\eta}{dt^2} - 2\omega\left(\sin\lambda\frac{d\xi}{dt} + \cos\lambda\frac{d\zeta}{dt}\right) &= 0, \\ \frac{d^2\zeta}{dt^2} + 2\omega\cos\lambda\frac{d\eta}{dt} &= -g. \end{aligned}\right\} \tag{58}$$

If $\omega = 0$, these equations express the ordinary case of a projectile's motion.

Now (58) admit of integration. Let $u =$ the velocity of projection, and let α, β, γ be the direction-angles of the line of projection in reference to the moving axes;

$$\left.\begin{aligned} \therefore\quad \frac{d\xi}{dt} + 2\omega\eta\sin\lambda &= u\cos\alpha, \\ \frac{d\eta}{dt} - 2\omega(\xi\sin\lambda + \zeta\cos\lambda) &= u\cos\beta, \\ \frac{d\zeta}{dt} + 2\omega\eta\cos\lambda &= u\cos\gamma - gt; \end{aligned}\right\} \tag{59}$$

which assign the components of the velocity at any point of the path.

Again, if we substitute for $\frac{d\xi}{dt}$ and $\frac{d\zeta}{dt}$ from the first and last

of these equations in the second of (58), and omit the terms involving the product of ω^2 and of one of the relative coordinates, then we have

$$\frac{d^2\eta}{dt^2} - 2u\omega(\cos\alpha\sin\lambda + \cos\gamma\cos\lambda) + 2\omega g t\cos\lambda = 0;$$

therefore

$$\eta = ut\cos\beta + u\omega(\cos\alpha\sin\lambda + \cos\gamma\cos\lambda)t^2 - \omega g\cos\lambda\frac{t^3}{3}; \quad (60)$$

and replacing η in the first and last of (59) by this value, and omitting terms involving products of ω^2 and of one of the relative coordinates, and integrating,

$$\xi = ut\cos\alpha - u\omega\sin\lambda\cos\beta\, t^2, \quad (61)$$

$$\zeta = ut\cos\gamma - \left(\frac{g}{2} + u\omega\cos\lambda\cos\beta\right)t^2; \quad (62)$$

which three equations express the motion of the projectile to the degree of approximation attainable by the preceding equations of motions.

From these values of ξ and η, the place of the particle at the time t may be found in reference to any other system of axes in the horizontal plane.

If $\omega = 0$, the results are the same as those which have already been found in Art. 349, Vol. III; viz.,

$$\xi = ut\cos\alpha, \qquad \eta = ut\cos\beta, \qquad \zeta = ut\cos\gamma - \frac{1}{2}g t^2.$$

On comparing these quantities with the preceding, the variation of the range on the horizontal plane, and also the deviation, as due to the earth's rotation, can easily be calculated; generally however it appears that if the particle or ball is projected from a place in the northern hemisphere, in a direction westwards of the meridian, both the vertical height of it and its distance southwards from the parallel of latitude are diminished by the earth's rotation; and that if it is projected eastwards of the meridian, that is, in the direction in which the earth is going, both these quantities are increased. As to the three terms of which η consists, only the first, viz., $ut\cos\beta$, depends on the line of projection being eastwards or westwards; and consequently the increase or diminution of η will depend also on the sign of the other two terms which involve t.

The apparent path of the projectile may be determined by the elimination of t; which will give the equations to two surfaces,

the line of intersection of which is its path: it is evident that the path will generally be a curve of double curvature.

412.] Let us however consider certain particular cases and results of these equations.

(1) Let the body fall down a mine, or from the top of a tower, without any initial velocity; then

$$u = 0; \qquad \cos\alpha = \cos\beta = 0; \qquad \cos\gamma = -1;$$

$$\therefore \quad \xi = 0, \qquad \eta = -\omega g \cos\lambda \frac{t^3}{3}, \qquad \zeta = -\frac{1}{2} g t^2. \qquad (63)$$

The first equation shews that there is no deviation in the line of the meridian; from the second we infer a deviation towards the east; that is, in the direction towards which the earth is moving, which varies as the cube of the time of falling; and that this deviation is greatest at the equator, where $\lambda = 0$; and the last equation shews that the earth's rotation does not produce any alteration in the vertical motion.

If we eliminate t, and take ζ downwards to be positive,

$$\eta^2 = -\frac{8\,\omega^2 (\cos\lambda)^2}{9g} \zeta^3;$$

which is the equation to a semicubical parabola; and shews that the square of the deviation towards the east varies as the cube of the space through which the particle has fallen.

(2) Let the particle be projected vertically upwards; then $\cos\alpha = \cos\beta = 0$; $\cos\gamma = 1$; and

$$\xi = 0, \quad \eta = u\omega \cos\lambda t^2 - \omega g \cos\lambda \frac{t^3}{3}, \quad \zeta = ut - \frac{1}{2} g t^2; \quad (64)$$

the last equation shews that the vertical motion is the same as it would be if the earth did not rotate; and consequently if h is the height to which the particle ascends, and T is the whole time of ascent and descent, $u^2 = 2gh$, and $\text{T} = \frac{2u}{g}$. The first equation shews that there is no deviation in the line of the meridian; the second shews that the deviation along the parallel of latitude is westwards when t is less than $\frac{3u}{g}$; that it vanishes when $t = \frac{3u}{g}$; and is eastwards when t is greater than $\frac{3u}{g}$; but as the greatest value of T is $\frac{2u}{g}$ (unless the particle, after having descended to its original horizontal plane, continues to fall),

the deviation along the parallel of latitude is always westwards. When the particle, after its ascent, strikes the earth,

$$\eta = \frac{4\omega u^3 \cos\lambda}{3g^2};$$

which is the deviation westwards of the point of impact on the ground; and varies as the cube of the velocity of projection.

(3) Let the particle be projected due westwards at an angle of elevation equal to θ; then

$$\cos\alpha = 0, \quad \cos\beta = \cos\theta, \quad \cos\gamma = \sin\theta;$$

and

$$\left.\begin{aligned} \xi &= -u\omega \sin\lambda \cos\theta\, t^2, \\ \eta &= ut\cos\theta + u\omega\sin\theta\cos\lambda\, t^2 - \omega g\cos\lambda \frac{t^3}{3}, \\ \zeta &= ut\sin\theta - \left(\frac{g}{2} + u\omega\cos\lambda\cos\theta\right) t^2; \end{aligned}\right\} \quad (65)$$

the first of which equations shews that the projectile generally deviates northwards; when the projectile strikes the ground, $\zeta = 0$; in which case

$$t = \frac{2u\sin\theta}{g + 2u\omega\cos\theta\cos\lambda} = \frac{2u\sin\theta}{g}\left\{1 - \frac{2u\omega}{g}\cos\theta\cos\lambda\right\},$$

omitting those terms which involve ω^2: in this case

$$\xi = -\frac{4u^3\omega\sin\lambda(\sin\theta)^2\cos\theta}{g^2}, \quad (66)$$

$$\eta = \frac{u^2}{g}\sin 2\theta + \frac{4u^3\omega\cos\lambda}{3g^2}\{(\sin\theta)^2 - 3(\cos\theta)^2\}; \quad (67)$$

which are the approximate coordinates of the point of impact on the ground. The terms involving ω denote the effects due to the earth's rotation: the former gives the deviation northwards; and the latter shews that the range measured westward is increased or diminished according as θ is greater or less than 60°.

(4) If the particle is projected due eastwards at an angle of elevation equal to θ, all the preceding results are true if we replace θ by $180° - \theta$; so that (66) and (67) become

$$\xi = \frac{4u^3\omega\sin\lambda(\sin\theta)^2\cos\theta}{g^2}; \quad (68)$$

$$\eta = -\frac{u^2}{g}\sin 2\theta + \frac{4u^3\omega\cos\lambda}{3g^2}\{(\sin\theta)^2 - 3(\cos\theta)^2\}; \quad (69)$$

so that in this case the deviation of the projectile is southwards; and the range is increased or diminished according as the angle of elevation is less than or greater than 60°.

(5) Let the particle be projected due southwards at an angle of elevation equal to θ; then

$$\cos\alpha = \cos\theta, \qquad \cos\beta = 0, \qquad \cos\gamma = \sin\theta;$$

and

$$\left.\begin{aligned} \xi &= ut\cos\theta, \\ \eta &= u\omega\sin(\theta+\lambda)t^2 - \omega g\cos\lambda\frac{t^3}{3}, \\ \zeta &= ut\sin\theta - \frac{gt^2}{2}; \end{aligned}\right\} \tag{70}$$

from the first and the last of these equations we infer, that neither the time nor the range on the meridian is altered by the rotation of the earth. But when $\zeta = 0$, that is, when the projectile strikes the ground, $t = \frac{2u\sin\theta}{g}$; in which case

$$\eta = \frac{4u^3\omega(\sin\theta)^2}{3g^2}\{\sin\theta\cos\lambda + 3\cos\theta\sin\lambda\}; \tag{71}$$

and therefore the point where the projectile strikes the ground is always west of the meridian.

If θ is replaced by $180° - \theta$, we have the case where the particle is projected due northwards at an elevation of θ.

Now we shall hereafter prove that these results, which have herein been applied to the motion of a material particle, are also true of that of the centre of gravity of a body. Neglecting therefore the resistance of the air, and the action due to the rotation of a ball or bolt, we have the following results as to rifle and cannon practice:

When the shot is fired due north or south, the range in that direction is not altered; but there is a deviation of the shot, the value of which at the point of impact on the ground is given in (71); and this deviation is westwards, vanishes, or is eastwards according as θ is less than, équal to, or greater than

$$180° - \tan^{-1}(3\tan\lambda).$$

When the shot is fired due east, the range eastwards is increased or diminished according as the angle of elevation of the gun is less than or greater than 60°; and there is deviation southwards for all places in the northern hemisphere, and northwards for all places in the southern hemisphere, the value of which is given in (68).

When the shot is fired due west, the range is increased or diminished according as the angle of elevation is greater than or less than 60°; and there is a deviation northwards for all

places in the northern hemisphere, and southwards for all places in the southern hemisphere.

So that for firing from a place in a direction coincident with the parallel of latitude, and with an elevation less than 60°, the range is increased or diminished according as we fire eastwards or westwards; and the difference between the two ranges

$$= \frac{8\,u^3\omega\cos\lambda}{3g^2}\{3(\cos\theta)^2-(\sin\theta)^2\};$$

and if the place is in the northern hemisphere, the deviation parallel to the meridian is north or south, according as we fire west or east.

And for places in the northern hemisphere for all directions lying west of the meridian, the deviation parallel to the meridian is northwards; and for all directions lying east of the meridian, the deviation parallel to the meridian is southwards.

413.] The expressions (60), (61), and (62), which have been explained in the preceding Article, are deduced from equations of motion, whose form is simplified on the assumption that products of ω^2 and one of the relative coordinates of m are small quantities, and are to be neglected. Let us now retain these quantities in the equations of motion, and assume that products of ω^3 and of a small variable are to be neglected; and that all small quantities of a lower order are to be retained. In this case the equations of motion are

$$\left.\begin{aligned}
&\frac{d^2\xi}{dt^2}-\xi\omega^2(\sin\lambda)^2-\zeta\omega^2\sin\lambda\cos\lambda+2\omega\sin\lambda\frac{d\eta}{dt}=0,\\
&\frac{d^2\eta}{dt^2}-\eta\omega^2-2\omega\left(\sin\lambda\frac{d\xi}{dt}+\cos\lambda\frac{d\zeta}{dt}\right)\qquad\qquad=0,\\
&\frac{d^2\zeta}{dt^2}-\xi\omega^2\sin\lambda\cos\lambda-\zeta\omega^2(\cos\lambda)^2+2\omega\cos\lambda\frac{d\eta}{dt}=-g.
\end{aligned}\right\}\qquad(72)$$

Of these equations, the values of ξ, η, ζ, given in (60), (61), (62) are approximate solutions of the first order, and may be employed to find approximate solutions of (72).

In the second of (72), in the term $\omega^2\eta$, let η be replaced by $ut\cos\beta$ from (60); then integrating, we have

$$\frac{d\eta}{dt}-u\cos\beta-u\omega^2\cos\beta\frac{t^2}{2}-2\omega(\xi\sin\lambda+\zeta\cos\lambda)=0;\qquad(73)$$

and substituting for ξ and ζ their values given in (61) and (62), and integrating again,

$$\eta = ut\cos\beta + u\omega(\cos a\sin\lambda + \cos\gamma\cos\lambda)t^2 \\ -\omega g\cos\lambda\frac{t^3}{3} - u\omega^2\cos\beta\frac{t^3}{2}. \qquad (74)$$

Again, in the first and third of (72), in the terms involving $\omega^2\xi$ and $\omega^2\zeta$, let ξ and ζ be respectively replaced by

$$ut\cos a, \quad \text{and} \quad ut\cos\gamma - \frac{1}{2}gt^2;$$

then integrating, we have

$$\frac{d\xi}{dt} - u\cos a - u\omega^2\sin\lambda(\cos a\sin\lambda + \cos\gamma\cos\lambda)\frac{t^2}{2} \\ + \omega^2 g\sin\lambda\cos\lambda\frac{t^3}{6} + 2\omega\sin\lambda\,\eta = 0;$$

$$\frac{d\zeta}{dt} - u\cos\gamma - u\omega^2\cos\lambda(\cos a\sin\lambda + \cos\gamma\cos\lambda)\frac{t^2}{2} \\ + \omega^2 g(\cos\lambda)^2\frac{t^3}{6} + 2\omega\cos\lambda\,\eta = -gt;$$

substituting in the last terms of these the value of η, given in (74), and integrating, we have

$$\xi = ut\cos a - u\omega\sin\lambda\cos\beta t^2 \\ - u\omega^2\sin\lambda(\cos a\sin\lambda + \cos\gamma\cos\lambda)\frac{t^3}{2} + g\omega^2\sin\lambda\cos\lambda\frac{t^4}{8}; \qquad (75)$$

$$\zeta = ut\cos\gamma - \frac{1}{2}gt^2 - u\omega\cos\lambda\cos\beta t^2 \\ - u\omega^2\cos\lambda(\cos a\sin\lambda + \cos\gamma\cos\lambda)\frac{t^3}{2} + g\omega^2(\cos\lambda)^2\frac{t^4}{8}; \qquad (76)$$

which expressions for ξ, η, ζ are correct as far as terms involving ω^2 inclusive.

414.] Explanations might be given of particular cases of these equations, similar to those of the last Article. I will only take two cases:

(1) Let the body fall without any initial velocity; then $u = 0$, $\cos a = \cos\beta = 0$; $\cos\gamma = -1$;

$$\left.\begin{aligned} \xi &= \omega^2 g\sin\lambda\cos\lambda\frac{t^4}{8}, \\ \eta &= -\omega g\cos\lambda\frac{t^3}{3}, \\ \zeta &= -\frac{1}{2}gt^2 + \omega^2 g(\cos\lambda)^2\frac{t^4}{8}. \end{aligned}\right\}$$

The first equation shews that there is a deviation of the falling particle in the plane of the meridian towards the south; and the

second shews that the deviation in the parallel of latitude is towards the east; so that the resulting deviation of the falling body is towards the south-east. From the last equation it appears, that the space due to a given time is less than it would be if there were no rotation. Hereby then we have corrections of the results explained in the first case of the preceding Article.

(2) Let the body be projected due southwards at an angle of elevation equal to θ, so that $\cos\alpha = \cos\theta$; $\cos\beta = 0$; $\cos\gamma = \sin\theta$; then

$$\left.\begin{aligned}
\xi &= ut\cos\theta - u\,\omega^2\sin\lambda\sin(\lambda+\theta)\frac{t^3}{2} + g\,\omega^2\sin\lambda\cos\lambda\,\frac{t^4}{8},\\
\eta &= u\,\omega\sin(\lambda+\theta)\,t^2 - \omega\,g\cos\lambda\,\frac{t^3}{3},\\
\zeta &= ut\sin\theta - \frac{gt^2}{2} - u\,\omega^2\cos\lambda\sin(\lambda+\theta)\frac{t^3}{2} + g\,\omega^2(\cos\lambda)^2\,\frac{t^4}{8};
\end{aligned}\right\}$$

when the projectile strikes the ground, $\zeta = 0$; and approximately $t = \dfrac{2u\sin\theta}{g}$; in which case

$$\eta = \frac{4u^3\omega(\sin\theta)^2}{3g^2}\{\sin\theta\cos\lambda + 3\cos\theta\sin\lambda\};$$

so that the deviation along the parallel of latitude is westwards.

In the investigation of this problem, given by M. Poisson, Journal de l'Ecole Polytechnique, Cahier 26, p. 1, terms are introduced representing the resistance of the air. The equations, thus enlarged, do not admit of direct integration; the effect however of the resistance of the air is determined by the method of variation of parameters. The student desirous of knowing the extent to which mathematical analysis has been applied to balistics, must consult the memoirs of M. Poisson, contained in Cahiers 26, 27 of the aforesaid Journal.

415.] Although the following is a particular case of the general problem of projection of a heavy particle as treated in the preceding Articles and has been solved as such, yet it is of so much interest that it is desirable to give an independent consideration to it. The problem is this;

Determine the motion of a heavy particle falling from rest from a height h to the earth, taking account of the earth's rotation.

The equations of motion are (58); and as, when $t = 0$,

$$\frac{d\xi}{dt} = \frac{d\eta}{dt} = \frac{d\zeta}{dt} = 0; \quad \xi = \eta = 0, \quad \zeta = h;$$

$$\left.\begin{aligned} \therefore\quad & \frac{d\xi}{dt} + 2\omega\eta\sin\lambda = 0, \\ & \frac{d\eta}{dt} - 2\omega\{\xi\sin\lambda + (\zeta - h)\cos\lambda\} = 0, \\ & \frac{d\zeta}{dt} + 2\omega\eta\cos\lambda = -gt; \end{aligned}\right\} \qquad (77)$$

substituting the first and last of these in the second of (58), and omitting terms involving ω^2, we have

$$\frac{d^2\eta}{dt^2} + 2\omega g\cos\lambda\, t = 0;$$

$$\therefore\quad \eta = -\frac{\omega g\cos\lambda}{3}t^3;$$

and from the first and the last of (77), omitting the terms involving ω^2, we have

$$\zeta = h - \frac{1}{2}gt^2,$$

$$\xi = 0;$$

so that to the degree of approximation we have taken, the vertical motion of the particle is the same as if the earth did not rotate; no deviation takes place in the plane of the meridian, and the horizontal deviation is towards the east, and varies as the cube of the time during which the body has been falling. Since the time due to the height h is $\left(\frac{2h}{g}\right)^{\frac{1}{2}}$, the deviation towards the east of the point where the body strikes the ground

$$= \frac{2^{\frac{3}{2}}\omega h^{\frac{3}{2}}\cos\lambda}{3g^{\frac{1}{2}}},$$

and varies therefore as the square root of the cube of the height from which the body has fallen.

These results are in accordance with the statements made in Art. 322, Vol. III, respecting the motion of a particle falling from the top of a vertical tower.

The student desirous of further information on the subject of these Articles, in addition to the Memoirs of M. Poisson already alluded to, will consult with advantage (1) Benzenberg, Versuche über das Gesetz des Falls, &c., Dortmund, 1804; (2) G. L. Houel, De deviatione Meridionali corporum libere cadentium, &c., Utrecht, 1839. In both these treatises he will find the investi-

gations of Gauss, in which the resulting equations are carried to an approximation involving higher powers of ω than the second. In the latter too he will find an account of the experiments made by M. Reich in a mine near to Freiberg, in Saxony, in the year 1833.

416.] We can also by means of these equations investigate the oscillations of a pendulum, when its motion is affected by the rotation of the earth. And we shall arrive at the results which M. Foucault exhibited in his famous pendulum experiment before the Academy of Sciences in Paris on Feb. 3rd, 1851; and which have been repeated, and confirmed, in many parts of the earth.

We shall hereby have another ocular proof of the diurnal rotation of the earth; and perhaps a more striking one than any that had formerly existed; for our process will shew that the observed results are in accordance with the physical laws which cause them.

It will be convenient to make a slight change in the moving system of reference, and to take the point of suspension of the pendulum for the moving origin: let the axis of ζ be taken vertically downwards from it, so that the sign of it must be changed in the preceding equations; the axes of ξ and η being taken respectively southwards and westwards as heretofore; and let h be the vertical distance of the point of suspension from the earth's surface.

We shall assume the pendulum to be perfect; and shall take l to be its length, that is, to be the distance of the bob, considered as a particle of mass m, from the point of suspension.

Let (ξ, η, ζ) be the place of its bob at the time t; then

$$\xi^2+\eta^2+\zeta^2 = l^2; \tag{78}$$

and let the tension along the rod of the pendulum $= \text{T}$; let the components of T be introduced into the equations of motion (55); and let X′, Y′, Z′ be the axial components of the other impressed velocity-increments; then the equations of motion are

$$\frac{d^2\xi}{dt^2}-\omega^2\xi(\sin\lambda)^2+(\zeta-l)\,\omega^2\sin\lambda\cos\lambda+2\,\omega\sin\lambda\frac{d\eta}{dt} = -\frac{\text{T}}{m}\frac{\xi}{l},$$

$$\frac{d^2\eta}{dt^2}-\omega^2\eta-2\,\omega\left(\sin\lambda\frac{d\xi}{dt}-\cos\lambda\frac{d\zeta}{dt}\right) = -\frac{\text{T}}{m}\frac{\eta}{l},$$

$$\frac{d^2\zeta}{dt^2}+\omega^2\xi\sin\lambda\cos\lambda-(\zeta-l)\,\omega^2(\cos\lambda)^2-2\,\omega\cos\lambda\frac{d\eta}{dt} = -\frac{\text{T}}{m}\frac{\zeta}{t}+g.$$

These equations represent accurately the motion of the pendulum; but as they do not admit of complete integration, we must have recourse to methods of approximation, as in the preceding Articles. We shall suppose the extent of oscillation to be very small, so that ξ, η, and $l-\zeta$ are always small quantities; and as ω^2 is a very small fraction, we shall neglect products of them and it; that is, the effects due to the force of transference are so small, that they may be omitted, but those due to the compound centrifugal force have an appreciable value, and are to be retained: thus the equations become

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2}+2\omega\sin\lambda\frac{d\eta}{dt} &= -\frac{\text{T}}{m}\frac{\xi}{l}, \\ \frac{d^2\eta}{dt^2}-2\omega\left(\sin\lambda\frac{d\xi}{dt}-\cos\lambda\frac{d\zeta}{dt}\right) &= -\frac{\text{T}}{m}\frac{\eta}{l}, \\ \frac{d^2\zeta}{dt^2}-2\omega\cos\lambda\frac{d\eta}{dt} &= -\frac{\text{T}}{m}\frac{\zeta}{l}+g. \end{aligned}\right\} \qquad (79)$$

417.] Various methods have been chosen by different mathematicians of dealing with these equations. If the rotation of the earth is neglected, $\omega = 0$, and the equations become those which express the motion of a conical pendulum, and which have already been discussed in Articles 440 and 441 of Vol. III. We may take the solution of these simplified equations to be *in form* the solution of our actual equations; the former will contain four undetermined constants depending on the initial values of the velocity and coordinates of the place of the bob of the pendulum; these constants may be considered variable, according to Lagrange's method of variation of parameters; and the differential equations of motion will enable us to determine these in terms of the time, whereby we shall obtain variable elements, which will at any time fix the position of the place of the pendulum. This method has been adopted by M. Quet, in a memoir of great ability in Liouville's Journal, Vol. XVIII. Paris, 1853. Other mathematicians have followed the same process under a different form: they have considered the terms involving ω to arise from a certain disturbing function, the ξ-, η-, ζ-partial differentials of which are severally,

$$-2\omega\sin\lambda\frac{d\eta}{dt}, \quad 2\omega\left(\sin\lambda\frac{d\xi}{dt}-\cos\lambda\frac{d\zeta}{dt}\right), \quad 2\omega\cos\lambda\frac{d\eta}{dt};$$

and then they have pursued the method indicated by Sir W. R.

Hamilton and Jacobi. This process has been developed by M. Dumas in an Academical Dissertation, "De Motu Penduli Sphærici rotatione Terræ perturbato," Königsberg, March, 1854, in which the results are expressed in terms of the higher elliptic transcendents.

Again, other mathematicians have adopted a method of approximation depending on the successive omission of small terms. The original investigation of M. Binet* was made on this principle; and it has subsequently been applied by Hansen, "Theorie der Pendelbewegung," Danzig, 1853. I have treated the equations in the following Articles by this process, because it is the most simple and the most natural, and indicates the principal results of the equations with the least labour.

418.] Let the equations (79) be multiplied respectively by $2d\xi$, $2d\eta$, $2d\zeta$, and added; then, since by (78),

$$\xi d\xi + \eta\, d\eta + \zeta d\,\zeta = 0, \tag{80}$$

we have

$$d.\left\{\frac{d\xi^2 + d\eta^2 + d\zeta^2}{dt^2}\right\} = 2g\,d\zeta; \tag{81}$$

which is the equation of vis viva, and from which the effects of the compound centrifugal force have disappeared, because that force does no work along the tangent, to which its line of action is perpendicular.

Again, if we multiply the second of (79) by ξ, and the first by η, and subtract the latter from the former, we have

$$d.\left\{\frac{\xi d\eta - \eta\, d\xi}{dt}\right\} - \omega \sin\lambda\, d\,(\xi^2 + \eta^2) + 2\,\omega \cos\lambda\, \xi d\,\zeta = 0; \tag{82}$$

which is the equation of moments on the horizontal plane.

Now let us refer the place of the pendulum at the time t to the horizontal plane at the place of observation and to a vertical line which passes through the point of suspension. Let θ be the angle between this vertical line and the rod of the pendulum, and let ψ be the angle at which the vertical plane, in which the pendulum is at the time t, is inclined to the plane of (ξ, ζ), which is the meridian plane; ψ increasing positively as we move from the ξ-axis towards the η-axis, that is, as we revolve from south westwards, and on northwards, and so on towards the east: that

* See Comptes Rendus de l'Académie des Sciences de Paris, 1851, p. 197.

is, in a direction opposite to that in which the earth rotates. Also let ρ be the perpendicular distance from the bob of the pendulum to the vertical line through the point of suspension. Then if the path described by the bob is projected on the horizontal plane, ρ and ψ are the polar coordinates of it, the pole being the point directly beneath the point of suspension: thus we have

$$\left.\begin{array}{ll}\rho = l\sin\theta, & \zeta = l\cos\theta, \\ \xi = \rho\cos\psi = l\sin\theta\cos\psi, & \eta = \rho\sin\psi = l\sin\theta\sin\psi;\end{array}\right\} \quad (83)$$

$$\therefore \quad d\xi^2 + d\eta^2 + d\zeta^2 = l^2\{(d\theta)^2 + (\sin\theta)^2(d\psi)^2\}, \quad (84)$$

$$\xi\, d\eta - \eta\, d\xi = l^2(\sin\theta)^2 d\psi; \quad (85)$$

thus (81) and (82) become

$$\left.\begin{array}{r} d.\left\{\left(\frac{d\theta}{dt}\right)^2 + (\sin\theta)^2\left(\frac{d\psi}{dt}\right)^2 - \frac{2g}{l}\cos\theta\right\} = 0, \\ d.\left\{(\sin\theta)^2\left(\frac{d\psi}{dt} - \omega\sin\lambda\right)\right\} - 2\omega\cos\lambda(\sin\theta)^2\cos\psi\, d\theta = 0. \end{array}\right\} \quad (86)$$

As these two equations are deduced from (79) by a change of coordinates, they have lost none of their generality, and consequently they express the general motion of a pendulum to the same degree of accuracy; and that is, when terms involving the products of ω^2 and either ξ, η, or $l-\zeta$ are omitted.

419.] For our purpose however it will be sufficient to consider the oscillations of the pendulum to be small, and thus to assume the greatest angle of inclination of the pendulum to the vertical to be so small that cubes of it and all powers higher than the cubes may be omitted. Consequently θ is always such that θ^3 and higher powers of θ will be omitted; also $\frac{d\theta}{dt}$ is a small quantity. Let us replace $l\theta$ by ρ, as we may by means of (83); because we shall thereby obtain the polar equation of the curve in the horizontal plane into which the path described by the bob of the pendulum is projected. Then omitting θ^3 and higher powers of θ, from (38) we have $\rho = l\theta$;

$$\therefore \quad \frac{d\rho}{dt} = l\frac{d\theta}{dt}.$$

Also the last terms in the second equation of (86) must be omitted, because $(\sin\theta)^2 d\theta$ is a small term of an order higher than those which are to be retained. Thus (86) become

$$\left.\begin{aligned} d\,.\left\{\left(\frac{d\rho}{dt}\right)^2+\rho^2\left(\frac{d\psi}{dt}\right)^2+\frac{g}{l}\rho^2\right\}&=0,\\ d\,.\left\{\rho^2\left(\frac{d\psi}{dt}-\omega\sin\lambda\right)\right\}&=0. \end{aligned}\right\} \tag{87}$$

Let these equations be integrated; and let us suppose the pendulum to start from rest at a distance $\rho = a$ from the vertical line passing through the point of suspension; then

$$\left(\frac{d\rho}{dt}\right)^2+\rho^2\left(\frac{d\psi}{dt}\right)^2+\frac{g}{l}(\rho^2-a^2)=0; \tag{88}$$

$$\rho^2\frac{d\psi}{dt}-\omega\sin\lambda\,(\rho^2-a^2)=0; \tag{89}$$

eliminating $\frac{d\psi}{dt}$, we have

$$\rho^2\left(\frac{d\rho}{dt}\right)^2=-\left\{\frac{g}{l}+\omega^2(\sin\lambda)^2\right\}\rho^4+a^2\left\{\frac{g}{l}+2\,\omega^2(\sin\lambda)^2\right\}\rho^2 - \omega^2(\sin\lambda)^2a^4, \tag{90}$$

$$=\left\{\frac{g}{l}+\omega^2(\sin\lambda)^2\right\}\left\{-\rho^4+\frac{g+2l\,\omega^2(\sin\lambda)^2}{g+l\,\omega^2(\sin\lambda)^2}a^2\rho^2 - \frac{\omega^2 l(\sin\lambda)^2}{g+l\,\omega^2(\sin\lambda)^2}a^4\right\}. \tag{91}$$

Now the right-hand member of this equation is a quadratic expression in ρ^2, which has two roots, both of which are positive, and of which one is a^2 and is the greater root; let b^2 be the other: also, for convenience of expression, let

$$\frac{g}{l}+\omega^2(\sin\lambda)^2=n^2; \tag{92}$$

so that

$$b=\frac{\omega\sin\lambda}{n}a. \tag{93}$$

Then (91) becomes

$$\rho^2\left(\frac{d\rho}{dt}\right)^2=n^2(a^2-\rho^2)(\rho^2-b^2); \tag{94}$$

so that a and b are manifestly the greatest and least values of ρ. From (94) we have

$$\frac{\rho\,d\rho}{\{(a^2-\rho^2)(\rho^2-b^2)\}^{\frac{1}{2}}}=-n\,dt;$$

the negative side being taken, because we will suppose the pendulum at the time t to be approaching the vertical line through the point of suspension, and consequently ρ to be decreasing as t increases. Let also $t=0$, when $\rho=a$; then integrating, we have

$$\rho^2=\frac{a^2+b^2}{2}+\frac{a^2-b^2}{2}\cos 2nt; \tag{95}$$

which gives the value of ρ^2 in terms of t; and shews that (1) a^2 and b^2 are respectively the greatest and least values of ρ^2; (2) their values recur periodically; and (3) the periodic time

$$= \frac{2\pi}{n} = \frac{2\pi l^{\frac{1}{2}}}{\{g + l\omega^2 (\sin\lambda)^2\}^{\frac{1}{2}}}: \tag{96}$$

this result evidently agrees with that of the common simple pendulum, when $\omega = 0$.

To find the relation between ρ and ψ, we have from (89)

$$\frac{d\psi}{dt} - \omega \sin\lambda = -\frac{\omega a^2 \sin\lambda}{\rho^2}; \tag{97}$$

but from (93) $$b = \frac{\omega \sin\lambda}{n} a;$$

$$\therefore \frac{d\psi}{dt} - \omega \sin\lambda = -\frac{nab}{\rho^2}; \tag{98}$$

$$\therefore d\psi - \omega \sin\lambda\, dt = \frac{ab\, d\rho}{\rho\{(a^2-\rho^2)(\rho^2-b^2)\}^{\frac{1}{2}}}; \tag{99}$$

whence, by integration, with the assumption that $\psi = \psi_0$, and $\rho = a$, when $t = 0$,

$$\frac{1}{\rho^2} = \frac{a^2+b^2}{2a^2b^2} - \frac{a^2-b^2}{2a^2b^2} \cos 2(\psi - \psi_0 - \omega \sin\lambda t)$$

$$= \frac{\{\cos(\psi-\psi_0-\omega\sin\lambda t)\}^2}{a^2} + \frac{\{\sin(\psi-\psi_0-\omega\sin\lambda t)\}^2}{b^2}. \tag{100}$$

420.] If t is constant, this is an ellipse whose principal axes are respectively $2a$ and $2b$; so that the path described by the bob of the pendulum projected on the horizontal plane is an ellipse, the whole period being that given in (96). And since $\frac{d\psi}{dt}$, which is given in (97), is negative, the pendulum revolves in a direction opposite to that in which ψ increases; that is, the direction of its revolution is the same as that of the earth. And since (93) shews that the ratio of b to a varies nearly as ω, b is small compared with a, so that the eccentricity of the ellipse is very large; if $\omega = 0$, $b = 0$, in which case the minor axis vanishes, and the pendulum moves in a plane: this however cannot be the case when account is taken of the earth's rotation, except at the Equator, when $\sin\lambda = 0$.

Since however t varies, we may still consider (100) to represent an ellipse whose principal axes are $2a$ and $2b$; and whose major axis at the time t is inclined to the ξ-axis, which is measured

southwards along the meridian at an angle equal to $\psi_0 + \omega \sin \lambda t$: now this angle increases as t increases; and consequently the major axis revolves in azimuth with a constant angular velocity equal to $\omega \sin \lambda$ in the same direction in which ψ increases. Thus, if the path described by the bob of the pendulum is projected on the horizontal plane, it will be a revolving ellipse, whose major axis revolves in azimuth with an angular velocity equal to $\omega \sin \lambda$, in a direction opposite to that in which the earth moves: the actual path will thus be a spiral limited by two concentric circles whose radii are a and b, of which a is the greater; the spiral never extending beyond the former, nor coming within the latter; and the point where it meets the larger circle advancing with an angular velocity equal to $\omega \sin \lambda$, in a direction opposite to that of the earth's rotation, and opposite to that in which the pendulum itself moves.

This is the law which the experiment exhibited by M. Foucault confirms. We have already given a simple explanation of it in Art. 41; and that explanation appeared to M. Poinsot (see Comptes Rendus, Tome XXXII, p. 206) to be sufficient. The preceding investigation however shews that the result follows from the equations of motion, when small terms are omitted. This therefore is only the general effect; but there are sundry deviations, owing to the omitted terms, which this dynamical process will indicate if it is carried to a higher approximation, and which the other method fails to shew; but it is beyond our purpose to enter upon these small disturbances in this treatise. The several memoirs already alluded to contain further approximations, and to them I must refer the student. I should also mention that M. Poncelet, whose name must ensure attention from every mathematician, has written two memoirs on this subject, which are inserted in the Comptes-Rendus de l'Académie des Sciences de Paris, Vol. LI, 1860, and has arrived at results differing in some respects from the preceding.

421.] The motion whose circumstances we have investigated has been imagined to be that of a bob of a pendulum fixed by a rod of given length to a point fixed relatively to the earth and moving with it, and the effect of that rotation has been exhibited in the preceding equations. This motion is consequently that of a material particle moving on the lower concave surface of a sphere, whose radius is l, fixed to the earth and moving with

it; and the general equations are applicable to any other kind of constrained motion of a particle. Let us take another example.

A particle moves on a smooth inclined plane fixed to the earth and moving with it: it is required to determine the relative motion of the particle.

Let the plane pass through P, the place of observation, see Fig. 63, whose latitude is λ; and let the equation to it be

$$\xi \cos a + \eta \cos \beta + \zeta \cos \gamma = 0. \tag{101}$$

Let (ξ, η, ζ) be the place of m at the time t: we shall assume these coordinates to be so small in reference to r, the earth's radius, that products of them and of ω^2 may be omitted. Let R be the normal pressure on the plane: then the equations of motion are

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} + 2\omega \sin\lambda \frac{d\eta}{dt} &= \frac{\text{R}}{m}\cos a, \\ \frac{d^2\eta}{dt^2} - 2\omega\left(\sin\lambda \frac{d\xi}{dt} + \cos\lambda \frac{d\zeta}{dt}\right) &= \frac{\text{R}}{m}\cos\beta, \\ \frac{d^2\zeta}{dt^2} + 2\omega\cos\lambda\frac{d\eta}{dt} &= \frac{\text{R}}{m}\cos\gamma - g. \end{aligned}\right\} \tag{102}$$

Although it is convenient to retain a, β, and γ, yet we shall require their values in terms of (θ) the inclination of the plane to the horizontal plane of (ξ, η), and of the angle (ψ) between the ξ-axis, which is southwards, and the line of intersection (the line of nodes) of the plane with the horizontal plane. In reference to these

$$\left.\begin{aligned} \cos a &= \sin\theta\psi, \\ \cos\beta &= -\sin\theta\psi, \\ \cos\gamma &= \cos\theta. \end{aligned}\right\} \tag{103}$$

As the results of the force of transference do not appear in the preceding equations, and as the line of action of R is perpendicular to the relative path of m, the principle of vis viva is applicable. Let v be the relative velocity of m, and let us suppose the particle to be at rest at (ξ_0, η_0, ζ_0) when $t = 0$. Then multiplying (102) severally by $d\xi, d\eta, d\zeta$, adding and integrating, we have

$$v^2 = 2g(z - \zeta_0); \tag{104}$$

which shews that the relative velocity of the moving particle is the same as if the rotation of the earth was not brought into account.

From the last two equations of (102) we have

$$\cos\gamma\frac{d^2\eta}{dt^2}-\cos\beta\frac{d^2\zeta}{dt^2}+2\omega(\cos\lambda\cos\alpha-\sin\lambda\cos\gamma)\frac{d\xi}{dt}=g\cos\beta; \quad (105)$$

therefore

$$\cos\gamma\frac{d\eta}{dt}-\cos\beta\frac{d\zeta}{dt}+2\omega(\cos\lambda\cos\alpha-\sin\lambda\cos\gamma)(\xi-\xi_0)=gt\cos\beta. \quad (106)$$

Similarly

$$\cos\alpha\frac{d\zeta}{dt}-\cos\gamma\frac{d\xi}{dt}+2\omega(\cos\lambda\cos\alpha-\sin\lambda\cos\gamma)(\eta-\eta_0)=-gt\cos\alpha; \quad (107)$$

$$\cos\beta\frac{d\xi}{dt}-\cos\alpha\frac{d\eta}{dt}+2\omega(\cos\lambda\cos\alpha-\sin\lambda\cos\gamma)(\zeta-\zeta_0)=0. \quad (108)$$

Again, multiplying (102) severally by $\cos\alpha$, $\cos\beta$, and $\cos\gamma$; adding, and omitting the terms which vanish by reason of the differential of (105), we have

$$-2\omega\sin\lambda\left(\cos\beta\frac{d\xi}{dt}-\cos\alpha\frac{d\eta}{dt}\right)+2\omega\cos\lambda\left(\cos\gamma\frac{d\eta}{dt}-\cos\beta\frac{d\zeta}{dt}\right)=\frac{R}{m}-g\cos\gamma; \quad (109)$$

substituting in which from (106) and (108), and omitting terms involving products of ω^2 and the coordinates, we have

$$\frac{R}{m}=g\cos\gamma+2\omega\cos\lambda\cos\beta gt, \quad (110)$$

$$=g\cos\theta-2\omega gt\cos\lambda\sin\theta\cos\psi; \quad (111)$$

which assigns the pressure on the plane; and shews that it is diminished or increased by the earth's rotation according as the line of nodes lies in the S.W. and N.E. quadrants, or in the N.W. and S.E. quadrants; and that this increase or diminution vanishes when the line of nodes lies E. and W. It vanishes at the pole, and is, cæteris paribus, a maximum at the equator; and it also vanishes when the plane is horizontal. It also varies as the time during which the particle has been moving. Since $\omega\cos\lambda$ is the component of the earth's angular velocity along the tangent to the meridian, that is, along the N. and S. line on the horizontal plane, the change of pressure on the plane is due to that component only, and not to the component along the vertical.

Substituting in the first and third equations of (102) the value of R, given in (110), and integrating, we have

$$\left.\begin{aligned}\frac{d\xi}{dt}+2\omega\sin\lambda(\eta-\eta_0)&=\quad gt\cos\alpha\cos\gamma+\omega\cos\lambda\cos\alpha\cos\beta gt^2,\\ \frac{d\zeta}{dt}+2\omega\cos\lambda(\eta-\eta_0)&=-gt(\sin\gamma)^2+\omega\cos\lambda\cos\beta\cos\gamma gt^2;\end{aligned}\right\} \quad (112)$$

and substituting these values in the second of (102), and omitting the terms involving products of ω^2 and the coordinates, we have

$$\frac{d^2\eta}{dt^2}+2\omega\cos\alpha\{\cos\lambda\cos\alpha-\sin\lambda\cos\gamma\}gt$$
$$-2\omega^2\cos\lambda\cos\beta\{\cos\alpha\sin\lambda+\cos\gamma\cos\lambda\}gt^2=g\cos\beta\cos\gamma;$$

$$\eta-\eta_0=\cos\beta\cos\gamma\frac{gt^2}{2}-\omega\cos\alpha\{\cos\lambda\cos\alpha-\sin\lambda\cos\gamma\}\frac{gt^3}{3}$$
$$-\omega^2\cos\lambda\cos\beta\{\cos\alpha\sin\lambda+\cos\gamma\cos\lambda\}\frac{gt^4}{6}.$$

Now this equation shews that terms involving t^2 are of the order as $\eta-\eta_0$; so that as the approximation is made on the assumption that products of ω^2 and the coordinates of m are to be omitted, we must omit products of ω^2 and t^2; and therefore

$$\eta-\eta_0=\cos\beta\cos\gamma\frac{gt^2}{2}-\omega\cos\alpha\{\cos\lambda\cos\alpha-\sin\lambda\cos\gamma\}\frac{gt^3}{3}; \quad (113)$$

and substituting this value of $\eta-\eta_0$ in (112), and integrating,

$$\xi-\xi_0=\cos\alpha\cos\gamma\frac{gt^2}{2}+\omega\cos\beta\{\cos\lambda\cos\alpha-\sin\lambda\cos\gamma\}\frac{gt^3}{3}; \quad (114)$$

$$\zeta-\zeta_0=-(\sin\gamma)^2\frac{gt^2}{2}; \quad (115)$$

so that in terms of θ and ψ,

$$\left.\begin{aligned}\xi&=\xi_0+\sin\theta\cos\theta\sin\psi\frac{gt^2}{2}-\omega\sin\theta\cos\psi\{\cos\lambda\sin\theta\cos\psi-\sin\lambda\cos\theta\}\frac{gt^3}{3},\\ \eta&=\eta_0-\sin\theta\cos\theta\cos\psi\frac{gt^2}{2}-\omega\sin\theta\sin\psi\{\cos\lambda\sin\theta\cos\psi-\sin\lambda\cos\theta\}\frac{gt^3}{3},\\ \zeta&=\zeta_0-(\sin\theta)^2\frac{gt^2}{2};\end{aligned}\right\} \quad (116)$$

which assign the position of the particle at the time t. Whence it appears, that if we omit terms involving products of ω^2 and the coordinates of m, the vertical distance through which m falls in the time t is not affected by the earth's rotation.

To determine the curve which the particle describes, let us refer its place at the time t to the point (ξ_0, η_0, ζ_0) as origin; and to two axes in the plane, one of which, that of ξ', is parallel to, and the other, that of η', is perpendicular to the line of nodes; so that

$$\left.\begin{aligned}\xi'&=(\xi_0-\xi)\cos\psi+(\eta_0-\eta)\sin\psi,\\ \eta'\sin\theta&=\zeta_0-\zeta;\end{aligned}\right\} \quad (117)$$

$$\therefore\quad\left.\begin{aligned}\xi'&=\omega\sin\theta\{\cos\lambda\sin\theta\cos\psi-\sin\lambda\cos\theta\}\frac{gt^3}{3},\\ \eta'&=\sin\theta\frac{gt^2}{2}.\end{aligned}\right\} \quad (118)$$

If we eliminate t from these equations, we obtain the equation

$$\xi'^2 = \frac{8\omega^2}{9g\sin\theta}\{\cos\lambda\sin\theta\cos\psi - \sin\lambda\cos\theta\}^2\eta'^3; \qquad (119)$$

which is the equation to the semicubical parabola, and is that to the path of the particle.

If $\omega = 0$, $\xi' = 0$, and the particle falls down the plane in a straight line perpendicular to the line of nodes; but if the rotation of the earth is considered, there is a lateral deviation from the rectilineal path, which varies as the cube of the time of falling.

It will be observed that I have supposed the particle to start from rest from (ξ_0, η_0, ζ_0); if it were projected from that point on the plane with a given velocity, other terms, which can easily be found, would be introduced into the preceding equations; and if $\omega = 0$, the resulting equations would of course represent a parabola.

422.] If the plane on which the particle moves is horizontal, the equations of motion are

$$\left.\begin{aligned}\frac{d^2\xi}{dt^2} + 2\omega\sin\lambda\frac{d\eta}{dt} &= 0,\\ \frac{d^2\eta}{dt^2} - 2\omega\sin\lambda\frac{d\xi}{dt} &= 0;\end{aligned}\right\}$$

and if we suppose the particle to be projected from the origin along the plane with a velocity u in a line inclined at an angle β to the axis of ξ; then, integrating the preceding equations, we have

$$\left.\begin{aligned}\frac{d\xi}{dt} + 2\omega\sin\lambda\,\eta &= u\cos\beta,\\ \frac{d\eta}{dt} - 2\omega\sin\lambda\,\xi &= u\sin\beta;\end{aligned}\right\}$$

which equations assign the relative velocity of the particle at the point (ξ, η); and by elimination of t and subsequent integration we have

$$\left(\xi + \frac{u\sin\beta}{2\omega\sin\lambda}\right)^2 + \left(\eta - \frac{u\cos\beta}{2\omega\sin\lambda}\right)^2 = \frac{u^2}{4\omega^2(\sin\lambda)^2};$$

which is the equation to a circle. Consequently the particle moves in a circle whose radius is $\frac{u}{2\omega\sin\lambda}$; whose centre is at the point $\left(-\frac{u\sin\beta}{2\omega\sin\lambda}, \frac{u\cos\beta}{2\omega\sin\lambda}\right)$; and the periodic time $= \frac{\pi}{\omega\sin\lambda} = \frac{43082''}{\sin\lambda}$ = a mean solar day divided by twice the sine of the latitude.

Another problem, which may be solved by these general equations, is the motion of a particle on the surface of a right circular cone, whose vertex is at P, the place of observation, and whose axis coincides with the vertical at that place.

SECTION 3.—*The relative motion of a material system.*

423.] The equations of relative motion which have been found refer only to the motion of a single material particle. Those however of certain material systems may be deduced from them by means of D'Alembert's principle.

Suppose m to be the type-particle of a system, to the motion of which equations (29) refer; and suppose I to be the type of an impressed momentum-increment due to an internal force, see Art. 69, acting on m, of which $\text{I}\cos\lambda$, $\text{I}\cos\mu$, $\text{I}\cos\nu$ are the axial components; and let us suppose the system to be free from all constraint except that which exists amongst its own members; so that every particle is free to move as it is affected by the external forces acting on it, and by the internal forces of the system; then the equations of motion of the system in their most general forms are

$$\left.\begin{aligned} \Sigma.m\left\{\text{X}'-\text{X}_t-\text{F}\cos\alpha-\frac{d^2\xi}{dt^2}\right\}-\Sigma.\text{I}\cos\lambda&=0,\\ \Sigma.m\left\{\text{Y}'-\text{Y}_t-\text{F}\cos\beta-\frac{d^2\eta}{dt^2}\right\}-\Sigma.\text{I}\cos\mu&=0,\\ \Sigma.m\left\{\text{Z}'-\text{Z}_t-\text{F}\cos\gamma-\frac{d^2\zeta}{dt^2}\right\}-\Sigma.\text{I}\cos\nu&=0.\end{aligned}\right\}\qquad(120)$$

If the material system is a rigid body, or is invariable in form, or otherwise is such that the internal forces taken throughout it disappear, then these equations become

$$\left.\begin{aligned} \Sigma.m\left\{\text{X}'-\text{X}_t-\text{F}\cos\alpha-\frac{d^2\xi}{dt^2}\right\}&=0,\\ \Sigma.m\left\{\text{Y}'-\text{Y}_t-\text{F}\cos\beta-\frac{d^2\eta}{dt^2}\right\}&=0,\\ \Sigma.m\left\{\text{Z}'-\text{Z}_t-\text{F}\cos\gamma-\frac{d^2\zeta}{dt^2}\right\}&=0;\end{aligned}\right\}\qquad(121)$$

and it is the motion of a system of this kind which for the most part we shall consider.

424.] From these the equations of the axial components of the moments of the couples are to be formed: let us take that whose axis is the moving axis of ξ; then we have

$$\Sigma . m \left\{ \eta \left(\mathrm{Z}' - \mathrm{Z}_t - \mathrm{F} \cos \gamma - \frac{d^2 \zeta}{dt^2} \right) - \zeta \left(\mathrm{Y}' - \mathrm{Y}_t - \mathrm{F} \cos \beta - \frac{d^2 \eta}{dt^2} \right) \right\} = 0 ;$$

and replacing $\mathrm{F} \cos \beta$ and $\mathrm{F} \cos \gamma$ by their values given in (19), this becomes

$$\Sigma . m \left\{ \eta \left(\mathrm{Z}' - \mathrm{Z}_t - \frac{d^2 \zeta}{dt^2} \right) - \zeta \left(\mathrm{Y}' - \mathrm{Y}_t - \frac{d^2 \eta}{dt^2} \right) \right\}$$

$$- 2 \omega_\xi \Sigma . m \frac{\eta \, d\eta + \zeta \, d\zeta}{dt} + 2 \omega_\eta \Sigma . m \eta \frac{d\xi}{dt} + 2 \omega_\zeta \Sigma . m \, \zeta \frac{d\xi}{dt} = 0 ;$$

or

$$\frac{d}{dt} \Sigma . m \left(\eta \frac{d\zeta}{dt} - \zeta \frac{d\eta}{dt} \right) = \Sigma . m \{ \eta (\mathrm{Z}' - \mathrm{Z}_t) - \zeta (\mathrm{Y}' - \mathrm{Y}_t) \}$$

$$- \omega_\xi \frac{d}{dt} \Sigma . m (\eta^2 + \zeta^2) + 2 \omega_\eta \Sigma . m \eta \frac{d\xi}{dt} + 2 \omega_\zeta \Sigma . m \, \zeta \frac{d\xi}{dt} = 0 ; \quad (122)$$

and the similar equations for the other axes are

$$\frac{d}{dt} \Sigma . m \left(\zeta \frac{d\xi}{dt} - \xi \frac{d\zeta}{dt} \right) = \Sigma . m \{ \zeta (\mathrm{X}' - \mathrm{X}_t) - \xi (\mathrm{Z}' - \mathrm{Z}_t) \}$$

$$- \omega_\eta \frac{d}{dt} \Sigma . m (\zeta^2 + \xi^2) + 2 \omega_\zeta \Sigma . m \zeta \frac{d\eta}{dt} + 2 \omega_\xi \Sigma . m \, \xi \frac{d\eta}{dt} = 0 ; \quad (123)$$

$$\frac{d}{dt} \Sigma . m \left(\xi \frac{d\eta}{dt} - \eta \frac{d\xi}{dt} \right) = \Sigma . m \{ \xi (\mathrm{Y}' - \mathrm{Y}_t) - \eta (\mathrm{X}' - \mathrm{X}_t) \}$$

$$- \omega_\zeta \frac{d}{dt} \Sigma . m (\xi^2 + \eta^2) + 2 \omega_\xi \Sigma . m \, \xi \frac{d\zeta}{dt} + 2 \omega_\eta \Sigma . m \, \eta \frac{d\zeta}{dt} = 0 ; \quad (124)$$

and by means of these six equations the relative motion of a material system of invariable form may be determined.

425.] These six equations of relative motion may be combined into a single equation by means of the principle of virtual velocities. For suppose δs to be any arbitrary geometrical displacement of the place of m at the time t, which is consistent with the geometrical relations of the system; and let $\delta \xi$, $\delta \eta$, $\delta \zeta$ be the axial projections of δs; and let all these quantities be type-quantities; then the equations of motion may be expressed by means of the single equation,

$$\Sigma . m \left\{ \left(\mathrm{X}' - \mathrm{X}_t - \mathrm{F} \cos \alpha - \frac{d^2 \xi}{dt^2} \right) \delta \xi + \left(\mathrm{Y}' - \mathrm{Y}_t - \mathrm{F} \cos \beta - \frac{d^2 \eta}{dt^2} \right) \delta \eta \right.$$

$$\left. + \left(\mathrm{Z}' - \mathrm{Z}_t - \mathrm{F} \cos \gamma - \frac{d^2 \zeta}{dt^2} \right) \delta \zeta \right\} = 0. \quad (125)$$

This equation is indeed equivalent to the six equations by

reason of the arbitrariness of $\delta\xi$, $\delta\eta$, $\delta\zeta$; for these quantities in their most general forms involve six displacements, which are independent of each other; viz., three of translation and three of rotation; and the coefficients of these separately vanish. If the relative motion of one or more of the particles of the system is constrained, these displacements are thus far subject to certain conditions, and consequently are not independent; and all that has been said in Art. 78 is, mutatis mutandis, to be applied to this case.

426.] Let us suppose that the conditions to which the system is subject do not involve the time explicitly; then we may take for the virtual arbitrary displacement of the place of m that which actually takes place in the time dt by reason of the motion of the system, and of the forces acting on it; so that in equation (125) we may put

$$\delta\xi = d\xi, \qquad \delta\eta = d\eta, \qquad \delta\zeta = d\zeta; \tag{126}$$

then, since from (28),

$$\cos\alpha\, d\xi + \cos\beta\, d\eta + \cos\gamma\, d\zeta = 0, \tag{127}$$

(125) becomes

$$\Sigma . m \left\{ \frac{d^2\xi}{dt^2} d\xi + \frac{d^2\eta}{dt^2} d\eta + \frac{d^2\zeta}{dt^2} d\zeta \right\}$$
$$= \Sigma . m \left\{ (X' - X_t) d\xi + (Y' - Y_t) d\eta + (Z' - Z_t) d\zeta \right\}; \tag{128}$$

so that if v is the relative velocity of m at the time t,

$$d . \Sigma . m v^2 = 2\, \Sigma . m \{ (X' - X_t) d\xi + (Y' - Y_t) d\eta + (Z' - Z_t) d\zeta \}. \tag{129}$$

Let us moreover suppose that X', Y', Z', X_t, Y_t, Z_t do not contain t explicitly, but are functions of ξ, η, ζ only; then by integration

$$\Sigma . m v^2 - \Sigma . m v_0^2 = 2 \int_0^1 \Sigma . m \, (X' d\xi + Y' d\eta + Z') d\zeta$$
$$- 2 \int_0^1 \Sigma . m (X_t d\xi + Y_t d\eta + Z_t d\zeta); \tag{130}$$

wherein v_0 is the initial value of v, and 1 and 0 denote the limiting values of the relative coordinates of the place of m, corresponding to the terminal and the initial values of the left-hand member of the equation.

Equation (130) is that of the relative vis viva of the material system; and if we consider it in its elemental form in (129), it gives the increment of the relative vires vivæ of all the particles of the system in the time dt, and shews that it is equal to the

excess of twice the sum of the products of the impressed momentum-increment of each particle and the space through which it has acted over the sum of the products of the momentum-increment due to the force of transference of the coordinate system (see Art. 395), and the space through which this latter force has acted.

It will be seen that F, the compound centrifugal force, has wholly disappeared in (129) and (130); and rightly so; because its line of action is perpendicular to that of the relative velocity of m; whereas into the equation of vis viva only those forces enter whose lines of action are in, or have a component along, that of the motion of m at the time t.

427.] It is expedient to mention certain particular forms which the preceding general equations take in special cases, because in these simplified forms they are frequently applicable to the solution of problems.

(1) Let us suppose the origin of the moving system to move along a given curve in the plane of (x, y); and the system to have no motion of rotation: then $a_1 = b_2 = c_3 = 1$, and all the other direction-cosines vanish; so that from (21), (22), and (23),

$$\mathrm{X}_t = \frac{d^2 x_0}{dt^2}, \qquad \mathrm{Y}_t = \frac{d^2 y_0}{dt^2}, \qquad \mathrm{Z}_t = \frac{d^2 z_0}{dt^2} = 0;$$

and (124), being the equation of relative moments about the axis of ζ, becomes

$$\frac{d}{dt}\Sigma.m\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) = \Sigma.m(\xi\mathrm{Y}' - \eta\mathrm{X}') - \Sigma.m\xi\frac{d^2 y_0}{dt^2} + \Sigma.m\eta\frac{d^2 x_0}{dt^2}$$

$$= \Sigma.m(\xi\mathrm{Y}' - \eta\mathrm{X}') - \frac{d^2 y_0}{dt^2}\Sigma.m\xi + \frac{d^2 x_0}{dt^2}\Sigma.m\eta \quad (131)$$

$$= \mathrm{N} - \frac{d^2 y_0}{dt^2}\Sigma.m\xi + \frac{d^2 x_0}{dt^2}\Sigma.m\eta, \quad (132)$$

if N is the moment of the couple of the impressed forces whose axis is the moving ζ-axis.

If the material system is of invariable form, and is fixed to the moving origin; and if r is the distance of m from an axis perpendicular to the plane of (ξ, η) through the origin, and $\frac{d\theta}{dt}$ is the angular velocity of the body at the time t; then (132) becomes

$$\frac{d^2\theta}{dt^2}\Sigma.mr^2 = \mathrm{N} - \frac{d^2 y_0}{dt^2}\Sigma.m\xi + \frac{d^2 x_0}{dt^2}\Sigma.m\eta; \quad (133)$$

by which equation the relative angular velocity of the body may be determined.

(2) If the origin of the moving axes is fixed at the fixed origin, and the moving axes revolve about a fixed axis, say the z-axis with an uniform angular velocity $= \omega$ (say); if θ is the angle between the axes of x and ξ at the time t; so that

$$\left.\begin{array}{lll} a_1 = \cos\theta, & b_1 = -\sin\theta, & c_1 = 0, \\ a_2 = \sin\theta, & b_2 = \cos\theta, & c_2 = 0, \\ a_3 = 0; & b_3 = 0; & c_3 = 1; \end{array}\right\} \qquad (134)$$

and $\qquad \mathrm{X}_t = -\omega^2 \xi, \quad \mathrm{Y}_t = -\omega^2 \eta, \quad \mathrm{Z}_t = 0;$

then (124) becomes

$$\frac{d}{dt}\Sigma.m\left(\xi\frac{d\eta}{dt}-\eta\frac{d\xi}{dt}\right) = \Sigma.m(\xi\mathrm{Y}'-\eta\mathrm{X}')-\omega\frac{d}{dt}\Sigma.m(\xi^2+\eta^2) \quad (135)$$

$$= \mathrm{N}-\omega\frac{d}{dt}\Sigma.m(\xi^2+\eta^2). \quad (136)$$

Therefore integrating, we have

$$\Sigma.mr^2\frac{d\theta}{dt}-\Sigma.m\left(r^2\frac{d\theta}{dt}\right)_0 = \int_0^1 \mathrm{N}\,dt-\omega(\mathrm{H}-\mathrm{H}_0),$$

if H and H_0 are the moments of inertia of the system about the fixed axis at the times t and 0 respectively, where N is the relative moment of the impressed couple about the fixed axis.

The equations (132) and (136) may also be derived directly from (35), without the intervention of the general forms given in (124).

If the body is rigid, and the origin is a fixed point of it, then $\Sigma.m(\xi^2+\eta^2)$ is independent of the time, and (136) becomes

$$\frac{d\omega}{dt}\Sigma.mr^2 = \mathrm{N}; \quad (137)$$

which is the same equation as that which expresses the rotation of a rigid body about a fixed or an instantaneous axis.

It is also to be observed that (132) is reduced to (137): (1) when (x_0, y_0), the place of the moving origin, is fixed absolutely: (2) when this moving origin has an uniform motion; so that $\frac{d^2x_0}{dt^2} = \frac{d^2y_0}{dt^2} = 0$; (3) when the moving ζ-axis passes through the mass-centre of the body, because in that case

$$\Sigma.m\xi = \Sigma.m\eta = 0.$$

Also the equation of relative vis viva which is given in (130) becomes in this case

$$\Sigma.mv^2-\Sigma.mv_0^2 = 2\int_0^1 \Sigma.m(\mathrm{X}'d\xi+\mathrm{Y}'d\eta+\mathrm{Z}'d\zeta)+\omega^2(\mathrm{H}-\mathrm{H}_0).$$

428.] From these general equations we may deduce theorems

similar to those of absolute motion which have been already demonstrated in Chap. III. Section 2, of the present Volume. In the first place, the relative motion of the mass-centre of a material system of invariable form, or in which the internal forces mutually destroy each other, is the same as if the whole mass of the system were collected into it, and all the momentum due to the external forces, the forces of transference, and the compound centrifugal forces, the last two with their directions changed, was thereat applied in lines parallel to the actual lines of action.

Let $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ be the place of the mass-centre of the system at the time t relatively to the moving axes; and let (ξ', η', ζ') be the place of m at the same time relatively to a system of parallel axes originating at the mass-centre: then we have

$$\left.\begin{aligned} \xi &= \bar{\xi}+\xi', \\ \eta &= \bar{\eta}+\eta', \\ \zeta &= \bar{\zeta}+\zeta'. \end{aligned}\right\} \qquad (138)$$

Also let M denote the mass of the whole moving system: then the newly introduced coordinates are subject to the following conditions:

$$\Sigma.m\xi' = \Sigma.m\eta' = \Sigma.m\zeta' = 0; \qquad (139)$$

$$\Sigma.m\xi = \text{M}\bar{\xi}, \quad \Sigma.m\eta = \text{M}\bar{\eta}, \quad \Sigma.m\zeta = \text{M}\bar{\zeta}. \qquad (140)$$

On referring to the analytical values of the momentum due to the forces of transference given in (21), (22), and (23), it appears that the values of $\Sigma.m\text{X}_t$, $\Sigma.m\text{Y}_t$, $\Sigma.m\text{Z}_t$ are not changed; but they may be expressed as

$$\text{M}\bar{\text{X}}_t, \qquad \text{M}\bar{\text{Y}}_t, \qquad \text{M}\bar{\text{Z}}_t;$$

where $\bar{\text{X}}_t$, $\bar{\text{Y}}_t$, $\bar{\text{Z}}_t$ are the values of X_t, Y_t, Z_t, when ξ, η, ζ are replaced by $\bar{\xi}, \bar{\eta}, \bar{\zeta}$, so that the momentum due to the forces of transference may be applied to a mass M at the mass-centre, along lines parallel to their original lines of action. A similar theorem is also true of $\Sigma.m\text{F}\cos\alpha$, $\Sigma.m\text{F}\cos\beta$, $\Sigma.m\text{F}\cos\gamma$, which may be replaced by $\text{M}\bar{\text{F}}\cos\alpha$, $\text{M}\bar{\text{F}}\cos\beta$, $\text{M}\bar{\text{F}}\cos\gamma$; so that the equations (121) become, after all reductions,

$$\left.\begin{aligned} \frac{\Sigma.m\text{X}'}{\text{M}} - \bar{\text{X}}_t - \bar{\text{F}}\cos\alpha - \frac{d^2\bar{\xi}}{dt^2} &= 0, \\ \frac{\Sigma.m\text{Y}'}{\text{M}} - \bar{\text{Y}}_t - \bar{\text{F}}\cos\beta - \frac{d^2\bar{\eta}}{dt^2} &= 0, \\ \frac{\Sigma.m\text{Z}'}{\text{M}} - \bar{\text{Z}}_t - \bar{\text{F}}\cos\gamma - \frac{d^2\bar{\zeta}}{dt^2} &= 0; \end{aligned}\right\} \qquad (141)$$

and these equations prove the theorem which has been enunciated. The theorem of relative motion analogous to that of Art. 83 may be framed in the same manner. Thus the relative motion of a material system, such as we have considered, may be resolved into the motion of translation of its mass-centre, all the forces being supposed to act on the whole mass condensed into that point, and into a motion of rotation about an axis passing through the mass-centre. Consequently the investigations of the preceding section are not limited to the motion of a material particle: they are also applicable to that of the mass-centre of a material system, of which the internal forces vanish. Thus they apply to the relative motion of translation of the mass-centre of planets, of shot, of pendulums with large balls, &c.; except that in these cases the resistance of the medium through which the bodies pass must be taken account of; so that other terms enter into the equations beside those which we have considered. It remains then only to investigate the rotation of the body about an axis passing through the mass-centre considered as a fixed point in reference to a system of moving axes. It is true, as we have heretofore remarked, that the point through which the rotation-axis passes need not be the mass-centre; for the general motion may always be resolved into a motion of translation of *any* point, and a motion of rotation about an axis passing through that point: but the mass-centre is the only point at which the mass may be supposed to be condensed and the forces may be applied each in its own intensity and direction, and the translation will be the same as it is in the motion of the whole system. In the following Articles I shall take the general case, and shall suppose the fixed point, about which the rotation is estimated to be any point, and not necessarily the mass-centre.

429.] At this point I shall assume three systems of reference, and subsequently of coordinate-axes, to originate. (1) A system the lines of which are parallel to the analogous lines in the system absolutely fixed, so that all angles will be the same in both; and this system may also be regarded as fixed: (2) the system of axes to which the motion of the body is to be referred; this is a moving system, and its motion with reference to the fixed system is given, and the elements of it are, as heretofore, data of the problem; these two systems are connected by the

scheme of cosines &c. which are involved in (1) of the present Chapter: (3) another system of rectangular axes, fixed in the body and moving with it, which I shall take to be a principal system at the point.

In reference to these three systems respectively I shall take the place of m at the time t to be (x, y, z), (ξ, η, ζ), (ξ', η', ζ'); and these last two I shall take to be connected by the following scheme of direction-cosines;

	ξ	η	ζ
ξ'	a_1	a_2	a_3
η'	β_1	β_2	β_3
ζ'	γ_1	γ_2	γ_3

(142)

so that

$$\left.\begin{aligned} \xi &= a_1\xi'+\beta_1\eta'+\gamma_1\zeta', \\ \eta &= a_2\xi'+\beta_2\eta'+\gamma_2\zeta', \\ \zeta &= a_3\xi'+\beta_3\eta'+\gamma_3\zeta'; \end{aligned}\right\} \tag{143}$$

To determine the relative motion, these nine direction-cosines must be expressed in terms of t: as only three are independent, it will be eventually more convenient to determine the position &c. of the body by means of Euler's three angles, θ, ϕ, ψ, according to the process of Articles 3 and 4: the relations between these three angles, and the nine direction-cosines being those of (20), (21), (22), Art. 4.

430.] Now we have two modes of estimating the angular velocity of the body, of which one is absolute, and the other is relative to the moving system of axes; let us resolve these along the principal axes at the time t; let ω_1', ω_2', ω_3' be the axial components of the former, and let ω_1, ω_2, ω_3 be those of the latter. The difference between them is evidently due to the angular velocity of the moving system: and consequently if we resolve this latter along the principal axes, we may equate each component to the corresponding excess of the absolute over the relative angular velocity. Thus we have

$$\left.\begin{aligned} \omega_1' &= \omega_1+a_1\,\omega_\xi+a_2\,\omega_\eta+a_3\,\omega_\zeta, \\ \omega_2' &= \omega_2+\beta_1\,\omega_\xi+\beta_2\,\omega_\eta+\beta_3\,\omega_\zeta, \\ \omega_3' &= \omega_3+\gamma_1\,\omega_\xi+\gamma_2\,\omega_\eta+\gamma_3\,\omega_\zeta; \end{aligned}\right\} \tag{144}$$

as ω_1, ω_2, ω_3, θ, ϕ, ψ are all employed relatively to the moving system of axes, they are connected by the equations given in (120), (121), (122), Article 64.

As ω_1', ω_2', ω_3' depend on the constitution of the body, on its initial circumstances, and on the forces which act on it, they must be determined from equations of motion, in terms of θ, ϕ, ψ, and t, and their values substituted in (144): hereby we shall have three equations in terms of θ, ϕ, ψ, and t, and their differentials: from these, by integration, θ, ϕ, and ψ may be expressed as functions of t, and the relative position of the body will be given.

Since ω_1', ω_2', ω_3' are the components along the principal axes of the absolute angular velocity, they may be determined by Euler's three equations of motion: and we have

$$\left.\begin{aligned} \mathrm{A}\frac{d\omega_1'}{dt}+(\mathrm{C}-\mathrm{B})\,\omega_2'\,\omega_3' &= \mathrm{L}, \\ \mathrm{B}\frac{d\omega_2'}{dt}+(\mathrm{A}-\mathrm{C})\,\omega_3'\,\omega_1' &= \mathrm{M}, \\ \mathrm{C}\frac{d\omega_3'}{dt}+(\mathrm{B}-\mathrm{A})\,\omega_1'\,\omega_2' &= \mathrm{N}; \end{aligned}\right\} \qquad (145)$$

where A, B, C are the principal moments, and L, M, N are the moments of the couples of the whole impressed momentum-increments on the body about the principal axes.

431.] By means of these relations we can express the general equations of relative vis viva and of moments in terms of the angular velocities about the axes of ξ', η', ζ' which are fixed in the body and move with it: and as the position of these axes is arbitrary we may take for them the principal axes at the moving origin, and thereby simplify the equations of motion. For this purpose let us take the t-differentials of (143), bearing in mind that ξ', η', ζ' do not vary with t. Then

$$\left.\begin{aligned} \frac{d\xi}{dt} &= \xi'\frac{d\alpha_1}{dt}+\eta'\frac{d\beta_1}{dt}+\zeta'\frac{d\gamma_1}{dt}, \\ \frac{d\eta}{dt} &= \xi'\frac{d\alpha_2}{dt}+\eta'\frac{d\beta_2}{dt}+\zeta'\frac{d\gamma_2}{dt}, \\ \frac{d\zeta}{dt} &= \xi'\frac{d\alpha_3}{dt}+\eta'\frac{d\beta_3}{dt}+\zeta'\frac{d\gamma_3}{dt}; \end{aligned}\right\} \qquad (146)$$

squaring, adding them, and taking the sum of them for every element of the moving system,

$$\Sigma . m \frac{d\xi^2 + d\eta^2 + d\zeta^2}{dt^2} = \Sigma . m\xi'^2 \left\{ \left(\frac{d\alpha_1}{dt}\right)^2 + \left(\frac{d\alpha_2}{dt}\right)^2 + \left(\frac{d\alpha_3}{dt}\right)^2 \right\}$$
$$+ \Sigma . m\eta'^2 \left\{ \left(\frac{d\beta_1}{dt}\right)^2 + \left(\frac{d\beta_2}{dt}\right)^2 + \left(\frac{d\beta_3}{dt}\right)^2 \right\}$$
$$+ \Sigma . m\zeta'^2 \left\{ \left(\frac{d\gamma_1}{dt}\right)^2 + \left(\frac{d\gamma_2}{dt}\right)^2 + \left(\frac{d\gamma_3}{dt}\right)^2 \right\}$$
$$+ 2\Sigma . m\eta'\zeta' \left\{ \frac{d\beta_1}{dt}\frac{d\gamma_1}{dt} + \frac{d\beta_2}{dt}\frac{d\gamma_2}{dt} + \frac{d\beta_3}{dt}\frac{d\gamma_3}{dt} \right\}$$
$$+ 2\Sigma . m\zeta'\xi' \left\{ \frac{d\gamma_1}{dt}\frac{d\alpha_1}{dt} + \frac{d\gamma_2}{dt}\frac{d\alpha_2}{dt} + \frac{d\gamma_3}{dt}\frac{d\alpha_3}{dt} \right\}$$
$$+ 2\Sigma . m\xi'\eta' \left\{ \frac{d\alpha_1}{dt}\frac{d\beta_1}{dt} + \frac{d\alpha_2}{dt}\frac{d\beta_2}{dt} + \frac{d\alpha_3}{dt}\frac{d\beta_3}{dt} \right\}; \quad (147)$$

then using the notation with respect to the summations, which are given in (25), (26), (27), Art. 150, and the equivalences given in (91) and (93), Art. 59, this is

$$\Sigma . m \frac{d\xi^2 + d\eta^2 + d\zeta^2}{dt^2} = \text{A}\omega_1^2 + \text{B}\omega_2^2 + \text{C}\omega_3^2 - 2\text{D}\omega_2\omega_3 - 2\text{E}\omega_3\omega_1 - 2\text{F}\omega_1\omega_2; \quad (148)$$

and thus the equation of relative vis viva given in (130) becomes

$$d . (\text{A}\omega_1^2 + \text{B}\omega_2^2 + \text{C}\omega_3^2 - 2\text{D}\omega_2\omega_3 - 2\text{E}\omega_3\omega_1 - 2\text{F}\omega_1\omega_2)$$
$$= 2\Sigma . m\{(\text{X}' - \text{X}_t)d\xi + (\text{Y}' - \text{Y}_t)d\eta + (\text{Z}' - \text{Z}_t)d\zeta\}; \quad (149)$$

the compound centrifugal force having no place in it, because it is a normal force, and does no work along the line of the tangent to a particular path. Equation (148) is, as it will be observed, the same as (108) in Art. 111.

This equation of relative vis viva is general; as however our system of axes of (ξ', η', ζ') is principal, $\text{D} = \text{E} = \text{F} = 0$; and thus

$$d . (\text{A}\omega_1^2 + \text{B}\omega_2^2 + \text{C}\omega_3^2) = 2\Sigma . m\{(\text{X}' - \text{X}_t)d\xi + (\text{Y}' - \text{Y}_t)d\eta + (\text{Z}' - \text{Z}_t)d\zeta\}. \quad (150)$$

If $\omega_1, \omega_2, \omega_3$ are replaced by their values as given in (120), (121), and (122), Art. 64, the expression for the vis viva of the system will be given in terms of θ, ϕ, ψ and their t-differentials.

432.] As to the equations of moments which are given in (122), (123), and (124), the transformation of them into equivalents in terms of $\omega_1, \omega_2, \omega_3$ may be effected by direct substitution of ξ', η', ζ', and the angular velocities about these axes for ξ, η, ζ; or by special processes; as the latter method is the shorter, we will take it, and transform separately the several terms of the equations. Let us take first the left-hand members of (122).

Let h_1, h_2, h_3 be, as in Articles 94 and 219, the moments of the momenta of the body or system of particles about the axes of ξ', η', ζ', so that

$$\left.\begin{aligned} h_1 &= \quad \text{A}\,\omega_1 - \text{F}\,\omega_2 - \text{E}\,\omega_3 = \left(\frac{d\text{T}}{d\omega_1}\right), \\ h_2 &= -\text{F}\,\omega_1 + \text{B}\,\omega_2 - \text{D}\,\omega_3 = \left(\frac{d\text{T}}{d\omega_2}\right), \\ h_3 &= -\text{E}\,\omega_1 - \text{D}\,\omega_2 + \text{C}\,\omega_3 = \left(\frac{d\text{T}}{d\omega_3}\right); \end{aligned}\right\} \qquad (151)$$

if $2\text{T} = \text{A}\,\omega_1^2 + \text{B}\omega_2^2 + \text{C}\,\omega_3^2 - 2\text{D}\,\omega_2\omega_3 - 2\,\text{E}\omega_3\,\omega_1 - 2\,\text{F}\,\omega_1\,\omega_2$; then, as the axes of ξ', η', ζ' are principal, $\text{D} = \text{E} = \text{F} = 0$, and

$$h_1 = \text{A}\,\omega_1, \quad h_2 = \text{B}\,\omega_2, \quad h_3 = \text{C}\,\omega_3. \qquad (152)$$

As $\Sigma.m\left(\eta\frac{d\zeta}{dt} - \zeta\frac{d\eta}{dt}\right)$ is the moment of the momenta about the axis of ξ, of which the direction-cosines with reference to the system of (ξ', η', ζ') are a_1, β_1, γ_1,

$$\Sigma.m\left(\eta\frac{d\zeta}{dt} - \zeta\frac{d\eta}{dt}\right) = h_1 a_1 + h_2\beta_1 + h_3\gamma_1$$

$$\left.\begin{aligned} &= \text{A}\,\omega_1\,a_1 + \text{B}\,\omega_2\beta_1 + \text{C}\,\omega_3\,\gamma_1: \\ \text{similarly} \quad \Sigma.m\left(\zeta\frac{d\xi}{dt} - \xi\frac{d\zeta}{dt}\right) &= \text{A}\omega_1\,a_2 + \text{B}\,\omega_2\beta_2 + \text{C}\omega_3\gamma_2, \\ \Sigma.m\left(\xi\frac{d\eta}{dt} - \eta\frac{d\xi}{dt}\right) &= \text{A}\,\omega_1 a_3 + \text{B}\,\omega_2\beta_3 + \text{C}\,\omega_3\gamma_3. \end{aligned}\right\} \qquad (153)$$

As to the terms of the right-hand members of (122), (123), and (124), let

$$\left.\begin{aligned} \Sigma.m\,\{\eta\,(\text{Z}' - \text{Z}_t) - \zeta(\text{Y}' - \text{Y}_t)\} &= \text{L}' - \text{L}_t, \\ \Sigma.m\,\{\zeta(\text{X}' - \text{X}_t) - \xi\,(\text{Z}' - \text{Z}_t)\} &= \text{M}' - \text{M}_t, \\ \Sigma.m\,\{\xi(\text{Y}' - \text{Y}_t) - \eta(\text{X}' - \text{X}_t)\} &= \text{N}' - \text{N}_t; \end{aligned}\right\} \qquad (154)$$

the right-hand member in each case being the axial component of the excess of the couple due to the impressed forces over that due to the forces of transference.

As to the second terms in the right-hand members, it is evident by the properties of moments of inertia, see (109), Art. 179, that their equivalents are as follows :

$$\left.\begin{aligned} \Sigma.m\,(\eta^2 + \zeta^2) &= \text{A}\,a_1^2 + \text{B}\beta_1^2 + \text{C}\gamma_1^2, \\ \Sigma.m\,(\zeta^2 + \xi^2) &= \text{A}\,a_2^2 + \text{B}\beta_2^2 + \text{C}\gamma_2^2, \\ \Sigma.m\,(\xi^2 + \eta^2) &= \text{A}\,a_3^2 + \text{B}\beta_3^2 + \text{C}\gamma_3^2. \end{aligned}\right\} \qquad (155)$$

And as to the third and fourth terms in the right-hand members,

$$\left.\begin{aligned} \Sigma . m\eta\frac{d\xi}{dt} &= \Sigma . m\left\{(a_2\xi'+\beta_2\eta'+\gamma_2\zeta')\left(\xi'\frac{da_1}{dt}+\eta'\frac{d\beta_1}{dt}+\zeta'\frac{d\gamma_1}{dt}\right)\right\}, \\ &= \text{A}' a_2\frac{da_1}{dt}+\text{B}'\beta_2\frac{d\beta_1}{dt}+\text{C}'\gamma_2\frac{d\gamma_1}{dt}, \\ \Sigma . m\zeta\frac{d\xi}{dt} &= \text{A}' a_3\frac{da_1}{dt}+\text{B}'\beta_3\frac{d\beta_1}{dt}+\text{C}\,\gamma_3\frac{d\gamma_1}{dt}, \end{aligned}\right\} \quad (156)$$

omitting the terms containing the products of the coordinates.

Substituting these values in (122), we have

$$\begin{aligned} \frac{d}{dt}(\text{A}\omega_1 a_1+\text{B}\omega_2\beta_1+\text{C}\omega_3\gamma_1) &= \text{L}'-\text{L}_t-\omega_\xi\frac{d}{dt}(\text{A}a_1^2+\text{B}\beta_1^2+\text{C}\gamma_1^2) \\ &+2\omega_\eta\left(\text{A}'a_2\frac{da_1}{dt}+\text{B}'\beta_2\frac{d\beta_1}{dt}+\text{C}'\gamma_2\frac{d\gamma_1}{dt}\right) \\ &+2\omega_\zeta\left(\text{A}'a_3\frac{da_1}{dt}+\text{B}'\beta_3\frac{d\beta_1}{dt}+\text{C}'\gamma_3\frac{d\gamma_1}{dt}\right); \quad (157) \end{aligned}$$

similarly from (123) and (124) we have

$$\begin{aligned} \frac{d}{dt}(\text{A}\omega_1 a_2+\text{B}\omega_2\beta_2+\text{C}\omega_3\gamma_2) &= \text{M}'-\text{M}_t-\omega_\eta\frac{d}{dt}(\text{A}a_2^2+\text{B}\beta_2^2+\text{C}\gamma_2^2) \\ &+2\omega_\zeta\left(\text{A}'a_3\frac{da_2}{dt}+\text{B}'\beta_3\frac{d\beta_2}{dt}+\text{C}'\gamma_3\frac{d\gamma_2}{dt}\right) \\ &+2\omega_\xi\left(\text{A}'a_1\frac{da_2}{dt}+\text{B}'\beta_1\frac{d\beta_2}{dt}+\text{C}'\gamma_1\frac{d\gamma_2}{dt}\right); \quad (158) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(\text{A}\omega_1 a_3+\text{B}\omega_2\beta_3+\text{C}\omega_3\gamma_3) &= \text{N}'-\text{N}_t-\omega_\zeta\frac{d}{dt}(\text{A}a_3^2+\text{B}\beta_3^2+\text{C}\gamma_3^2) \\ &+2\omega_\xi\left(\text{A}'a_1\frac{da_3}{dt}+\text{B}'\beta_1\frac{d\beta_3}{dt}+\text{C}'\gamma_1\frac{d\gamma_3}{dt}\right) \\ &+2\omega_\eta\left(\text{A}'a_2\frac{da_3}{dt}+\text{B}'\beta_2\frac{d\beta_3}{dt}+\text{C}'\gamma_2\frac{d\gamma_3}{dt}\right); \quad (159) \end{aligned}$$

the last three groups of terms in the right-hand members of each of these equations being due to the compound centrifugal force.

433.] As these equations severally express the moments about the axes of ξ, η, ζ respectively at the time t in terms of the angular velocities &c. about the principal axes of ξ', η', ζ', we can deduce from them the equations of moments about the principal axes, by taking the components of them along these axes. Thus to determine the equation of moments, that is of rotation, about the axis of ξ', let the equations (157), (158), (159) be multiplied respectively by a_1, a_2, a_3, and added: then the sum of the left-hand members is

$$\text{A}\frac{d\omega_1}{dt}+(\text{C}-\text{B})\omega_2\omega_3;$$

and of the right-hand members, if L_1 and L_t' are the moments about the principal axis of ξ' of the couples due to the impressed forces and to the force of transference respectively, the sums of the first two terms are respectively L_1 and L_t': and if in the other terms we replace A, B, C respectively by $B'+C'$, $C'+A'$, $A'+B'$, and employ the equivalences given in (88), Art. 58, the sum of the terms is

$$2C'\omega_2(\omega_\xi\gamma_1+\omega_\eta\gamma_2+\omega_\zeta\gamma_3)-2B'\omega_3(\omega_\xi\beta_1+\omega_\eta\beta_2+\omega_\zeta\beta_3); \quad (161)$$

and the equation becomes

$$A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\omega_3 = L_1-L_t'$$
$$+2C'\omega_2(\omega_\xi\gamma_1+\omega_\eta\gamma_2+\omega_\zeta\gamma_3)-2B'\omega_3(\omega_\xi\beta_1+\omega_\eta\beta_2+\omega_\zeta\beta_3). \quad (162)$$

Operating by similar processes for the equations of moments about the axes of η' and ζ', we have

$$B\frac{d\omega_2}{dt}+(A-C)\,\omega_3\omega_1 = M_1-M_t'$$
$$+2A'\omega_3(\omega_\xi\alpha_1+\omega_\eta\alpha_2+\omega_\zeta\alpha_3)-2C'\omega_1(\omega_\xi\gamma_1+\omega_\eta\gamma_2+\omega_\zeta\gamma_3), \quad (163)$$

$$C\frac{d\omega_3}{dt}+(B-A)\,\omega_1\omega_2 = N_1-N_t'$$
$$+2B'\omega_1(\omega_\xi\beta_1+\omega_\eta\beta_2+\omega_\zeta\beta_3)-2A'\omega_2(\omega_\xi\alpha_1+\omega_\eta\alpha_2+\omega_\zeta\alpha_3). \quad (164)$$

Since ω_ξ, ω_η, ω_ζ are the angular velocities of the moving system of reference about the axes of ξ, η, ζ respectively,

$$\omega_\xi\alpha_1+\omega_\eta\alpha_2+\omega_\zeta\alpha_3, \quad \omega_\xi\beta_1+\omega_\eta\beta_2+\omega_\zeta\beta_3, \quad \omega_\xi\gamma_1+\omega_\eta\gamma_2+\omega_\zeta\gamma_3$$

are the angular velocities of the moving system about the axes of ξ', η', ζ' respectively; if we denote these quantities by $\omega_{\xi'}$, $\omega_{\eta'}$, $\omega_{\zeta'}$, respectively, (162), (163), (164) will be expressed in the form

$$\left.\begin{aligned} A\frac{d\omega_1}{dt}+(C-B)\,\omega_2\omega_3 &= L_1-L_t'+2C'\omega_2\omega_{\zeta'}-2B'\omega_3\omega_{\eta'}, \\ B\frac{d\omega_2}{dt}+(A-C)\,\omega_3\omega_1 &= M_1-M_t'+2A'\omega_3\omega_{\xi'}-2C'\omega_1\omega_{\zeta'}, \\ C\frac{d\omega_3}{dt}+(B-A)\,\omega_1\omega_2 &= N_1-N_t'+2B'\omega_1\omega_{\eta'}-2A'\omega_2\omega_{\xi'}. \end{aligned}\right\} \quad (165)$$

These equations for rotatory motion are evidently analogous to Euler's equations which determine the angular velocity of a body rotating about an axis passing through a fixed point. The left-hand members are identical in the two systems; and to the right-hand members in Euler's equations are added terms which express the effects of the fictitious force of transference and of the fictitious compound centrifugal force.

434.] I propose now to apply these equations of relative vis viva and of moments about the principal axes, viz. (150), (165), to the motion of a body as affected by the diurnal rotation of the earth. I shall suppose the origin of the moving system of axes to be a point on the surface of the earth, and this point or place to have no other motion than that due to this diurnal rotation.

Let us take a plane, whose position for the present is arbitrary, so as to admit of subsequent determination, passing through the fixed point to be that of (ξ, η): this plane being fixed to and moving with the earth. Through the fixed point let a straight line be drawn parallel to the earth's polar axis; and let it be projected on the plane of (ξ, η): this line we shall take to be the ξ-axis, and reckon it positive in such a way that when the plane is horizontal that direction shall be southwards; and the positive direction of the η-axis we shall take to be such that that direction may be westwards when the plane is horizontal: hereby, if the plane of (ξ, η) is horizontal at the place of observation, we shall have the same arrangement and the same system of axes as in Fig. 63. Let the positive direction of $\text{o}\zeta$ be so taken as to be *away from* the earth's surface when the plane of (ξ, η) is horizontal; and let ν be the angle at which the ζ-axis is inclined to the earth's polar axis; so that when the ζ-axis is vertical, ν is the co-latitude of the place of observation.

Let ω, as heretofore, be the diurnal angular velocity of the earth. Then, taking account of its direction, which is from West to East,

$$\omega_{\xi} = \omega \sin \nu, \qquad \omega_{\eta} = 0, \qquad \omega_{\zeta} = -\omega \cos \nu.$$

Let us suppose the body to move under the action of the earth's attraction, whatever are the other forces which also act on it; then, so far as this attraction is concerned, $X'-X_t$, $Y'-Y_t$, $Z'-Z_t$ are the components of gravity (as it is commonly called) along the axes of ξ, η, ζ at the time t. Let us moreover suppose the dimensions of the body to be such that the earth's action is the same on all particles of it of equal mass; then thus far these components are constant, and may be replaced by constants E, F, G, where

$$X'-X_t = E, \qquad Y'-Y_t = F, \qquad Z'-Z_t = G.$$

Hence, if $\overline{M}$ is the mass of the body, and $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ the place of its mass-centre at the time t in reference to the axes of (ξ, η, ζ),

$$\Sigma . m\{(X'-X_t)\,d\xi + (Y'-Y_t)\,d\eta + (Z'+Z_t)\,d\zeta\} = \overline{M}(E\,d\bar{\xi} + F\,d\bar{\eta} + G\,d\bar{\zeta}); \quad (166)$$

and the right-hand member being substituted in (150), the expression may be integrated, and the value of the relative vis viva at the time t, or the increase of vis viva in the passage from one position to another, may be obtained.

If the axis of ζ' passes through the mass-centre, and h is the distance of the mass-centre from the fixed point, then $\bar{\xi} = h\gamma_1$, $\bar{\eta} = h\gamma_2$, $\bar{\zeta} = h\gamma_3$, and

$$\overline{\text{M}}\,(\text{E}\,d\bar{\xi}+\text{F}\,d\bar{\eta}+\text{G}\,d\bar{\zeta}) = \overline{\text{M}}\,h\,d\,(\text{E}\gamma_1+\text{F}\gamma_2+\text{G}\gamma_3). \qquad (167)$$

Also as to the equations of moments,

$$\text{L}'-\text{L}_t=\overline{\text{M}}\,h\,(\text{G}\gamma_2-\text{F}\gamma_3),\ \text{M}'-\text{M}_t=\overline{\text{M}}\,h\,(\text{E}\gamma_3-\text{G}\gamma_1),\ \text{N}'-\text{N}_t=\overline{\text{M}}\,h\,(\text{F}\gamma_1-\text{E}\gamma_2),$$

$$\left.\begin{aligned} \text{L}_1-\text{L}_t' &= -\overline{\text{M}}\,h\,(\text{E}\beta_1+\text{F}\beta_2+\text{G}\beta_3), \\ \text{M}_1-\text{M}_t' &= \ \ \overline{\text{M}}\,h\,(\text{E}a_1+\text{F}a_2+\text{G}a_3), \\ \text{N}_1-\text{N}_t' &= 0. \end{aligned}\right\} \qquad (169)$$

435.] Furthermore, let us suppose the constitution of the body, which has one point fixed but is otherwise unconstrained, to be such that $\text{A} = \text{B}$, and C to be so small that it may be omitted in comparison of A and B. Then $\text{A}' = \text{B}'$; and $\text{C}' = \text{A}$, because

$$2\text{C}' = \text{A}+\text{B}-\text{C} = 2\text{A}.$$

Let the plane of (ξ, η) be horizontal and the axis of ζ vertical, so that if λ is the latitude of the place, $\nu = 90-\lambda$; and $\text{E} = \text{F} = 0$, $\text{G} = -g$. As the only other force acting on the body is the stress at the fixed point, it does not enter either into the equation of relative vis viva or into the equations of moments.

Hence the equation of vis viva becomes

$$\text{A}\,d\,.\,(\omega_1{}^2+\omega_2{}^2) = -2\,\overline{\text{M}}\,g\,h\,d\gamma_3 = -2\,\overline{\text{M}}\,g\,h\,d\,.\cos\theta;$$

and if $\text{A} = \overline{\text{M}}\,k^2$, this equation in terms of θ and ψ becomes

$$\frac{d\theta^2}{dt^2}+(\sin\theta)^2\frac{d\psi^2}{dt^2} = \frac{2gh}{k^2}(\cos\theta_0-\cos\theta), \qquad (170)$$

if θ_0 is the value of θ, when the body is at relative rest.

Also taking equation (159), which is that of moments about the ζ-axis, and reducing it to terms involving θ and ψ, we have

$$\frac{d}{dt}\left\{(\sin\theta)^2\frac{d\psi}{dt}\right\} = \omega\sin\lambda\frac{d}{dt}(\sin\theta)^2+2\omega\cos\lambda\sin\theta\sin\psi\frac{d.\cos\theta}{dt}$$

$$= 2\,\omega\,(\sin\lambda\cos\theta-\cos\lambda\sin\theta\sin\psi)\sin\theta\frac{d\theta}{dt}. \quad (171$$

And taking equation (164), which is that of moments about the principal axis of ζ', and reducing it to terms involving θ and ψ, we have

$$\frac{d\,\omega_3}{dt} = \omega \cos\lambda\,(\omega_1\beta_1 - \omega_2\alpha_1) - \omega\sin\lambda\,(\omega_1\beta_3 - \omega_2\alpha_3)$$

$$= \omega\cos\lambda\left(-\sin\psi\cos\theta\frac{d\theta}{dt} - \cos\psi\sin\theta\frac{d\psi}{dt}\right) - \omega\sin\lambda\sin\theta\frac{d\theta}{dt}$$

$$= -\omega\cos\lambda\frac{d}{dt}(\sin\psi\sin\theta) + \omega\sin\lambda\frac{d.\cos\theta}{dt}.$$

$$\therefore \frac{d}{dt}\left\{\frac{d\phi}{dt} + \cos\theta\frac{d\psi}{dt} + \omega\cos\lambda\sin\psi\sin\theta - \omega\sin\lambda\cos\theta\right\} = 0. \quad (172)$$

These three equations (170), (171), and (172) are sufficient for the complete solution of the Problem, which is that of Foucault's Pendulum. But as full explanation of similar equations has been given in Chapter VI. it is unnecessary to repeat it in this place.

436.] It is however desirable to consider a more general problem than the preceding, in application of these principles and equations, and so I propose to take that of the Gyroscope of M. Foucault. This is a problem of great interest, inasmuch as it exhibits by means of instrumental observation the diurnal rotation of the earth, and its incidents, and also the latitude of the place; for if a rapid rotation is given to the disc of the instrument, the position of the axis retains a fixed direction, that is, points to a fixed star in the heavens, and is not affected by the earth's rotation; the relative effect of this fact is to show an angular motion of the axis of the instrument on any plane, that angular velocity depending of course on the latitude of the place of observation and the position of the plane at it. A diagram of the instrument has been given in Fig. 21, and the construction and arrangement have been described in Art. 177. The centre of gravity of the whole machine, which coincides with those of the several parts of it, is at the centre of the rotating disc; and this point remains at relative rest, whatever are the rotations of the disc and of the metallic circles. At this point therefore the systems of axes originate.

Now M. Foucault contrived in some of his experiments that the axis of the disc should be constrained to move according to a given law, fixed relatively to the earth: we will in the first place consider the case in which the axis is constrained to move in a given plane, and investigate the phenomena which the machine presents under that constraint to an observer moving with the earth. The problem in its dynamical form is this:

A heavy body of revolution rotates rapidly about its axis of figure; its centre of gravity is fixed relatively to the earth, and the rotation-axis of the body can move only in a plane, which is likewise relatively fixed: it is required to determine the motion of this moving axis, regard being had to the diurnal rotation of the earth.

We will take the fixed plane, in which only the rotation-axis of the body can move, to be the plane of (ξ, η); so that ν is the angle at which this plane is inclined to that of the terrestrial equator. As the rotating disc is a solid of revolution, $A = B$; and if the axis of the disc is that of ζ', C is the principal moment relative to it, and we will suppose it to be greater than A. Also, as the rotation-axis is in the plane of (ξ, η), $\theta = 90°$, and consequently $\frac{d\theta}{dt} = 0$. Hence, by Art. 64,

$$\omega_1 = \sin\phi \frac{d\psi}{dt}, \qquad \omega_2 = \cos\phi \frac{d\psi}{dt}, \qquad \omega_3 = \frac{d\phi}{dt}; \qquad (173)$$

$$\left.\begin{array}{lll} a_1 = \cos\phi\cos\psi, & a_2 = \cos\phi\sin\psi, & a_3 = \sin\phi, \\ \beta_1 = -\sin\phi\cos\psi, & \beta_2 = -\sin\phi\sin\psi, & \beta_3 = \cos\phi, \\ \gamma_1 = \sin\psi; & \gamma_2 = -\cos\psi; & \gamma_3 = 0. \end{array}\right\} \qquad (174)$$

As to the circumstances of constraint. The body moves subject to the restraint of its axis being always in the plane of (ξ, η) and to the pressures at the two bearings of the axis at points in this plane; now the action lines of these pressures are always perpendicular to the planes of (ξ, η), and are thus parallel to the axis of ζ. Thus they produce a couple, whose axis is in the plane of (ξ, η) and is perpendicular to the axis of the gyroscope, which is the axis of ζ'. Let its moment be H; then H will appear in the equations of moments relative to the axes of ξ, η, ξ', η', but will not appear in those relative to the axes of ζ and ζ'. Also the forces which produce H will not appear in the equation of relative vis viva, because their points of application either are at relative rest, or move in a line perpendicular to their own line of action, and consequently do not appear in that equation, and no work is done by these forces.

As to the impressed forces, the only acting force is the earth's attraction; and if we suppose the dimensions of the gyroscope to be small, so that the earth's attraction may be considered to be the same at all points of the instrument, we may consider the centre of gravity to be the place of application of this force; and

consequently if the gyroscope is so constructed that there is no relative motion of its centre of gravity, the earth's attraction will not appear in the equations either of relative vis viva, or of moments about the axes of either set of coordinates.

Also the compound centrifugal forces will not appear in the equation of relative vis viva, because their lines of action are perpendicular to the relative line of motion of each particle.

These circumstances are also shewn in the equations of motion of the instrument, which are thus determined.

Let us take equations (55), Art. 408, which refer to the motion of a single particle, and replacing λ by $90°-\nu$, as ν is the angle at which the plane of (ξ, η) is inclined to the plane of the earth's equator, and extending the equations to the system of particles of the instrument, we have

$$\left.\begin{aligned}
\Sigma.m\left\{\frac{d^2\xi}{dt^2}-\omega^2 r\sin\nu\cos\nu-\omega^2\xi(\cos\nu)^2-\omega^2\zeta\sin\nu\cos\nu-2\omega\cos\nu\frac{d\eta}{dt}\right\}&=\Sigma.m\mathrm{X}', \\
\Sigma.m\left\{\frac{d^2\eta}{dt^2}-\omega^2\eta+2\omega\left(\cos\nu\frac{d\xi}{dt}+\sin\nu\frac{d\zeta}{dt}\right)\right\}&=\Sigma.m\mathrm{Y}', \\
\Sigma.m\left\{\frac{d^2\zeta}{dt^2}-\omega^2 r(\sin\nu)^2-\omega^2\xi\sin\nu\cos\nu-\omega^2\zeta(\sin\nu)^2-2\omega\sin\nu\frac{d\eta}{dt}\right\}&=\Sigma.m\mathrm{Z}';
\end{aligned}\right\} \quad (175)$$

the right-hand members in these equations denoting the impressed momenta which arise from gravity and from the pressure at the two bearings of the axis of the instrument.

437.] From these equations the equation of relative vis viva may be deduced; for from them we have

$$\Sigma.m\left\{\frac{d^2\xi}{dt^2}d\xi+\frac{d^2\eta}{dt^2}d\eta+\frac{d^2\zeta}{dt^2}d\zeta\right\}$$
$$=\Sigma.m\omega^2\{(r\sin\nu+\xi\cos\nu+\zeta\sin\nu)\cos\nu\, d\xi+\eta\, d\eta$$
$$+(r\sin\nu+\xi\cos\nu+\zeta\sin\nu)\sin\nu\, d\zeta\},$$

the other terms disappearing; and $\Sigma.m\,d\xi=\Sigma.m\,d\zeta=0$, as the centre of gravity has no relative motion; therefore integrating,

$$\Sigma.m v^2-\Sigma.m v_0^2=\omega^2\Big[\Sigma.m\{(\cos\nu)^2\xi^2+\eta^2+(\sin\nu)^2\zeta^2+2\sin\nu\cos\nu\,\xi\zeta\}\Big]_0^1. \quad (176)$$

Now the quantity within the brackets of limits is the moment of inertia of the system about an axis passing through the centre of gravity and parallel to the earth's axis, as appears from the value which is given in Art. 179. Let this quantity be H; then (176) becomes

$$\Sigma.m v^2-\Sigma.m v_0^2=\omega^2(\mathrm{H}-\mathrm{H}_0). \quad (177)$$

Also we may express the vis viva as follows: transforming the right-hand member of (176) into its equivalent in terms of ξ', η', ζ', and consequently of A, B, C the principal moments of the instrument, and taking the direction-cosines which are given in (174),

$$\Sigma . m\{(\cos\nu)^2\xi^2+\eta^2+(\sin\nu)^2\zeta^2 + 2\sin\nu\cos\nu\,\xi\zeta\}$$
$$= \Sigma . m\{(\cos\nu)^2(a_1{}^2\xi'^2+\beta_1{}^2\eta'^2+\gamma_1{}^2\zeta'^2)+a_2{}^2\xi'^2+\beta_2{}^2\eta'^2+\gamma_2{}^2\zeta'^2$$
$$+(\sin\nu)^2(a_3{}^2\xi'^2+\beta_3{}^2\eta'^2+\gamma_3{}^2\zeta')+2\sin\nu\cos\nu(a_1a_3\xi'^2+\beta_1\beta_3\eta'^2+\gamma_1\gamma_3\zeta'^2)\}$$
$$=(\cos\nu)^2\text{A}'(\cos\psi)^2+\text{C}'(\sin\psi)^2+\text{A}'(\sin\psi)^2+\text{C}'(\cos\psi)^2+\text{A}'(\sin\nu)^2$$
$$=\text{C}'+\text{A}'-(\text{C}'-\text{A}')(\sin\nu\sin\psi)^2$$
$$=\text{A}+(\text{C}-\text{A})(\sin\nu\sin\psi)^2; \tag{178}$$
$$\therefore\ \Sigma . mv^2-\Sigma . mv_0{}^2=(\text{C}-\text{A})(\omega\sin\nu)^2\{(\sin\psi)^2-(\sin\psi_0)^2\}; \tag{179}$$

and expressing the left-hand member in terms of ψ and ϕ, this becomes

$$\text{A}\Big\{\Big(\frac{d\psi}{dt}\Big)^2-\Big(\frac{d\psi}{dt}\Big)_0^2\Big\}+\text{C}\Big\{\Big(\frac{d\phi}{dt}\Big)^2-\Big(\frac{d\phi}{dt}\Big)_0^2\Big\}$$
$$=(\text{C}-\text{A})(\omega\sin\nu)^2\{(\sin\psi)^2-(\sin\psi_0)^2\}, \tag{180}$$

which is the equation of relative vis viva. Since $\omega\sin\nu$ is the component about the axis of ξ of the earth's diurnal rotation, $\omega\sin\nu\sin\psi$ is the component of that angular velocity about the axis of ζ', that is, about the axis of the gyroscope. Thus (179) shews that the increase of vis viva of the instrument depends on the increase in the square of the component of the earth's angular velocity about the axis of the instrument. It also appears that if C = A the vis viva of the instrument is constant, but that otherwise, the variation of it is periodical. This equation is that of the conservation of energy.

438.] The next step in the solution of this problem is the formation of the equations of moments, about the axes of ξ, η, ζ, ξ', η', ζ' respectively of which the general forms are given in Articles 432 and 433; and we will first take those of Art. 432. Now bearing in mind that by reason of the symmetry of the instrument with respect to the plane of (ξ,η) in which the axis of the instrument always is, $\Sigma . m\,\xi\zeta=\Sigma . m\,\eta\,\zeta=0$,

$$\text{L}'-\text{L}_t=\Sigma . m\{\eta(\text{Z}'-\text{Z}_t)-\zeta(\text{Y}'-\text{Y}_t)\}$$
$$=-\Sigma . m(\eta\,\text{Z}_t-\zeta\,\text{Y}_t)$$
$$=\omega^2\sin\nu\cos\nu\,\Sigma . m\,\xi\eta;$$

By similar processes we can shew that

$$\text{M}' - \text{M}_t = \omega^2 \sin\nu \cos\nu\, \Sigma.m(\zeta^2 - \xi^2);$$

$$\text{N}' - \text{N}_t = (\omega \sin\nu)^2\, \Sigma.m\,\xi\eta:$$

here $\Sigma.m\,\xi\eta = \Sigma.m(\alpha_1\alpha_2\xi'^2 + \beta_1\beta_2\eta'^2 + \gamma_1\gamma_2\zeta'^2)$

$$= (\text{A}' - \text{C}')\sin\psi\cos\psi = (\text{C} - \text{A})\sin\psi\cos\psi;$$

and similarly $\Sigma.m(\zeta^2 - \xi^2) = (\text{C} - \text{A})(\sin\psi)^2$;

$$\left.\begin{aligned} \therefore\ \text{L}' - \text{L}_t &= \omega^2(\text{C} - \text{A})\sin\nu\cos\nu\sin\psi\cos\psi,\\ \text{M}' - \text{M}_t &= \omega^2(\text{C} - \text{A})\sin\nu\cos\nu(\sin\psi)^2,\\ \text{N}' - \text{N}_t &= \omega^2(\text{C} - \text{A})(\sin\nu)^2\sin\psi\cos\psi. \end{aligned}\right\} \quad (181)$$

To the right-hand members of the first two of these equations terms are to be added to express the moments of the couples which arise from the pressures at the bearings of the axes; but as I do not propose to investigate these pressures, but only the circumstances of the motion of the axis of the gyroscope, it is unnecessary to add them. As the action lines of these pressures are parallel to the ζ-axis, they do not enter into the equation of moments about that axis. That equation is (159), which, when adapted to the special circumstances, takes the following value,

$$\text{A}\frac{d^2\psi}{dt^2} = (\text{C} - \text{A})(\omega\sin\nu)^2\sin\psi\cos\psi + \text{C}\,\omega\sin\nu\cos\psi\frac{d\phi}{dt}. \quad (182)$$

Also the equation of moments about the ζ'-axis is (164) which, when adapted to the special circumstances of the problem, takes the following form

$$\text{C}\frac{d\omega_3}{dt} = (\text{L}' - \text{L}_t)\sin\psi - (\text{M}' - \text{M}_t)\cos\psi - \text{C}\,\omega\sin\nu\cos\psi\frac{d\psi}{dt};$$

but $(\text{L}' - \text{L}_t)\sin\psi - (\text{M}' - \text{M}_t)\cos\psi = 0$:

$$\therefore\ \frac{d\omega_3}{dt} = -\omega\sin\nu\cos\psi\frac{d\psi}{dt};$$

therefore, if $\omega_3 = \Omega$, when $\psi = \psi_0$,

$$\omega_3 - \Omega = \omega\sin\nu(\sin\psi_0 - \sin\psi); \quad (183)$$

which gives the angular velocity of the instrument about its own axis in every position of it. The expression is evidently periodic, and shews that $\omega_3 = \Omega\sin\nu$ whenever $\sin\psi = \sin\psi_0$. Also, since $\omega\sin\nu\sin\psi$ is the component of the earth's diurnal rotation about the axis of the gyroscope, (183) shews that the sum of the relative angular velocity of the instrument about its own

axis and of the component about the same axis of the earth's angular velocity is constant throughout the motion.

And since $\omega_3 = \frac{d\phi}{dt}$,

$$\frac{d\phi}{dt} = \Omega + \omega \sin \nu (\sin \psi_0 - \sin \psi); \tag{184}$$

substituting this value in (182), we have

$$\text{A}\frac{d^2\psi}{dt^2} = -\text{A}(\omega \sin \nu)^2 \sin \psi \cos \psi + \text{C}\omega \sin \nu (\Omega + \omega \sin \nu \sin \psi_0) \cos \psi;$$

$$\therefore \quad \text{A}\left\{\left(\frac{d\psi}{dt}\right)^2 - \left(\frac{d\psi}{dt}\right)^2\right\} = \text{A}(\omega \sin \nu)^2 \{(\sin \psi_0)^2 - (\sin \psi)^2\}$$
$$+ 2\text{C}\omega \sin \nu (\Omega + \omega \sin \nu \sin \psi_0)(\sin \psi - \sin \psi_0), \tag{185}$$

which equation gives the angular velocity of the axis of the instrument in the plane of (ξ, η) about the axis of ζ, and does not admit of further integration.

These equations, viz. (180), (184), (185), are sufficient for the determination of ϕ and ψ in terms of t, and consequently of the general motion of the instrument; they may however be simplified without loss of generality by the following arrangement of the axes at the initial epoch.

439.] Let us suppose the frame of the instrument to be set in a position of relative rest, and an angular velocity Ω to be given to the rotating disc or ball about its axis, this Ω being, as above, the initial value of ω_3. Then the axes of ξ, η, ζ being arranged as already explained, let the axes of ξ', η' be so taken that the axis of ξ' is at the initial time in the plane of (ξ, η), the axis of ζ' being in that plane and at an angle $\psi_0 - 90°$ from the axis of ξ; then

$$\phi_0 = 0, \qquad \left(\frac{d\phi}{dt}\right)_0 = \Omega, \qquad \left(\frac{d\psi}{dt}\right)_0 = 0;$$

and the equations (184), (185), (180) take the following forms

$$\frac{d\phi}{dt} = \Omega + \omega \sin \nu (\sin \psi_0 - \sin \psi); \tag{186}$$

$$\text{A}\frac{d\psi^2}{dt^2} = \text{A}(\omega \sin \nu)^2 \{(\sin \psi_0)^2 - (\sin \psi)^2\}$$
$$+ 2\text{C}\omega \sin \nu (\Omega + \omega \sin \nu \sin \psi_0)(\sin \psi - \sin \psi_0); \tag{187}$$

$$\text{A}\frac{d\psi^2}{dt^2} + \text{C}\frac{d\phi^2}{dt^2} - \text{C}\Omega^2 = (\text{C} - \text{A})(\omega \sin \nu)^2 \{(\sin \psi)^2 - (\sin \psi_0)^2\}. \tag{188}$$

If in this last equation $\frac{d\phi}{dt}$ is replaced by its value given in

(186), we obtain (187), thus shewing that the three equations are not independent, but that they are equivalent to only two; these however are sufficient for the problem as it has only two degrees of freedom, viz. the motion of the axis of the instrument in its plane of constraint and the angular velocity of the disc or ball about that axis; and these angular velocities are $\frac{d\psi}{dt}$ and $\frac{d\phi}{dt}$ respectively. The equations shew that these velocities are generally of a periodic character, and that the motion is generally oscillatory. The complete solution requires ψ and ϕ to be expressed in terms of t; this however is impossible in the most general form as the equations are not integrable; we are obliged therefore, as in other problems, to have recourse to approximations, the circumstances of the instrument affording means of estimating small terms, higher powers of which may be neglected in comparison of those of lower powers.

440.] If a very rapid rotation about its axis is given to the disc or ball of the instrument, Ω is very large in comparison of $\omega \sin \nu$, which is the angular velocity of the earth about the axis of ξ; and if the friction of the bearings and the loss of momentum due to the motion of the instrument taking place in a resisting medium be diminished as far as possible, or if the rotation of the disc is kept up to Ω by means of a souffleur or other instrument which will not interfere with the proper action of the gyroscope, then certain terms in the right-hand of (187) may be omitted, and the motion is approximately expressed by the equation

$$\text{A} \frac{d\psi^2}{dt^2} = 2\text{C}\Omega\omega \sin \nu (\sin \psi - \sin \psi_0). \qquad (189)$$

In experiments with the gyroscope a rapid rotation is given to the disc, and this is maintained for a time sufficiently long to exhibit the phenomena of the instrument.

It will be convenient to make a slight change in the form of this equation. ψ is the angle measured in the plane of (ξ, η) between the axis of ξ and the line of intersection of the planes of (ξ, η) and (ξ', η'). The axis of ζ' is also in the plane of (ξ, η) and is perpendicular to this line of intersection: if therefore v is the angle between the axes of ξ and ζ',

$$\psi = v + 90^\circ; \qquad (190)$$

and substituting v in (189), we have

$$\text{A} \frac{dv^2}{dt^2} = 2 \, \text{C} \, \Omega \, \omega \sin \nu \, (\cos v - \cos v_0). \qquad (191)$$

Now this equation is the same in form as that which expresses the motion of a pendulum under the action of a constant force whose line of action is parallel to that of the line of the rod of the pendulum when it is at rest. Let us then compare the motion of the rotation-axis of the body as expressed by (191) with that of the pendulum; and let us assume the length of the pendulum to be unity; so that the constant force, under the action of which the rotation-axis may be supposed to move, is $\frac{\text{C} \Omega \omega \sin \nu}{\text{A}}$, the line of which is the ξ-axis, and is the projection southwards on the plane of (ξ, η) of the earth's polar axis. Now let us suppose v_0 to be a small angle, and the angle between the axis of the gyroscope and the axis of ξ always to be small; then, as on a similiar supposition, the pendulum vibrates through small arcs to equal distances on either side of the vertical line, so will the rotation-axis of the disc vibrate over equal small angles on either side of the ξ-axis. And as the pendulum remains always at rest in a vertical line, if it is ever at rest in it; so will the rotation-axis always be at relative rest along the ξ-axis, if it is ever at rest in it. If therefore the rotation-axis is on the ξ-axis when $\frac{d\psi}{dt} = 0$, it always remains on it, and has no oscillation.

Also, as the pendulum has two positions of rest, one of stable equilibrium, when it hangs vertically downwards from its point of suspension, and another when it is balanced on its point of suspension, the centre of gravity having its lowest and its highest position in the two cases respectively; so are there two positions of relative rest of the rotation-axis of the disc, one of which is of stable, and the other of unstable rest. Now if ω and Ω have the signs given to them in (191); that is, if the direction of Ω is contrary to that of the earth, then the rest of the rotation-axis will be stable or unstable, according as the axis coincides with the positive or negative direction of the ξ-axis. And the contrary will be the case, when the direction of Ω is the same as that of the earth.

If the rotation-axis is in its position of stable rest, and is slightly disturbed therefrom by an extraneous force, it oscillates

in the plane of (ξ, η) over a small angle on either side of the line of rest; and if T is the time of an oscillation,

$$T = \pi \left\{ \frac{A}{C \Omega \omega \sin \nu} \right\}^{\frac{1}{2}}. \tag{192}$$

If however the rotation-axis is in its position of unstable rest, and is slightly displaced therefrom, it goes farther from that position and does not return to it, until it has passed through 360° in its plane of motion.

441.] In particular cases these results take forms which are of considerable interest.

(1) If the plane of (ξ, η), in which the rotation-axis of the body is constrained to move, is horizontal at the place of observation: and if the latitude of the place is λ; then $\sin \nu = \cos \lambda$; and from (192),

$$T = \pi \left\{ \frac{A}{C \Omega \omega \cos \lambda} \right\}^{\frac{1}{2}}. \tag{193}$$

In this case the ξ-axis is the meridian line, of which the positive direction is that towards the south. If the direction of the rotation of the body is the same as that of the earth, the position of rest of the rotation-axis will be of stable or unstable equilibrium according as it is drawn from the point towards the north or towards the south; and if the rotation is contrary to that of the earth, the rotation-axis will be in stable or unstable rest according as its direction is due south or due north.

(2) If the plane of (ξ, η), in which the rotation-axis is constrained to move, is the meridian plane at the place of observation; then $\nu = 90°$; and the line of relative equilibrium of the rotation-axis is parallel to the earth's polar axis; and the equilibrium of the axis is stable or unstable according as the direction of rotation is contrary to, or is the same as, that of the earth. In this case

$$T = \pi \left\{ \frac{A}{C \Omega \omega} \right\}^{\frac{1}{2}}; \tag{194}$$

so that, cæteris paribus, the time of oscillation is less in this case than it is when the rotation-axis moves in the horizontal plane; and generally the oscillations in the meridian plane are quicker than in any other plane.

(3) These last results however are not limited to the meridian plane; for $\sin \nu = 1$ for all planes drawn at the place of observation parallel to the earth's polar axis.

(4) If the plane of (ξ, η) is perpendicular to the earth's polar axis, $\nu = 0$; and $\mathrm{T} = \infty$; so that the rotation-axis of the gyroscope is at rest for all positions in that plane.

(5) If the number of oscillations of the rotation-axis in the meridian plane is determined by observation, T is known; and consequently, from (194),

$$\omega = \frac{\pi^2 \mathrm{A}}{\mathrm{C}\Omega\mathrm{T}^2}; \tag{195}$$

and thus the angular velocity of the earth may be determined.

(6) If T and T′ are the times of oscillation of the rotation-axis in the horizontal and the meridian planes respectively at a given place, corresponding to the same value of Ω, then

$$\cos\lambda = \frac{\mathrm{T}'^2}{\mathrm{T}^2}.$$

From all these theorems we conclude, that if the phenomena of the gyroscope are observed with sufficient care, we can by them determine the meridian line and the altitude of the pole at the place; and consequently the latitude: we can determine also the direction of the diurnal rotation of the earth, and, from (195), the mean length of the sidereal day. All these results then are confirmations, if they are required, of the evidence of that motion of the earth which astronomical phenomena suggest to us. And although the proof of the diurnal rotation, thus acquired, may not be as palpable as that afforded by astronomical observation, yet it is not to be rejected as useless, nor is its investigation to be regarded as idle speculation; for evidence supporting theories of cosmical phenomena is cumulative; and the value of any addition to it increases in geometrical ratio.

442.] Also in the gyroscope, as ordinarily constructed, certain parts can be clamped so that the axis of rotation of the disc or ball can move only on the surface of a right circular cone, the vertex of which is at the mass-centre of the instrument. We will take the axis of this cone to be the ζ-axis, the plane of (ξ, η) being that which is perpendicular to it, and passes through the fixed point; the axes of ξ and η being so placed that if the axis of the cone is the vertical line at the fixed point, they become respectively the lines drawn southwards and westwards. Let ν be the angle between the axis of the cone and the earth's polar axis; and let a be the semi-vertical angle of the cone, so

that $\theta = \alpha =$ a constant, and consequently, $\frac{d\theta}{dt} = 0$. Hence, by Art. 64,

$$\omega_1 = \sin\alpha \sin\phi \frac{d\psi}{dt}, \quad \omega_2 = \sin\alpha\cos\phi\frac{d\psi}{dt}, \quad \omega_3 = \frac{d\phi}{dt} + \cos\alpha\frac{d\psi}{dt}; \quad (196)$$

$$\left.\begin{aligned} \alpha_1 &= \cos\phi\cos\psi - \sin\phi\sin\psi\cos\alpha, \\ \beta_1 &= -\sin\phi\cos\psi - \cos\phi\sin\psi\cos\alpha, \\ \gamma_1 &= \sin\psi\sin\alpha; \end{aligned}\right\} \quad \left.\begin{aligned} \alpha_2 &= \cos\phi\sin\psi + \sin\phi\cos\psi\cos\alpha, \\ \beta_2 &= -\sin\phi\sin\psi + \cos\phi\cos\psi\cos\alpha, \\ \gamma_2 &= -\cos\psi\sin\alpha; \end{aligned}\right\} \quad (197)$$

$$\alpha_3 = \sin\phi\sin\alpha, \quad \beta_3 = \cos\phi\sin\alpha, \quad \gamma_3 = \cos\alpha. \qquad (198)$$

We have to express the equations of relative vis viva and of angular velocity of the disc about the axis of ζ', which is its axis of rotation, in terms of ϕ and ψ and their t-differentials.

As to the equation of relative vis viva; since all that has been said in Art. 436 as to the pressures at the bearings of the axis is applicable to the case, we may take equation (176) as its expression: then if we suppose Ω to be the initial rotation of the disc, and its axis to be initially at relative rest, the left-hand member becomes, by means of (196),

$$\text{A}(\sin\alpha)^2\frac{d\psi^2}{dt^2} + \text{C}\left(\frac{d\phi}{dt} + \cos\alpha\frac{d\psi}{dt}\right)^2 - \text{C}\Omega^2;$$

and the coefficient of ω^2 in the right-hand member, when transformed by means of (197) and (198) as in Art. 437, takes the form $\omega^2(\text{C}-\text{A})\{(\sin\nu\sin\alpha)^2\{(\sin\psi)^2 - (\sin\psi_0)^2\}$

$$+ 2\sin\alpha\cos\alpha\sin\nu\cos\nu(\sin\psi - \sin\psi_0)\};$$

so that the equation of relative vis viva is

$$\text{A}(\sin\alpha)^2\frac{d\psi^2}{dt^2} + \text{C}\left(\frac{d\phi}{dt} + \cos\alpha\frac{d\psi}{dt}\right)^2 - \text{C}\Omega^2$$
$$= \omega^2(\text{C}-\text{A})\sin\nu\sin\alpha\{\sin\nu\sin\alpha\{(\sin\psi)^2 - (\sin\psi_0)^2\}$$
$$+ 2\cos\nu\cos\alpha(\sin\psi - \sin\psi_0)\}. \quad (199)$$

Also the equation of moments about the ζ'-axis is as follows; taking the equation as given in (164), we have

$$\text{C}\frac{d\omega_3}{dt} = \Sigma.m(\xi'\text{Y}_1 - \eta'\text{X}_1) + \text{C}\omega\sin\nu(\beta_1\omega_1 - \alpha_1\omega_2) - \text{C}\omega\cos\nu(\beta_3\omega_1 - \alpha_3\omega_2), \quad (200)$$

the pressures at the bearings of this axis not entering into the equation, as their lines of action are parallel to the axis.

Now

$$\Sigma.m(\xi'\text{Y}_1 - \eta'\text{X}_1)$$
$$= \Sigma.m\{(\beta_1\xi' - \alpha_1\eta')(\text{X}' - \text{X}_t) + (\beta_2\xi' - \alpha_2\eta')(\text{Y}' - \text{Y}_t) + (\beta_3\xi' - \alpha_3\eta')(\text{Z}' - \text{Z}_t)\}$$
$$= -\Sigma.m\{(\beta_1\xi' - \alpha_1\eta')\text{X}_t + (\beta_2\xi' - \alpha_2\eta')\text{Y}_t + (\beta_3\xi' - \alpha_3\eta')\text{Z}_t\} = 0, \quad (201)$$

omitting the terms which disappear by reason of the axes of ξ', η', ζ' being central principal axes. Also

$$\beta_1\omega_1 - \alpha_1\omega_2 = \sin\alpha\frac{d\psi}{dt}(\beta_1\sin\phi - \alpha_1\cos\phi) = -\sin\alpha\cos\psi\frac{d\psi}{dt},$$

$$\beta_3\omega_1 - \alpha_3\omega_2 = \sin\alpha\frac{d\psi}{dt}(\beta_3\sin\phi - \alpha_3\cos\phi) = 0;$$

so that (200) becomes

$$\text{C}\frac{d\omega_3}{dt} = -\text{C}\,\omega\sin\nu\sin\alpha\cos\psi\frac{d\psi}{dt};$$

$$\therefore\quad \omega_3 - \Omega = -\omega\sin\nu\sin\alpha(\sin\psi - \sin\psi_0); \qquad (202)$$

and replacing ω_3 by its value as given in (196), we have

$$\frac{d\phi}{dt} + \cos\alpha\frac{d\psi}{dt} = \Omega - \omega\sin\nu\sin\alpha(\sin\psi - \sin\psi_0). \qquad (203)$$

This equation and that of vis viva given in (199) are together sufficient for the determination of the problem, as there are only two degrees of freedom, and these are two independent equations.

Substituting in (199) the value of $\frac{d\phi}{dt}$ which is given in (203), we have

$$\text{A}(\sin\alpha)^2\frac{d\psi^2}{dt^2} = 2\omega^2\sin\alpha\sin\nu\{\text{C}\sin\alpha\sin\nu\sin\psi_0 + (\text{C}-\text{A})\cos\alpha\cos\nu\}(\sin\psi - \sin\psi_0)$$
$$-\text{A}(\omega\sin\alpha\sin\nu)^2\{(\sin\psi)^2 - (\sin\psi_0)^2\} + 2\text{C}\Omega\omega\sin\alpha\sin\nu(\sin\psi - \sin\psi_0). \quad (204)$$

It is evident that these equations do not admit of integration in their general forms as given in (203) and (204). They shew however that $\frac{d\phi}{dt}$ and $\frac{d\psi}{dt}$ have periodical values; $\frac{d\psi}{dt}$ being equal to zero whenever $\sin\psi = \sin\psi_0$, and $\frac{d\phi}{dt}$ in that case being equal to Ω. If $\alpha = 90°$, the cone on the surface of which the axis of the gyroscope moves becomes a plane, and we have the problem of Article 439; and (203) and (204) become (186) and (188) respectively.

443.] If, as in Art. 440, a very rapid rotation is given to the disc and that rotation is maintained, ω is initially and continues very small in comparison of Ω, and the terms involving ω^2 may be omitted in (204), so that we have the equation

$$\text{A}(\sin\alpha)\frac{d\psi^2}{dt^2} = 2\,\text{C}\,\Omega\,\omega\sin\nu(\sin\psi - \sin\psi_0); \qquad (205)$$

which is of the same form as (189) in Art. 440, and may be treated in a similar manner. The results and the interpretation of the results are of course similar, and it is unnecessary to repeat

them. It may be observed that the time of a small oscillation of the rotation-axis of the disc

$$= \pi \left\{ \frac{\text{A} \sin \alpha}{\text{C} \Omega \omega \sin \nu} \right\}^{\frac{1}{2}}; \tag{206}$$

this quantity vanishes when $\alpha = 0$ and becomes infinite when $\sin \nu = 0$.

444.] Let us now consider the motion of the gyroscope, when its axis is entirely unconstrained.

Let the line passing through the centre of gravity of the instrument and parallel to the axis of the earth be the ζ-axis; so that the plane of (ξ, η) is parallel to that of the earth's equator; let the line of intersection of this plane with the meridian plane of the place be the ξ-axis reckoned positively away from the earth, and let the η-axis be taken positively westwards; then, ω being as heretofore the angular velocity of the earth,

$$\omega_{\xi} = \omega_{\eta} = 0; \quad \omega_{\zeta} = -\omega; \tag{207}$$

$$\text{X}_t = -\omega^2 (r \cos \lambda + \xi); \quad \text{Y}_t = \text{Z}_t = 0. \tag{208}$$

The nine direction-cosines which connect the principal system of the instrument with the system of ξ, η, ζ as determined above, and also $\omega_1, \omega_2, \omega_3$ have their most general values; hence the system will have three degrees of relative freedom, and consequently three independent equations will be required for the solution of the problem; these we will take to be the equations of (1) relative vis viva; (2) moments about the axis of ζ; (3) moments about the axis of ζ'; and they will contain $\frac{d\theta}{dt}, \frac{d\psi}{dt}, \frac{d\phi}{dt}$; the further values of θ, ψ, ϕ to be obtained by integration determining the position of the axis at any time t.

To determine the equation of relative vis viva, we have the following values, supposing Ω to be the angular velocity initially communicated to the disc, and the axis of the instrument to be initially at relative rest,

$$\text{A}\left\{ (\sin\theta)^2 \frac{d\psi^2}{dt^2} + \frac{d\theta^2}{dt^2} \right\} + \text{C}\,(\omega_3{}^2 - \Omega^2)$$

$$= 2\int_0^1 \Sigma . m \left\{ (\text{X}' - \text{X}_t)\, d\xi + (\text{Y}' - \text{Y}_t)\, d\eta + (\text{Z}' - \text{Z}_t)\, d\zeta \right\}$$

$$= 2\omega^2 \int_0^1 \Sigma . m (r \cos\lambda + \xi)\, d\xi = \omega^2 \Big[\Sigma . m\, \xi^2 \Big]_0^1$$

$$= -\omega^2 (\text{C} - \text{A}) \left\{ (\sin\psi)^2 (\sin\theta)^2 - (\sin\psi_0)^2 (\sin\theta_0)^2 \right\}, \tag{209}$$

where ψ_0 and θ_0 are the initial values of ψ and θ.

And for moments about the axis of ζ, taking equation (159), we have

$$\frac{d}{dt}\{\text{A}(\omega_1\alpha_3+\omega_2\beta_3)+\text{C}\,\omega_3\gamma_3\} = \Sigma.m\,\{\xi\,(\text{Y}'-\text{Y}_t)-\eta\,(\text{X}'-\text{X}_t)\} + \omega\,\frac{d}{dt}\{\text{A}(\alpha_3{}^2+\beta_3{}^2)+\text{C}\gamma_3{}^2\};$$

but
$$\omega_1\alpha_3+\omega_2\beta_3 = \frac{d\psi}{dt}(\sin\theta)^2;$$
$$\Sigma.m\{\zeta(\text{Y}'+\text{Y}_t)-\eta\,(\text{X}'+\text{X}_t)\} = 0;$$
$$\text{A}(\alpha_3{}^2+\beta_3{}^2)+\text{C}\gamma_3{}^2 = \text{A}+(\text{C}-\text{A})\gamma_3{}^2 = \text{A}+(\text{C}-\text{A})(\cos\theta)^2;$$
so that the equation becomes
$$\frac{d}{dt}\{\text{A}(\sin\theta)^2\frac{d\psi}{dt}+\text{C}\,\omega_3\cos\theta\} = \omega\,(\text{C}-\text{A})\frac{d}{dt}(\cos\theta)^2; \tag{210}$$
and integrating between the given limits
$$\text{A}(\sin\theta)^2\,\frac{d\psi}{dt}+\text{C}(\omega_3\cos\theta-\Omega\cos\theta_0)=\omega(\text{C}-\text{A})\{(\cos\theta)^2-(\cos\theta_0)^2\}. \tag{211}$$
Also for moments about the axis of ζ', taking equation (164), we have
$$\text{C}\frac{d\omega_3}{dt} = 2\text{A}'\omega\,(\omega_2\alpha_3-\omega_1\beta_3)$$
$$= -\text{C}\,\omega\sin\theta\frac{d\theta}{dt};$$
$$\therefore\ \ \omega_3-\Omega = \omega\,(\cos\theta-\cos\theta_0). \tag{212}$$
These three equations, viz. (209), (211) and (212), are independent, and are sufficient for the solution of the problem. It is evident that in their general forms they do not admit of further integration; they give rise however to the following equations.

Eliminating ω_3 by means of (211) and (212), we have
$$\text{A}\,(\sin\theta)^2\frac{d\psi}{dt} = (\cos\theta_0-\cos\theta)\{\text{A}\,\omega(\cos\theta+\cos\theta_0)+\text{C}(\Omega-\omega\cos\theta_0)\}; \tag{213}$$
and substituting this value of $\frac{d\psi}{dt}$ in (209), we obtain a value of $\frac{d\theta}{dt}$ in terms of θ only, from which, theoretically, θ may be found in terms of t; these expressions shew that the values of θ and ψ are generally oscillatory, and have nutatory values in the plane of the meridian and in the plane parallel to that of the earth's equator. Let us however take some particular cases.

445.] If the disc of the gyroscope is not initially put into rotation, so that $\Omega = 0$, then $\frac{d\theta}{dt} = 0$; thus θ does not vary but

is constantly equal to θ_0; hence (213) shews that $\frac{d\psi}{dt} = 0$, and hence also $\omega_3 = 0$; so that notwithstanding the diurnal rotation of the earth the axis of the disc remains at relative rest.

If, however, we suppose Ω to be very great in comparison of ω, as is ordinarily the case with the gyroscope, we may omit all powers of ω above the first, retaining the first however as otherwise the effect of the earth's rotation would not appear in the equations; then eliminating $\frac{d\psi}{dt}$ between (209) and (213) we have

$$\text{A}(\sin\theta)^2\frac{d\theta^2}{dt^2} = \text{C}\Omega(\cos\theta_0 - \cos\theta)$$
$$\{2\text{A}\omega(\sin\theta_0)^2 + \text{C}(\Omega - 2\omega\cos\theta_0)(\cos\theta - \cos\theta_0)\}. \quad (214)$$

To simplify the form of this equation, let

$$\left.\begin{aligned} \text{C}^2\Omega(\Omega - 2\omega\cos\theta_0) &= \text{A}^2k^2, \\ \text{C}\Omega\{\text{C}\cos\theta_0(\Omega - 2\omega\cos\theta_0) - 2\text{A}\omega(\sin\theta_0)^2\} &= \text{A}^2k^2\cos\alpha; \end{aligned}\right\} \quad (215)$$

whence
$$\cos\alpha = \cos\theta_0 - \frac{2\text{A}\omega}{\text{C}(\Omega - 2\omega\cos\theta_0)}(\sin\theta_0)^2; \quad (216)$$

which gives a real value to α in the ordinary case of the gyroscope. Making these substitutions in (214), we have

$$(\sin\theta)^2\frac{d\theta^2}{dt^2} = k^2(\cos\theta_0 - \cos\theta)(\cos\theta - \cos\alpha); \quad (217)$$

so that θ always lies between α and θ_0, which, as shown in (216), are two angles nearly equal; consequently the inclination of the rotation-axis of the disc to the earth's polar axis is nearly constant, only varying between limits which are very nearly equal to each other.

Integrating (217) we have

$$\cos\theta = \frac{\cos\theta_0 + \cos\alpha}{2} + \frac{\cos\theta_0 - \cos\alpha}{2}\cos kt; \quad (218)$$

and replacing $\cos\alpha$ by its value, and omitting $2\omega\cos\theta_0$ when subtracted from Ω, we have

$$\cos\theta = \cos\theta_0 - \frac{\text{A}\omega}{\text{C}\Omega}(\sin\theta_0)^2\left(1 - \cos\text{A}\frac{\text{C}\Omega}{\text{A}\omega}t\right). \quad (219)$$

If we substitute this value of $\cos\theta$ in (213) and omit terms involving the square and higher powers of ω,

$$\frac{d\psi}{dt} = \omega\left(1 - \cos\frac{\text{C}\Omega}{\text{A}}t\right). \quad (220)$$

$$\therefore\ \psi - \psi_0 = \omega t - \frac{\text{A}\omega}{\text{C}\Omega}\sin\frac{\text{C}\Omega}{\text{A}}t, \quad (221)$$

where ψ_0 is the value of ψ, when $t = 0$.

If then we consider the first terms in (219) and (220), which are the principal terms, it appears that the rotation-axis of the disc revolves uniformly round an axis parallel to the earth's axis in a direction contrary to that of the earth, and that it is inclined to this axis at an angle almost constant: and besides this general precessional motion the axis has also motion of nutation both parallel to and perpendicular to the plane of the earth's equator; and that the periodic time of these nutations $= \frac{2\pi A}{C\Omega}$; so that, the greater the initial angular velocity of the disc, the shorter is the periodic time.

446.] From this investigation then the following results follow:

(1) If the disc of the gyroscope has not any initial angular velocity, it remains at relative rest with the earth, whether the earth rotates or not.

(2) If the disc rotates with a very rapid angular velocity, and its axis is placed in a position of relative equilibrium with the earth, then that equilibrium would continue if the earth did not rotate; but if the earth rotates, the axis of disc has a relative motion.

(3) And the direction in which this motion takes place does not depend on that of the angular velocity of the disc, but is always opposite to that of the angular velocity of the earth; and consequently, if it is observed, it indicates the direction in which the earth rotates; and thus its motion supplies evidence of the rotation of the earth.

(4) The angle of inclination of the rotation-axis of the disc to the axis of the earth is nearly constant throughout the motion of the disc: there are however small nutational variations of this angle as well as of the precessional velocity of the axis, the periodic time of which decreases according as the angular velocity of the disc increases.

For further investigations of the subject of the gyroscope, I must refer the student to a memoir by M. Quet contained in Liouville's Journal, Vol. XVIII, and to another Mathematical investigation by M. Yvon Villarceau, p. 343, Vol. XIV, Nouvelles Annales des Mathématiques, Paris, 1855.

CHAPTER IX.

THE MOTION OF ELASTIC BODIES.

447.] THE principles and laws of motion have thus far been applied to rigid bodies, and to systems of rigid bodies, the constituent molecules of which have been assumed to be in a state of relative rest during the motion; and the equations of motion by which problems have heretofore been solved have been deduced from these principles thus restricted. Our purpose is to apply them more generally. Ere, however, we do so, there are two reasons why we should repeat as concisely as possible the modification of the equations which this assumption of the relative rest of the constituent molecules introduces. (1) Because we have come to the end of our investigations on that subject, and it is good once more prominently to restate the conspicuous principle of the process so frequently employed: and (2) because in the present chapter we shall investigate equations expressing the motion of a particle which is not at rest relatively to its neighbouring particles, all being constituent molecules of a body; and our research will include the varying form of flexible bodies, (as they are called,) the molecules of which move relatively to each other; and our conception of such motions will be more exact when they are contrasted with those of the molecules of a rigid body in their chief differences.

The equations of motion of a rigid body are found by the following process: Let dm be an element of the body, and let (x, y, z) be its place at the time t, relatively to a system of coordinate axes fixed in space. Now this particle is supposed to be under the action of certain external forces, whereby a certain velocity or velocity-increment is impressed on it. In consequence of this external force it would have a definite expressed velocity-increment if it were alone, and thus free from all constraint from its surrounding molecules. As it is not free, the constraints enter as other forces, which, affecting its motion, produce a change of its expressed velocity-increments: these constraints we consider as internal forces, which produce their own effects;

and these effects modify those which would otherwise take place. And consequently, if X, Y, Z are the axial components of the velocity-increment impressed on dm by the external forces, and if I is the resultant of the momentum-increment due to all the internal forces or constraints acting on the particle, of which α, β, γ are the direction-angles; then the equations of motion of dm are

$$\left.\begin{aligned} dm\left\{\mathrm{X}-\frac{d^2x}{dt^2}\right\}+\mathrm{I}\cos\alpha &= 0,\\ dm\left\{\mathrm{Y}-\frac{d^2y}{dt^2}\right\}+\mathrm{I}\cos\beta &= 0,\\ dm\left\{\mathrm{Z}-\frac{d^2z}{dt^2}\right\}+\mathrm{I}\cos\gamma &= 0;\end{aligned}\right\} \quad (1)$$

from which also arise three other equations, which express the rotation of dm; viz.

$$\left.\begin{aligned} dm\left\{y\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)-z\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)\right\}+\mathrm{I}\,(y\cos\gamma-z\cos\beta) &= 0,\\ dm\left\{z\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)-x\left(\mathrm{Z}-\frac{d^2z}{dt^2}\right)\right\}+\mathrm{I}\,(z\cos\alpha-x\cos\gamma) &= 0,\\ dm\left\{x\left(\mathrm{Y}-\frac{d^2y}{dt^2}\right)-y\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)\right\}+\mathrm{I}\,(x\cos\beta-y\cos\alpha) &= 0.\end{aligned}\right\} \quad (2)$$

Equations of the same form as those in (1) and (2) are true for every molecule of the body. Let these be formed; then we shall have a series of groups of equations expressing the motion of every molecule, the sum of which will express the motion of the whole body. And here enters the characteristic of the rigidity of the body: all the internal forces and their consequent velocity-increments enter in pairs, of which the directions are opposite to each other; every constraint, acting from (say) dm to dm', has an equal and opposite constraint acting from dm' to dm: the law of the equality of action and reaction is true in this case of every pair of molecules; so that

$$\Sigma.\mathrm{I}\cos\alpha = \Sigma.\mathrm{I}\cos\beta = \Sigma.\mathrm{I}\cos\gamma = 0. \quad (3)$$

$$\Sigma.\mathrm{I}(y\cos\gamma-z\cos\beta)=\Sigma.\mathrm{I}\,(z\cos\alpha-x\cos\gamma)=\Sigma.\mathrm{I}\,(x\cos\beta-y\cos\alpha)=0. \quad (4)$$

And therefore, adding together (1) and all its similar groups, and (2) and all its similar groups, we obtain the equations of motion of a rigid body which are given in (37) and (38), Art. 73.

The same process of reasoning is applicable to the motion in space of a system of rigid bodies moving relatively to each other, if the internal action of one on another is always accompanied

by an equal and opposite reaction; because these will disappear in the summation of the several equations, when that extends to and includes all the molecules of all the moving bodies.

448.] In the problem of the present chapter, however, the subject of motion is a body, the molecules of which move one relatively to another, and the bounding form of which hereby generally changes. A fine vibrating string, a thin vibrating membrane, a mass of quivering jelly or caoutchouc, are such bodies as we here contemplate. In these the form of the bounding surface will change from time to time; and so also will the relative arrangement of the constituent molecules. When the molecules move one relatively to another, internal forces are brought into action which affect their motion: these are generally called elastic forces. These forces vary from molecule to molecule, and also from time to time; so that if the body is referred to a system of axes fixed in space, and (x, y, z) is the place of dm at the time t, the elastic forces acting on dm are functions of x, y, z, and t. In the most general case we suppose external forces to act on the several molecules of the body; so that dm is acted on by these as well as by the elastic forces, and both will enter into its equations of motion. Thus, if I is the whole elastic force acting on dm at the time t, and α, β, γ are the direction-angles of its line of action, the equations of motion of dm are those given in (1) and (2). Similar equations will express the motion of every particle of the body. Now we cannot take the sum of all these, and thereby determine the motion of the whole body, as the process is in the case of a rigid body; (1) because our object is to determine the form of the body at any time, and to do this it is necessary to determine the place of every particle at that time; so that the set of equations corresponding to a given particle must be separately considered, and its place therefrom determined: and (2) because all the internal forces may not be in equilibrium amongst themselves; and consequently the conditions (3) and (4) may not be satisfied. These internal or elastic or molecular forces, as they are called, may enter in pairs of equal and opposite forces in the interior of the body, and thus far may disappear in the sum corresponding to the sum of all the particles; but at the bounding surface they may be counteracted by and thus be in equilibrium with certain external forces thereat acting; so that all will not disappear in

the sum of the groups of the equations corresponding to all the particles of the body. Herein then is the difference of the mode of formation of the equations of motion of a rigid body and of a molecule of an elastic body.

449.] But it is not my intention to enter into the consideration of the equilibrium and motion of the particles of an elastic system in its general form. The subject of elasticity is too large and too important to be treated satisfactorily in a single Chapter at the close of this work; no less than a volume is required for the full investigation in all its forms, developments and applications. There are however some special problems of elastic material systems in one and two (approximately) dimensions, that is of fine strings and thin membranes which admit of treatment on principles which have been expounded in this and the preceding volumes; and as they are of considerable interest, and as our treatise without them would hardly be complete, I propose to investigate them. I shall assume the truth of Hooke's Law, 'ut tensio sic vis,' without enquiring into its origin or its evidence, and by means of it form the equations of motion.

450.] I will, in the first place, consider the motion of the particles of a perfectly flexible fine thread or string, which is in the general case extensible and elastic; then the equations of motion are formed as follows: as the string is elastic, elastic forces or stresses are brought into action when either an extension or a compression of the string takes place; and supposing the change of figure or strain to be very small the consequent stress by Hooke's Law varies as the strain. The string is supposed to have been slightly displaced from its position of rest, and at the time t to be under the action of the stresses thus brought into existence, as well as of external forces. We will assume that its form is that of a curve of double curvature, and we refer it to a system of rectangular axes fixed in space.

Let dm be an element of its mass, whose place at the time t is (x, y, z). Let $ds =$ the length of this element, and let ρ $=$ its density; let $\omega =$ the area of a transverse section of the string: so that $dm = \rho\omega ds$. Let X, Y, Z be the axial components of the impressed velocity-increments on dm; and let $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ be the axial components of the expressed velo-

city-increments. Let $\text{T} =$ the tension of the thread at the point (x, y, z), which I take to be the beginning of ds: then, as the tangent is the line of action of T, the axial components of T are $\text{T}\frac{dx}{ds}$, $\text{T}\frac{dy}{ds}$, $\text{T}\frac{dz}{ds}$; and the axial components of the tension at $(x+dx,\ y+dy,\ z+dz)$, which is the other end of ds, are

$$\text{T}\frac{dx}{ds}+d.\text{T}\frac{dx}{ds}, \qquad \text{T}\frac{dy}{ds}+d.\text{T}\frac{dy}{ds}, \qquad \text{T}\frac{dz}{ds}+d.\text{T}\frac{dz}{ds}:$$

thus, according to (1), the equations of the motion of dm are

$$\left.\begin{aligned} \rho\omega\, ds\left\{\text{X}-\frac{d^2x}{dt^2}\right\}+d.\text{T}\frac{dx}{ds} &= 0, \\ \rho\omega\, ds\left\{\text{Y}-\frac{d^2y}{dt^2}\right\}+d.\text{T}\frac{dy}{ds} &= 0, \\ \rho\omega\, ds\left\{\text{Z}-\frac{d^2z}{dt^2}\right\}+d.\text{T}\frac{dz}{ds} &= 0. \end{aligned}\right\} \tag{5}$$

I may observe that these equations have been found before; viz. in Art. 79, where their determination has been given in illustration of the principle of virtual velocities. I have chosen however again to investigate them, in order that the meaning of all the symbols involved in them may be clearly understood.

451.] Of these I will first take a most simple example. Let the string be fastened at one end O to a fixed point, and let it pass over a small pulley A, where $\text{OA} = a$, and have a weight $=$ W attached to its other end: so that the tension of the string throughout is equal to W; and let W be so great that the weight of the string may in comparison of it be neglected. Let us also suppose $\text{X} = \text{Y} = \text{Z} = 0$; then, in its position of equilibrium, the string lies along the straight line OA; let it be slightly displaced by means of an external force; the displacement being so small that the angle at which any element of it is inclined to the line OA is infinitesimal. Our object is to investigate the law of the displacement of any particle which follows on this initial displacement. Let OA be the axis of x; and let (x, y, z) be the place of any element $(= \rho\omega\, ds)$ at the time t. Then, as the angle of inclination of ds to the axis of x is infinitesimal, we have approximately $ds = dx$. So that, neglecting infinitesimals of the second order, each element of the string moves in a plane perpendicular to the line OA, and consequently the point of the string at A does not move: thus there will be no motion along

the line O A, and $\frac{d^2 x}{dt^2} = 0$, for all elements of the string. Thus the first of (5) gives $\text{T} = \text{a constant}$; and the tension of the string is constant throughout its length and throughout the motion, and is equal to the weight W. Introducing these results into the last two equations of (5), they become

$$\left.\begin{aligned}\frac{d^2 y}{dt^2} - \frac{\text{W}}{\rho\omega}\frac{d^2 y}{dx^2} = 0,\\ \frac{d^2 z}{dt^2} - \frac{\text{W}}{\rho\omega}\frac{d^2 z}{dx^2} = 0.\end{aligned}\right\} \qquad (6)$$

If, for the sake of simplicity, we suppose the curve of the string in its initial displacement to be in one plane, we may take that to be the plane of (x, y); and then the equation which represents the subsequent motion of the particle is

$$\frac{d^2 y}{dt^2} - \frac{\text{W}}{\rho\omega}\frac{d^2 y}{dx^2} = 0; \qquad (7)$$

which is a partial-differential equation of the second order. The integration and interpretation of it I shall defer to the following Articles; because we shall again meet with equations of the same form. The homogeneity of it on the principles explained in Section 9, Chapter III, deserves notice.

452.] Next let us investigate the motion of the particles of a thin heavy elastic string, which is homogeneous and of the same thickness throughout its length, stretched between two given points O and A; see Fig. 64; where $\text{OA} = l$. Let T_0 be the tension of the string at rest: which we assume to be so great that the weight may be neglected without sensible error in comparison of it: thus the string lies in the straight line joining O and A, when it is in statical equilibrium.

Now let us suppose the string to be put into motion by some external force; as a pianoforte string by the blow of the hammer, or the string of a harp by the finger of the player: hereby the particles are displaced both relatively and in space; and elastic forces of tension are brought into action, tending to restore the string to its original condition; and although the force which produces the displacement ceases to act, yet the particles of the string continue to move, and the string vibrates about its rectilinear position. We will take the most simple form of the problem, and suppose no other force to act: so that in (5), $\text{X} = \text{Y} = \text{Z} = 0$.

Let us consider the string in its vibrating state at the time t,

and refer it to three rectangular axes originating at O, of which the x-axis coincides with OA. Let us take a particle ($= dm$) whose distance from O in its position of rest $= x$; and let ρ be its density: then if $\omega =$ the area of a transverse section of the string, $dm = \rho\,\omega\,dx$: let the place of this particle at the time t be $(x+\xi, \eta, \zeta)$, so that ξ, η, ζ are the actual displacements of the particle; and let ρ' be its density in its displaced state, and ω' the corresponding area of the transverse section of the string; and let ds be the length-element of the curve which dm occupies: then, as the mass of dm is unaltered,

$$dm = \rho\,\omega\,dx = \rho'\,\omega'\,ds. \tag{8}$$

As the displacement of the particle is very small, ξ, η, ζ are all small: they are functions of x and t, and are to be expressed in terms of these variables. It is however to be observed that x is not a function of t. Thus (5) become

$$\left.\begin{aligned} \rho\,\omega\,dx\frac{d^2\xi}{dt^2} - d\,.\,\mathrm{T}\frac{d(x+\xi)}{ds} &= 0, \\ \rho\,\omega\,dx\frac{d^2\eta}{dt^2} - d\,.\,\mathrm{T}\frac{d\eta}{ds} &= 0, \\ \rho\,\omega\,dx\frac{d^2\zeta}{dt^2} - d\,.\,\mathrm{T}\frac{d\zeta}{ds} &= 0. \end{aligned}\right\} \tag{9}$$

Now the length of the element $= dx$, under the action of the tension T_0: and $= ds$, under that of the tension T: consequently, if E is the modulus of elasticity, by Hooke's law, as explained in Art. 171, Vol. III,

$$ds = dx\left\{1 + \frac{\mathrm{T}-\mathrm{T}_0}{\mathrm{E}}\right\}; \tag{10}$$

where, it will be observed, E is a weight depending on the nature of the string.

Also $$ds^2 = (dx + d\xi)^2 + d\eta^2 + d\zeta^2;$$

and as the displacements of the molecules are small, $d\xi, d\eta, d\zeta$ are so small, that all powers of them higher than the first may be neglected. Consequently,

$$ds = dx + d\xi; \tag{11}$$

$$\therefore\ \mathrm{T} = \mathrm{T}_0 + \mathrm{E}\frac{d\xi}{dx}.$$

Let us substitute all these values in (9); then, omitting small terms, we have

$$\left.\begin{gathered} \rho\,\omega\frac{d^2\xi}{dt^2} = \mathrm{E}\frac{d^2\xi}{dx^2}, \\ \rho\,\omega\frac{d^2\eta}{dt^2} = \mathrm{T}_0\frac{d^2\eta}{dx^2}, \qquad \rho\,\omega\frac{d^2\zeta}{dt^2} = \mathrm{T}_0\frac{d^2\zeta}{dx^2}. \end{gathered}\right\} \tag{12}$$

Let $$\frac{\text{E}}{\rho\omega} = a^2, \qquad \frac{\text{T}_0}{\rho\omega} = b^2; \tag{13}$$

then the preceding equations become

$$\frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2}, \qquad \frac{d^2\eta}{dt^2} = b^2\frac{d^2\eta}{dx^2}, \qquad \frac{d^2\zeta}{dt^2} = b^2\frac{d^2\zeta}{dx^2}; \tag{14}$$

which are three partial linear differential equations of the second order with constant coefficients. As the variables in them are separated, we conclude that the vibrations of the string parallel to the three axes of ξ, η, ζ are independent of each other, and coexist without interference. The first equation expresses the vibrations along the string; these are called longitudinal vibrations; the last two express the displacement, which is perpendicular to the axis of x, as it is projected on the planes of (ξ, η) and (ξ, ζ) respectively, and consequently express curves in these planes: these vibrations are called transversal or lateral vibrations: the forms of all, as shewn by the preceding equations, are the same. If these lateral vibrations all take place in one plane, we may take that plane to be the plane of (ξ, η), so that $\zeta = 0$ throughout the motion.

In reference to the values of a and b which are given in (13), it will be noticed that they are of (1) dimension in space, and of (-1) dimension in time, and thus represent velocities.

453.] Let these equations be integrated by some one of the methods given in Vol. II; then we have

$$\left.\begin{aligned} \xi &= \text{F}_1(x+at)+f_1(x-at), \\ \eta &= \text{F}_2(x+bt)+f_2(x-bt), \\ \zeta &= \text{F}_3(x+bt)+f_3(x-bt); \end{aligned}\right\} \tag{15}$$

where F and f are symbols of arbitrary functions, as yet undetermined; before however we determine them by means of the initial or other circumstances of the string, we will make some general observations.

Suppose an element of the string to be displaced from its place of rest in the line OA by means of a disturbance or pulse passing along the string; and $(x, 0, 0)$ to be that place of rest: then, if v is the velocity of propagation of the pulse, and t is the time which has elapsed since the pulse passed through O, $vt-x$ is the distance along the axis of x through which the pulse has passed in the time during which the particle has been displaced to its relative place (ξ, η, ζ) at the time t; and these small displace-

ments are functions of that distance. This is the case when the pulse travels along OA in a positive direction; and as another pulse may also travel in an opposite direction, the displacement due to this will be a function of $-vt-x$, that is, of $vt+x$; and consequently the whole displacement will be the sum of these separate displacements. Moreover, also, as there may be many pulses travelling, some in one direction and some in the opposite direction, the resultant displacement will be the sum of all these, and will be correctly represented by such expressions as are given in (15). This result also follows from the principle of superposition of small motions, and from the linear form of the differential equations (14). These observations apply to the second and third of (15) as well as to the first, and from them it appears that a is the velocity along OA with which the pulse causing the longitudinal displacement is propagated, and b is the velocity of propagation of the pulse causing the lateral displacement. It appears then that the velocity of propagation of the pulse which causes the longitudinal vibration is different to that which causes the lateral vibration.

The character of the problem is such that the displacements are not only small but are also periodic, so that F and f represent periodic functions, such as sines and cosines, into which they may be developed by means of Fourier's Theorem; suppose τ to be the periodic time; then, as the velocity of propagation of the pulse is constant, being a for longitudinal and b for lateral displacements, the distances through which the pulse is propagated in the time τ are $a\tau$ and $b\tau$ respectively. Let us take the case of longitudinal vibration only, as it is representative, and let $a\tau = \lambda$: then $f(x-at)$, which expresses the displacement due to the positive pulse remains the same if x is increased by λ, and t is increased by τ; and the same result is true for successive similar and simultaneous increases: so that the string takes the form of a series of equal and similar curves. A similar result is of course true for the displacements due to a pulse moving in the opposite direction. The curves due to the transversal displacements are called ventral segments, the length of each of which on the line OA is $b\tau$. More will be said on this subject in future Articles.

454.] Now, supposing the points at O and A to be fixed, the form of the functions given in the right-hand members of (15)

may be determined by means of the initial and other circumstances of the string. We will take as our typical case the second of (15), and, omitting the subscripts, express it in the form

$$\eta = \text{F}(x+bt)+f(x-bt). \tag{16}$$

Let us suppose the string, when $t = 0$, to have been disturbed from its rectilinear position of rest, and to be in the form of a curve whose equation is

$$\eta = \Phi(x); \tag{17}$$

and let us suppose the velocity parallel to the y-axis of the particle at $(x+\xi, \eta, \zeta)$, when $t = 0$, to be given by the equation

$$\frac{d\eta}{dt} = b\,\phi'(x), \tag{18}$$

where $\phi'(x)$ is the x-derived function of $\phi(x)$; $\Phi(x)$ and $\phi(x)$ are supposed to be known functions for all values of x between $x = 0$, and $x = l$, and will be treated as such; they are also subject to the condition that each vanishes when $x = 0$ and when $x = l$.

Hence, when $t = 0$, from (16), (17), and (18), we have

$$\text{F}(x)+f(x) = \Phi(x),$$

$$\text{F}'(x)-f'(x) = \phi'(x);$$

$$\therefore\quad \text{F}(x)-f(x) = \phi(x);$$

$$\therefore\quad 2\,\text{F}(x) = \Phi(x)+\phi(x), \tag{19}$$

$$2f(x) = \Phi(x)-\phi(x); \tag{20}$$

and consequently $\text{F}(x)$ and $f(x)$ are known for all values of x for which $\Phi(x)$ and $\phi(x)$ are known; that is, for all values of x between $x = 0$, and $x = l$, when $t = 0$.

455.] The subject-variables however of F and f, as they are given in (16), are not limited by these values. The subject-variable of F is $x+bt$, and consequently varies, as t increases, through all positive values between 0 and ∞. And the subject-variable of f is $x-bt$, which has all values between l and $-\infty$; so that the complete solution requires the values of the functions corresponding to these values to be known.

Since the points O and A are fixed throughout the motion, $\eta = 0$, when $x = 0$ and when $x = l$; consequently, from (16), we have

$$\text{F}(bt)+f(-bt) = 0, \tag{21}$$

$$\text{F}(l+bt)+f(l-bt) = 0. \tag{22}$$

It appears from (21) that $f(-bt)$ and $\text{F}(bt)$ are equal and of contrary signs; so that if $\text{F}(bt)$ is known for all values of t

between 0 and ∞, $f(-bt)$ is also known between those same limits.

In (22) let bt be replaced by $l+bt$; then

$$\text{F}(2l+bt) = -f(-bt)$$
$$= \text{F}(bt); \tag{23}$$

which shews that the value of $\text{F}(bt)$ remains the same, when its subject-variable is increased by $2l$; consequently it is the same when the subject-variable is increased by $4l$, or $6l, \ldots$, or $2nl$, where n is a whole number. And therefore if the value of $\text{F}(bt)$ is known from $bt = 0$ to $bt = 2l$, the value is known for all values between $t = 0$ and $t = \infty$. Thus $2l$ is the length along the line OA, which has been denoted by λ in Art. 453, and is equal to $\lambda\tau$.

Again, in (22) let bt be replaced by $l-bt$, so that bt is less than l; then
$$\text{F}(2l-bt) = -f(bt); \tag{24}$$
but $f(bt)$ is known for all values of bt between 0 and l; consequently the value of $\text{F}(bt)$ is known for all values of bt between $bt = l$ and $bt = 2l$.

Hence the value of $\text{F}(a)$ is known for all values of a, from $a = 0$ to $a = \infty$; and these are the required limits.

Thus much as to F. And we have shewn above that all values of $f(\beta)$ are known from $\beta = 0$ to $\beta = -\infty$; and the initial equation (20) gives all values of $f(\beta)$ from $\beta = l$ to $\beta = 0$; so that all values of $f(\beta)$ are known within the required limits.

It is worth observing that the values of $\text{F}(a)$ for negative values of a, and that of $f(\beta)$ for values of β greater than l, have not been found in the preceding explanations; and they are not required; as the particular values of their subject-variables are not within the limits given by the problem.

The values of the functions which express the ζ-displacement may be found in a manner precisely similar.

Thus the form of the string in its displaced state, and the velocities of its several molecules parallel to the y- and z-axes at the time t will be known; and the problem will be completely solved, so far as the transversal vibrations are concerned.

Also, all that has been said on transversal vibrations is applicable, if we replace b by a, to the longitudinal vibrations of the string. In this case the initial equations will assign the position and the velocity along the x-axis of every particle of the string, when $t = 0$, between the limits $x = 0$ and $x = l$.

Thus the problem is completely solved. I propose however to interpret the result graphically, for the general motion of the string will be rendered clearer by means of a diagram. The results which will be exhibited might be derived from the equations just now discussed; but it will be more convenient to take a less general form, which will be equally expressive and more easily constructed.

456.] For this purpose I will assume that the string, having been disturbed, takes the form given by a known equation, such as (17), when $t = 0$; and that all its particles are then at rest; so that $\frac{d\eta}{dt} = 0$, when $t = 0$; consequently since

$$\eta = \text{F}(x+bt)+f(x-bt),$$

$$\frac{d\eta}{dt} = b\text{F}'(x+bt)-bf'(x-bt);$$

and therefore if $\frac{d\eta}{dt} = 0$, when $t = 0$, $\text{F}'(x) = f'(x)$;

$$\therefore \quad \text{F}(x) = f(x);$$

no constant being added, because we will assume $\text{F}(x) = f(x) = 0$, when $x = 0$, and

$$\eta = f(x+bt)+f(x-bt); \tag{25}$$

therefore, when $t = 0$, $\quad \eta = 2f(x). \quad$ (26)

Suppose however the equation to the curve of the string in its displaced state at rest, when $t = 0$, to be $\eta = \text{F}(x)$;

then, from (26), $\quad \text{F}(x) = 2f(x)$;

and thus (25) becomes

$$\eta = \frac{1}{2}\{\text{F}(x+bt)+\text{F}(x-bt)\}. \tag{27}$$

This then is the η-displacement of the particle which is at $(x, 0, 0)$, when the string is rectilineal.

The function whose symbol is F is subject to the following conditions, which are derived from equations (21)......(24) of the preceding Article:

$$\text{F}(x) = -\text{F}(-x), \tag{28}$$

$$\text{F}(l+x) = -\text{F}(l-x); \tag{29}$$

$$\text{F}(x) = \text{F}(2l+x) = \text{F}(4l+x) = \ldots = \text{F}(2nl+x), \tag{30}$$

$$\text{F}(x) = -\text{F}(2l-x) = -\text{F}(4l-x) = \ldots\ldots \tag{31}$$

These equations enable us to infer a correct notion of the motion of the several molecules of the string, from the form which it has

in its initial displaced state. From (28) it appears, that the curve represented by $y = \text{F}(x)$ is continued in similar forms on each side of O; the curve being on one side above, and on the other below the axis of x; and (29) shews that the curve is continued in similar forms on each side of A, the curve being on one side above, and on the other below the axis of x. Consequently the curve about the points O and A, and between these points, is of a form similar to that drawn in fig. 65, the plane of the paper being that of (x, y).

Again, to develope other properties of it: along the axis of x take from O in both directions a series of lengths, each of which is equal to l; viz., $\text{OA} = \text{AO}' = \text{O}'\text{A}' = \text{A}'\text{O}'' = \ldots = l$. Then equation (30) shews that whatever is the form of the curve between O and O′, it is the same between O′ and O″. And (31) shews that the form of the curve between O and A is the same as that between O′ and A, except that in the latter case it is inverted and lies below the axis of x. Thus the whole curve consists of similar portions drawn as in fig. 65.

Now this curve enables us to trace the motion of any particle of the vibrating string; and consequently the motion of the whole string. For the place P at the time t of the particle, which is at M ($\text{OM} = x$) when the string is straight, may thus be found: along OA take, on both sides of M, $\text{MN} = \text{MN}' = bt$; so that $\text{NQ} = \text{F}(x+bt)$, $\text{N}'\text{Q}' = \text{F}(x-bt)$, then, from (27),

$$2\,\text{MP} = \text{NQ} + \text{N}'\text{Q}'; \tag{32}$$

and by a similar process may the place of every element be determined at any time t.

457.] Now this molecule, and similarly every molecule of the string, and consequently the whole string, will oscillate; that is, the string will occupy a certain series of positions in succession, and will then be found in its original state; afterwards it will go through the same series again, and then return to its original state; and so on continually.

For if we assume $bt = 2l$, or $= 4l$, or $= 6l$, ..., or $= 2nl$, at the times corresponding to these intervals, we have, by means of (30) and (31),

$$\eta = \frac{1}{2}\{\text{F}(x) + \text{F}(x)\} = \text{F}(x);$$

which is the value of η, when $t = 0$; and as this result is true for every point of the string, the string comes back to its original

position at the times corresponding to these values. The interval between two successive states of like position is $\frac{2l}{b}$ which is therefore the periodic time of vibration.

When we take $bt = l$, or $= 3l$, or $= 5l$, ..., or $= (2n+1)l$, at the times corresponding to these intervals, we have

$$\eta = \frac{1}{2}\{\text{F}(x+l) + \text{F}(x-l)\} = \text{F}(x+l);$$

so that in the time $= \frac{l}{b}$, every molecule will have described one half of the whole course which it describes in going from a position to the same position again. Thus the form of the string at these times is exactly similar to what it was when $t = 0$, but in an inverted position; as the lower line in fig. 64; the greatest ordinate being now at the same distance from A as it was from O in the original form of the string. The time which is occupied in this change of figure is one half of that occupied in a complete vibration.

Similarly we may consider the positions of the curve when $t = \frac{l}{2b}$, or when $t = \frac{3l}{2b}$: in the former case, the length of time is one-fourth of that of a complete vibration; and in the latter case, is three-fourths of that of a complete vibration. Similar results are also true when each of these times is increased by $\frac{2l}{b}$. It is to be observed, that during this vibratory motion the string never becomes a straight line.

Similar results to these are also true for the ζ-displacement; for in the last equation of (14) the same constant coefficient enters as in that for the η-displacement. Thus the periodic time of the complete path will be the same in both displacements. The forms of the functions may be different in the two cases, although they are both subject to the conditions developed in Art. 455.

This vibratory motion would continue perpetually if there were no diminution of vis viva of the string. In the case however of pianoforte or harp strings vis viva is lost for two reasons. The points O and A, to which the ends of the string are fastened, are not points rigidly fixed; so that vibrations are communicated to them from the string, and continued by means of their sup-

ports or framework to the earth, and thereby lost to the string; the string also vibrates in air, to the particles of which vibrations are communicated, and thus vis viva is taken from the string. Owing to these two causes, the oscillations of the string become gradually feebler, and eventually cease. The periodic time of the vibrations however is not changed.

458.] If the original position of the string-curve had been that drawn in fig. 66, where B is the middle point of OA, and where the two branches of the curve are exactly equal and similar, though their positions are inverted, the string would vibrate so that the point B would always remain fixed; that is, each half will vibrate as if the string were fixed at O and B, and at B and A; and the periodic time of vibration will be only one-half of what it is in the case already discussed. So, again, if the original string-curve had consisted of three or more equal and similar portions, intersecting OA at points equally distant from one another and from O and A, and being alternately above and below the line OA, the string will vibrate as if it were three or more distinct strings, and the points of it at which it intersects OA will remain at rest during the motion.

These points are called nodes or nodal points, the curve between two consecutive nodes being a ventral segment.

459.] If τ is the time in which one complete transversal vibration of the string takes place,

$$\tau = \frac{2l}{b} = 2l\left(\frac{\rho\omega}{\mathrm{T}_0}\right)^{\frac{1}{2}}; \tag{33}$$

so that, for a string of constant thickness and density, the time of transversal vibration varies as the length of the string directly, and as the square root of the tension in its straight form inversely.

And if n is the number of transverse vibrations which take place in a second of time,

$$n = \frac{1}{\tau} = \frac{1}{2l}\left(\frac{\mathrm{T}_0}{\rho\omega}\right)^{\frac{1}{2}}; \tag{34}$$

so that for a string of constant thickness and density, n varies as the square root of the tension in its straight form directly, and as the length of the string inversely.

Again, let τ' be the periodic time of a complete longitudinal vibration; then

$$\tau' = \frac{2l}{a} = 2l\left(\frac{\rho\omega}{\mathrm{E}}\right)^{\frac{1}{2}}; \tag{35}$$

and let n' be the number of longitudinal vibrations which take place in a second of time,

$$n' = \frac{1}{\tau'} = \frac{1}{2l}\left(\frac{E}{\rho\omega}\right)^{\frac{1}{2}}. \qquad (36)$$

Now if the string suspended by one end is stretched by a weight at the other end, E is that weight by which the length of the string will be doubled; so that the number of longitudinal vibrations of the string in a second of time varies directly as the square root of this weight.

From the preceding equations we have

$$\frac{n}{n'} = \left(\frac{T_0}{E}\right)^{\frac{1}{2}}; \qquad (37)$$

but in the ordinary pianoforte strings, E is evidently very much greater than T_0, so that the number of longitudinal vibrations in a second of time is very much greater than that of the transversal vibrations.

The ratio of n to n' may also be expressed in the following way: suppose l to be the length of a string in its natural state, and Δl to be the increase of length when it is stretched by T_0: then, by Hooke's law, $l T_0 = E \Delta l$;

$$\therefore \frac{n}{n'} = \left(\frac{\Delta l}{l}\right)^{\frac{1}{2}}. \qquad (38)$$

460.] Let us briefly notice these results in reference to the theory of music. When a string vibrates, the vibrations are imparted to the molecules of the surrounding air, and are through the medium of the air communicated to the tympanum of the ear, the auditory nerves of which are excited, and the sensation of sound is created. The ear recognises three special properties of sound, (1) the pitch, (2) the intensity, (3) a peculiar quality which is in England called technically the "quality" of a note, and is in France called "le timbre." The origin of this last property, and indeed most of its affections, it is very difficult to account for; it probably arises from many causes, amongst which may possibly be the difference of periodic time in the longitudinal and the transversal vibrations. The quality of note given by a string of a violincello played with a bow is very different to that given by a pianoforte string struck by a hammer, when the note or pitch is the same in both. The second property depends on the amplitude of vibration, and varies as the vis viva of the particles put into motion. The first property is

that which is called the musical *tone* or *note*, and depends on the number of vibrations made by the string in a second of time. And thus the number of vibrations is taken as the measure of the pitch or note. The note is higher the greater the number of vibrations made in a second of time. Similarly as to the note due to the transversal vibrations, the note varies inversely as the length of the string, and directly as the square root of the tension. So that if the note of a given pianoforte string is too low, the string must be wound up, whereby the tension is increased.

The note given by a string when its two extremities are fixed, and when all the other points of the string vibrate in the motion, is called the *fundamental note* of the string: thus the fundamental note of a string is higher by one half than that of a string of which the length is twice as great. And thus, too, if a string is struck so that it has one node, the note in that case is twice as high as the fundamental note; and if it has two nodes, the note is thrice as high as the fundamental note; and so on. Hence the notes of strings are compared by means of their lengths and the distances between their nodes.

When two strings have equal periodic times, and vibrate together, they are, in musical language, said to be in unison. If two strings vibrate simultaneously, the resulting sound is most agreeable to the ear when they are in unison. Next to an unison the most agreeable concord is the octave, in which one string vibrates twice as fast as the other; that is, in which the times of vibration are as $1:2$; and the note produced by the former is said to be an octave above that of the latter. Thus if a given string vibrates so that its middle point is a node, it produces the octave to the fundamental note.

It is invariably found, for all ears, that when two notes not in unison are sounded together, the resulting sounds are most agreeable when the times of vibration of the individual notes are in some simple proportions; say $1:2$; $1:3$; $1:4$; $2:3$; and that the concord is more agreeable the less the difference between the terms of the ratio.

Thus the octave of the octave of a note, or the fifteenth, as it is called in music, is an agreeable concord; for it consists of vibrations of two strings whose periodic times are as the numbers $1:4$. Thus also two strings whose periodic times are in the

ratio of 1 : 8, or in the ratio of 1 : 16, and so on, produce pleasant concords, and seem to partake of the perfection of the octave.

The next most simple numerical ratio is that of 1:3, in which we have three vibrations of one string corresponding in time to one vibration of the other. This concord is called a twelfth. If we replace the string, which vibrates once, by its octave, which vibrates twice in the given time, the times of vibration of the two strings are as 2 : 3 : this also forms an agreeable concord, and is called a fifth.

It is however beyond the object of this treatise to enter further into this subject: and for other details, and for an exposition of the theory of music, founded on the preceding and other similar equations, I would refer the reader to the Treatise on Sound by Sir John F. W. Herschel, which was originally contained in the Encyclopædia Metropolitana.

461.] As the arbitrary functions which are in the right-hand member of (16) are both periodic, and the conditions of the motion of the string are such as to satisfy the conditions of development of arbitrary periodic functions in terms of sines and cosines as explained in Vol. II, Chap. VII, section 4, we may apply these theorems and thus obtain a solution of (16) in the form of a series: the process will also be instructive as shewing the way in which the displacement of a particle is made up of a series of separate displacements, or to each term of the series a distinct displacement corresponds.

The series (89), which is given in Vol. II, Art. 197, is that which is to be applied: the following is the complete form of it, viz.

$$f(x) = \frac{1}{2\pi}\int_a^b f(z)\,dz + \frac{1}{\pi}\Sigma.\int_a^b f(z)\cos n(x-z)\,dz; \qquad (39)$$

where $f(x)$ denotes a periodic function of which the period is 2π; and which $= \frac{f(a)}{2}$, when $x = a$; $= f(x)$ for all values of x from $x = a$ to $x = b$; and $= \frac{f(b)}{2}$, when $x = b$; $b-a$ being not greater than 2π. The summatory symbol denotes the sum of all terms corresponding to integer values of n from $n = 1$ to $n = \infty$, both inclusive. The first term in the right-hand member of (39) is non-periodic, and all the other terms are periodic, the period of each being 2π.

In applying this theorem to the present problem, the form and arrangement of the problem shews that there is no non-periodic term, because the effect of it would be only to shift the x-axis parallel to itself until it assumed a position relative to which the string-curve would be symmetrical above and below it; and the axis of x as now taken satisfies that condition; hence (16) takes the form

$$\eta = \frac{1}{\pi}\Sigma . \int_0^l \{F(z)\cos n(x+bt-z)+f(z)\cos n(x-bt-z)\}\,dz$$

$$= \frac{1}{\pi}\Sigma . \Big[\cos nx \int_0^l \{F(z)\cos n(bt-z)+f(z)\cos n(bt+z)\}\,dz$$

$$- \sin nx\int_0^l \{F(z)\sin n(bt-z)-f(z)\sin n(bt+z)\}\,dz\Big]. \quad (40)$$

Now $\eta = 0$, when $x = 0$, for all values of t; consequently the coefficient of $\cos nx$ is equal to zero; and

$$\eta = -\frac{1}{\pi}\Sigma . \sin nx\int_0^l \{F(z)\sin n(bt-z)-f(z)\sin n(bt+z)\}\,dz. \quad (41)$$

Also $\eta = 0$, when $x = l$, whatever is the value of t; therefore $nl = i\pi$, where i is any integer from 1 to ∞; so that

$$n = \frac{i\pi}{l};$$

$$\therefore \quad \eta = -\frac{1}{\pi}\Sigma . \sin\frac{i\pi x}{l}\int_0^l \Big\{F(z)\sin\frac{i\pi(bt-z)}{l} - f(z)\sin\frac{i\pi(bt+z)}{l}\Big\}\,dz. \quad (42)$$

It remains still to determine the form of the element-function in terms of the initial form and circumstances of the string, which are given by the equations

$$\eta = \Phi(x), \quad \text{and} \quad \frac{d\eta}{dt} = b\,\phi'(x),$$

when $t = 0$. Expanding the sines in (42), and putting terms involving t outside the signs of definite integration, we have

$$\eta = -\frac{1}{\pi}\Sigma . \sin\frac{i\pi x}{l}\Big[\sin\frac{i\pi bt}{l}\int_0^l \{F(z)-f(z)\}\cos\frac{i\pi z}{l}\,dz$$

$$-\cos\frac{i\pi bt}{l}\int_0^l \{F(z)+f(z)\}\sin\frac{i\pi z}{l}\,dz\Big]; \quad (43)$$

$$\therefore \quad \frac{d\eta}{dt} = -\frac{b}{l}\Sigma . \sin\frac{i\pi x}{l}\Big[i\cos\frac{i\pi bt}{l}\int_0^l \{F(z)-f(z)\}\cos\frac{i\pi z}{l}\,dz$$

$$+ i\sin\frac{i\pi bt}{l}\int_0^l \{F(z)+f(z)\}\sin\frac{i\pi z}{l}\,dz\Big]. \quad (44)$$

Let $t = 0$; then $\eta = \text{F}(x) + f(x) = \Phi(x)$, and

$$\frac{d\eta}{dt} = b\{\text{F}'(x) - f'(x)\} = b\,\phi'(x);$$

so that $\text{F}'(x) - f'(x) = \phi'(x)$, and consequently $\text{F}(x) - f(x) = \phi(x)$, as in Art. 456.

$$\therefore\quad \Phi(x) = \frac{1}{\pi}\Sigma.\sin\frac{i\pi x}{l}\int_0^l \Phi(z)\sin\frac{i\pi z}{l}\,dz;$$

$$\phi'(x) = -\frac{1}{l}\Sigma.\sin\frac{i\pi x}{l}\int_0^l i\phi(z)\cos\frac{i\pi z}{l}\,dz$$

$$= \frac{1}{\pi}\Sigma.\sin\frac{i\pi x}{l}\int_0^l \phi'(z)\sin\frac{i\pi z}{l}\,dz;$$

which are values to be satisfied by (43), when $t = 0$. Hence

$$\eta = \frac{1}{\pi}\Sigma.\sin\frac{i\pi x}{l}\left[\frac{l}{i\pi}\sin\frac{i\pi bt}{l}\int_0^l \phi'(z)\sin\frac{i\pi z}{l}\,dz + \cos\frac{i\pi bt}{l}\int_0^l \Phi(z)\sin\frac{i\pi z}{l}\,dz\right]; \quad (45$$

which gives the solution of the problem in a form free from arbitrary functions, but in the form of a series of which the several terms arise from giving to i successive integer values from i to ∞, both inclusive. Thus the number of terms is infinite. All the properties of the motion of the string which have been deduced in Art. 454 from the arbitrary functions F and f may be inferred equally as well from this equation. Similar values may also be found for ξ and ζ.

Thus, according to this method, the motion of the string is expressed by a series of terms, each of which might exist alone, and might be the complete solution of the equation, if it agreed with the initial state of the string. The most general motion of the string however results from the coexistence or superposition of an infinity of vibratory motions, and the resulting note from the coexistence of the several notes which are due to these several single motions.

Another problem of the same kind is that in which the string moves in a resisting medium, the resistance of which varies directly as the velocity; in this case, the equation of motion takes the form

$$\frac{d^2\eta}{dt^2} + 2b\frac{d\eta}{dt} - a^2\frac{d^2\eta}{dx^2} = 0;$$

and is integrable.

Other problems will be found with complete solutions in Donkin's Acoustics; Clarendon Press, Oxford, 1884.

462.] The following is another problem in which the method of expansion in a series of sines and cosines is applied.

A thin elastic string, whose unstretched length is l, resting on a smooth horizontal plane with one end fixed, is stretched in the direction of its length so as to be of the length l', and is then set free; it is required to determine the subsequent motion of the string.

Let O, the fixed end of the string, be the origin, and let x be the distance of a type-particle of it from O in its unstretched state; let ξ be the displacement; then, taking the notation of Art. 452,

$$\xi = \mathrm{F}(x+at)+f(x-at), \tag{46}$$

where a is the velocity of the pulse along the string, and t is the time elapsed from the instant at which the string was set free. It is evident from the nature and arrangement of the problem that no non-periodic term occurs in the expansion; hence the expansion takes the form

$$\xi = \frac{1}{\pi}\Sigma.\int_0^l \{\mathrm{F}(z)\cos n(x+at-z)+f(z)\cos n(x-at-z)\}\,dz,$$

the summatory symbol of which denotes the sum of all the definite integrals corresponding to values of n from $n=1$ to $n=\infty$, both inclusive. The period of these periodic functions is 2π. Hence, placing the terms involving x outside the sign of definite integration, we have

$$\xi = \frac{1}{\pi}\Sigma.\Big[\cos nx\int_0^l \{\mathrm{F}(z)\cos n(at-z)+f(z)\cos n(at+z)\}\,dz$$
$$-\sin nx\int_0^l \{\mathrm{F}(z)\sin n(at-z)-f(z)\sin n(at+z)\}\,dz\Big]; \tag{47}$$

this is a general expression, and we have to determine the form which it takes in its application to the particular problem.

Since $\xi=0$, when $x=0$, for all values of t, the coefficient of $\cos nx$ in this equation $=0$: that is

$$\int_0^l \{\mathrm{F}(z)\cos n(at-z)+f(z)\cos n(at+z)\}\,dz = 0.$$

$$\therefore\ \xi = -\frac{1}{\pi}\Sigma.\sin nx\int_0^l \{\mathrm{F}(z)\sin n(at-z)-f(z)\sin n(at+z)\}\,dz. \tag{48}$$

Also since $\frac{d\xi}{dx}=0$, when $x=l$, for all values of t, therefore $\cos nl = 0$; and

$$n = \frac{(2i+1)\pi}{2l}, \tag{49}$$

for all values of i from $i = 0$ to $i = \infty$. This equation then assigns a particular form to the values of n.

If $l' = l(1+\mu)$, then when $t = 0$, $\xi = \mu x$ in accordance with Hooke's law for all values of x from $x = 0$ to $x = l$: and therefore from (46), when $t = 0$, $\mu x = \text{F}(x) + f(x)$. (50)

Also expressing (48) in the form

$$\xi = -\frac{1}{\pi}\Sigma.\sin nx\Big[\sin nat\int_0^l \{\text{F}(z)-f(z)\}\cos nz\,dz$$
$$-\cos nat\int_0^l \{\text{F}(z)+f(z)\}\sin nz\,dz\Big]; \quad (51)$$

and observing that $\frac{d\xi}{dt} = 0$, when $t = 0$ for all values of x from $x = 0$ to $x = l$; and that by (50), $\text{F}(z)+f(z) = \mu z$, under these circumstances,

$$\xi = \frac{1}{\pi}\Sigma.\sin nx\cos nat\int_0^l \{\text{F}(z)+f(z)\}\sin nz\,dz$$
$$= \frac{\mu}{\pi}\Sigma.\sin nx\cos nat\int_0^l z\sin nz\,dz$$
$$= \frac{\mu}{\pi}\Sigma.\sin nx\cos nat\frac{\sin nl}{n^2}$$
$$= \frac{4\mu l^2}{\pi^3}\Sigma.\frac{1}{(2i+1)^2}\sin\frac{2i+1}{2}\frac{\pi x}{l}\cos\frac{2i+1}{2}\frac{\pi at}{l}\sin(2i+1)\frac{\pi}{2}, \quad (52)$$

when we replace n by its value in terms of i as given in (49). Also since the period $= 2\pi$, $l = 2\pi a$; and consequently

$$\xi = \frac{16\mu a^2}{\pi}\Sigma.\frac{1}{(2i+1)^2}\sin\frac{2i+1}{2}\frac{\pi x}{l}\cos\frac{2i+1}{2}\frac{\pi at}{l}\sin(2i+1)\frac{\pi}{2},$$

the last factor of which is $+1$ or -1, according to the form of $2i+1$.

Then expressing the series at length, as i includes all values from $i = 0$ to $i = \infty$, (52) becomes

$$\frac{4\mu l^2}{\pi^3}\left\{\sin\frac{\pi x}{2l}\cos\frac{\pi at}{2l} - \frac{1}{9}\sin\frac{3\pi x}{2l}\cos\frac{3\pi at}{2l} + \frac{1}{25}\sin\frac{5\pi x}{2l}\cos\frac{5\pi at}{2l} - \dots\right\}, \quad (53)$$

which is the solution of the problem in the form of a series. As the sum is convergent, the principal term is the first term, and that alone will give an approximate solution of the problem.

463.] Also in illustration of the general equations, let us take the following Problem. A thin heavy elastic string of constant thickness and density is suspended by one end from a fixed point,

and is slightly displaced in a vertical plane. It is required to find the equations which determine the motion of the string.

Let the fixed point be the origin, and the line of the string hanging vertically at rest be the axis of z; and let the plane in which the string moves be the plane of (z, x). Let l be the length of the string in its natural state and unstretched; and let l' be its length when suspended and stretched by its own weight. Let z be the distance of the particle $\rho\,\omega\,dz$ from the fixed end when unstretched, and z' be the distance of the same element when the string is stretched. Then, as in Ex. 1, Art. 172, Vol. III, by Hooke's Law,

$$dz' = dz\left\{1 + \frac{\rho g \omega}{\text{E}}(l - z)\right\};$$

$$\therefore \quad z' = z + \frac{\rho g \omega}{2\text{E}}(2lz - z^2); \tag{54}$$

$$l' = l + \frac{\rho g \omega l^2}{2\text{E}}. \tag{55}$$

Let ξ and ζ be the horizontal and vertical displacements of a particle of the string whose distance from the origin is z when the string is not stretched; then if T is the tension at the time t, when the string is stretched and in motion, the equations of motion are

$$\left.\begin{aligned} \rho\,\omega\,dz\frac{d^2\xi}{dt^2} - d\,.\,\text{T}\frac{d\xi}{ds} &= 0, \\ \rho\,\omega\,dz\frac{d^2\zeta}{dt^2} - d\,.\,\text{T}\frac{dz}{ds} &= \rho\,\omega\,g\,dz. \end{aligned}\right\} \tag{56}$$

Now $ds^2 = d\xi^2 + (dz + d\zeta)^2$; whence, omitting small terms of the second and higher orders,

$$ds = dz + d\zeta = dz\left(1 + \frac{d\zeta}{dz}\right);$$

$$\text{but } ds = dz\left(1 + \frac{\text{T}}{\text{E}}\right), \text{ by Hooke's Law.}$$

$$\therefore \quad \text{T} = \text{E}\frac{d\zeta}{dz}; \tag{57}$$

and consequently the second equation of (56) becomes

$$\frac{d^2\zeta}{dt^2} = \frac{\text{E}}{\rho\,\omega}\frac{d^2\zeta}{dz^2} + g. \tag{58}$$

Let $\frac{\text{E}}{\rho\,\omega} = a^2$; then as E is a weight, the left-hand member is of (2) dimensions in space, and of (-2) dimensions in time, and

consequently a denotes a velocity, which varies as the square-root of the modulus of elasticity; thus (58) takes the form

$$\frac{d^2\zeta}{dt^2} = a^2 \frac{d^2\zeta}{dz^2} + g. \tag{59}$$

Now as z does not vary with t, this equation can be put into the form

$$\frac{d^2}{dt^2}\left\{\zeta + \text{A}z + \frac{gz^2}{2a^2}\right\} = a^2 \frac{d^2}{dz^2}\left\{\zeta + \text{A}z + \frac{gz^2}{2a^2}\right\}, \tag{60}$$

where A is an undetermined constant. Then integrating in the usual way, the solution is

$$\zeta + \text{A}z + \frac{gz^2}{2a^2} = \text{F}(at+z) + f(at-z), \tag{61}$$

where F and f denote arbitrary periodic functions.

To determine A, when $t = 0$, we have by (54),

$$\zeta = \frac{\rho g \omega}{2\text{E}}(2lz - z^2) = \frac{g}{2a^2}(2lz - z^2). \tag{62}$$

Hence comparing this with (61), and taking non-periodic terms only, $\text{A} = -\frac{gl}{a^2}$; and therefore

$$\zeta - \frac{glz}{a^2} + \frac{gz^2}{2a^2} = \text{F}(at+z) + f(at-z), \tag{63}$$

which satisfies (59), and is its general integral.

But when $z = 0$, $\zeta = 0$, for all values of t; therefore

$$\text{F}(at) + f(at) = 0,$$

$$\therefore\ f(at) = -\text{F}(at);$$

$$\zeta - \frac{glz}{a^2} + \frac{gz^2}{2a^2} = \text{F}(at+z) - \text{F}(at-z), \tag{64}$$

and this expression gives the displacement of a disturbed particle of the string in the direction of its length. The terms in the right-hand member shew that the disturbance travels along the string with the velocity a, and that the displacement is due to two similar disturbances travelling in opposite directions with equal velocities; and the terms in the left-hand gives the displacement due to the weight of that portion of the string which lies below the element whose motion we are considering.

As the functions in (64) are arbitrary periodic functions, they may be expressed in a series of sines and cosines, as in the previous examples. Thus

$$\zeta = \frac{g}{2a^2}(2lz-z^2)+\frac{1}{\pi}\Sigma.\int_0^l \text{F}(u)\{\cos n(at+z-u)-\cos n(at-z-u)\}du$$

$$= \frac{g}{2a^2}(2lz-z^2)-\frac{2}{\pi}\Sigma.\sin nz\int_0^l \text{F}(u)\sin n(at-u)\,du \qquad (65)$$

$$= \frac{g}{2a^2}(2lz-z^2)-\frac{2}{\pi}\Sigma.\sin nz\Big[\sin nat\int_0^l \text{F}(u)\cos nu\,du$$

$$-\cos nat\int_0^l \text{F}(u)\sin nu\,du\Big]. \qquad (66)$$

Now when $t = 0$, $\frac{d\zeta}{dt} = 0$, for all values of z: consequently

$$\int_0^l \text{F}(u)\cos nu\,du = 0,$$

and therefore

$$\zeta = \frac{g}{2a^2}(2lz-z^2)+\frac{2}{\pi}\Sigma.\sin nz\cos nat\int_0^l \text{F}(u)\sin nu\,du. \qquad (67)$$

Also as $\text{T} = 0$, that is $\frac{d\zeta}{dz} = 0$, when $z = l$; therefore

$$\cos nz = 0, \text{ when } z = l;$$

$$\therefore \quad nl = (2i+1)\frac{\pi}{2},$$

where i has every integral value from $i = 0$ to $i = \infty$, both included: therefore

$$n = (2i+1)\frac{\pi}{2l}, \qquad (68)$$

which assigns the form of n.

And the definite integral in the last term of the right-hand member of (67) may be thus expressed. When $t = 0$, $\zeta = 0$ for all values of z, from $z = 0$ to $z = l$, and (67) must satisfy this condition. Hence

$$z^2-2lz = \frac{2}{\pi}\Sigma.\sin nz\int_0^l (u^2-2lu)\sin nu\,du$$

$$= -\frac{4}{\pi}\Sigma.\frac{\sin nz}{n^3};$$

$$\therefore \quad \zeta+\frac{g}{2a^2}(z^2-2lz) = -\frac{2g}{\pi a^2}\Sigma.\frac{\sin nz\cos nat}{n^3},$$

and substituting for n from (68), we have

$$\zeta = \frac{g}{2a^2}(2lz-z^2)-\frac{16gl^3}{\pi^4 a^2}\Sigma.\frac{\sin(2i+1)\frac{\pi z}{2l}\cos(2i+1)\frac{\pi at}{2l}}{(2i+1)^3}, \qquad (69)$$

which gives the relative displacement of the particle in the

direction of the length of the string, the summation reaching from $i=0$ to $i=\infty$. Hence

$$\zeta = \frac{g}{2a^2}(2lz-z^2) - \frac{16gl^3}{\pi^4 a^2}\left\{\sin\frac{\pi z}{2l}\cos\frac{\pi a t}{2l} + \frac{1}{3^3}\sin\frac{3\pi z}{2l}\cos\frac{3\pi a t}{2l} + \dots\right\}. \quad (70)$$

As to the horizontal motion of the element in the plane of (ζ, ξ), we have from the first of (56), replacing T by its value $\rho\omega g(l-z)$, and ds by dz

$$\frac{d^2\xi}{dt^2} - g\frac{d}{dz}\left\{(l-z)\frac{d\xi}{dz}\right\} = 0;$$

$$\therefore \quad \frac{d^2\xi}{dt^2} - g(l-z)\frac{d^2\xi}{dz^2} + g\frac{d\xi}{dz} = 0, \quad (71)$$

which is the differential equation of the transverse motion of an element of the string.

464.] Another case of motion of the elements of an elastic body which can be considered is that of the longitudinal vibrations of the elements of a fine elastic rod.

I shall assume the rod to be homogeneous, and in its natural state to be prismatic or cylindrical; so that if $\omega =$ the area of a transverse section, ω is constant throughout the length, and is infinitesimal because the rod is thin. I shall take the line which contains the mass-centres of all thin transverse slices to be the x-axis. Although the rod is thin, that is, although the lines of its transverse section are infinitesimal in comparison of the length of the rod, yet its thickness is such that the rod is not bent by the forces acting on it.

Now we suppose the particles of the rod to be displaced longitudinally by a force acting in the direction of the length of the rod, so that each particle is displaced through a small distance along the x-axis. Moreover, we suppose that every particle in a thin transverse section or slice perpendicular to the x-axis is displaced through an equal distance; so that all particles which were in a given transverse slice before the displacement are in a transverse slice after the displacement. We also suppose that by this displacement, due to an external force, certain elastic forces are brought into action whose lines of action are parallel to the x-axis, and that the molecules subsequently vibrate under the action of these forces. It is this subsequent motion, when all other forces have ceased to act, which we shall now investigate.

Let O and A be the ends of the bar in its original state of rest;

and let OA, its length, $= l$; let $\omega =$ the area of a transverse section, $\rho =$ the density. Let us consider the motion of a thin slice, whose distance from O $= x$, when $t = 0$, and whose thickness $= dx$; so that its mass $= \rho\,\omega\,dx$. This mass is unchanged during the motion.

Let $x+\xi$ be the distance of this elemental slice from O at the time t: ξ being a small quantity and evidently a function of both x and t. Let T be at the time t the tension drawing the slice towards O, and $\text{T}+d\text{T}$ that drawing it towards A: then, as no other force acts, the equation of motion is

$$\rho\,\omega\,dx\frac{d^2(x+\xi)}{dt^2} = d\text{T};$$

$$\therefore \quad \rho\,\omega\,dx\frac{d^2\xi}{dt^2} = d\text{T}. \tag{72}$$

We suppose the extension of the bar to vary according to Hooke's law: so that if E is the modulus of elasticity of the bar,

$$\text{T} = \text{E}\frac{d\xi}{dx};$$

and consequently, if $\text{E} = \rho\,\omega\,a^2$, (72) becomes

$$\frac{d^2\xi}{dt^2} = a^2\frac{d^2\xi}{dx^2}; \tag{73}$$

of which the solution is

$$\xi = \text{F}(x+at)+f(x-at), \tag{74}$$

where F and f denote arbitrary periodic functions. As $\frac{d\xi}{dt}$ is the velocity of the slice $\rho\,\omega\,dx$, and $\frac{d\xi}{dx}$ is the linear dilatation of unit-length of the bar, (73) shews that the molecules of the bar are so arranged at the time t, that the portions are alternately those of condensation and dilatation, the molecules being packed in the former more closely and in the latter less closely than they are in their natural constrained state; and the form of (74) shewing that there is a regular advance along the bar of the given state of, say, condensation in the positive direction as shewn by $f(x-at)$, and in the negative direction as shewn by $\text{F}(x+at)$, with the constant velocity a.

If the initial position of the several slices of the bar, and the initial velocities of them along the line of the bar are given, F and f may be expressed in terms of them. Thus I will suppose, when $t = 0$, $\xi = \Phi(x)$, $\frac{d\xi}{dt} = a\,\phi'(x)$, these values being true for all values of x from $x = 0$ to $x = l$; so that when $t = 0$,

$$\left.\begin{aligned} F(x)+f(x) &= \Phi(x), \\ F'(x)-f'(x) &= \phi'(x). \end{aligned}\right\} \qquad (75)$$

If a point of the bar is fixed, then at that point $\xi = 0$, and $\frac{d\xi}{dt} = 0$ during the whole motion, and T at that point is the stress which it has to bear. In a free end $\frac{d\xi}{dx} = 0$, for all values of t. Now let us replace the arbitrary functions by their values in terms of sines and cosines, as in (40); then

$$\xi = \frac{1}{\pi}\Sigma.\Big[\cos nx\int_0^l \{F(z)\cos n(at-z)+f(z)\cos n(at+z)\}\,dz$$
$$-\sin nx\int_0^l \{F(z)\sin n(at-z)-f(z)\sin n(at+z)\}\,dz\Big], \quad (76)$$

and is the complete expression for the displacement.

Now let us suppose both ends of the bar to be fixed, so that $\xi = 0$, when $x = 0$, and also when $x = l$, for all values of t. Then (76) evidently takes the form

$$\xi = -\frac{1}{\pi}\Sigma.\sin\frac{i\pi x}{l}\int_0^l \{F(z)\sin\frac{i\pi}{l}(at-z)-f(z)\sin\frac{i\pi}{l}(at+z)\}\,dz \quad (77)$$

$$= -\frac{1}{\pi}\Sigma.\sin\frac{i\pi x}{l}\Big[\sin\frac{i\pi at}{l}\int_0^l \{F(z)-f(z)\}\cos\frac{i\pi z}{l}\,dz$$
$$-\cos\frac{i\pi at}{l}\int_0^l \{F(z)+f(z)\}\sin\frac{i\pi z}{l}\,dz\Big]; \quad (78)$$

and this also satisfies the conditions that $\frac{d\xi}{dt} = 0$, when $x = 0$ and when $x = l$. Also since when $t = 0$, we have the values given in (75), this becomes

$$\xi = -\frac{1}{\pi}\Sigma.\sin\frac{i\pi x}{l}\Big\{\sin\frac{i\pi at}{l}\int_0^l \phi(z)\cos\frac{i\pi z}{l}\,dz - \cos\frac{i\pi at}{l}\int_0^l \Phi(z)\sin\frac{i\pi z}{l}\,dz\Big\}, \quad (79)$$

which is the complete answer and satisfies all the conditions of the problem.

Hence if τ is the time of the whole vibration of a slice of the bar, $\tau = \frac{2l}{a} = 2l\left(\frac{\rho\omega}{E}\right)^{\frac{1}{2}} = 2\left(\frac{l}{g}\right)^{\frac{1}{2}}\left(\frac{W}{E}\right)^{\frac{1}{2}}$, where W is the weight of the bar; so that the note of the bar is the same as that of a vibrating string of the same elasticity thickness, density, and length.

If, however, one end of the bar, say O, is fixed, and the end A is free, then $\xi = 0$, and $\frac{d\xi}{dt} = 0$, when $x = 0$;

$$\therefore \quad \xi = -\frac{1}{\pi}\Sigma.\sin nx\int_0^l \{F(z)\sin n(at-z)-f(z)\sin n(at+z)\}\,dz. \quad (80)$$

Also since $\frac{d\xi}{dx} = 0$, when $x = l$,

$$n = (2i+1)\frac{\pi}{2l}; \tag{81}$$

which determines the form of n. And assuming the initial conditions to be those given in (75), the displacement takes the form

$$\xi = -\frac{1}{\pi}\Sigma . \sin(2i+1)\frac{\pi x}{2l}\left\{\sin(2i+1)\frac{\pi a t}{2l}\int_0^l \phi(z)\cos(2i+1)\frac{\pi z}{2l}dz \right.$$
$$\left. -\cos(2i+1)\frac{\pi a t}{2l}\int_0^l \Phi(z)\sin(2i+1)\frac{\pi z}{2l}dz\right\}; \tag{82}$$

and this expression satisfies all the conditions of the problem.

As the values of ξ become the same whenever at is increased by a multiple of $4l$, so if τ is the time of a complete vibration of a slice of the bar,

$$\tau = \frac{4l}{a} = 4l\left(\frac{\rho\omega}{\text{E}}\right)^{\frac{1}{2}} = 4\left(\frac{l}{g}\right)^{\frac{1}{2}}\left(\frac{\text{W}}{\text{E}}\right)^{\frac{1}{2}}, \tag{83}$$

if W is the weight of the bar. Thus the time of vibration is twice as long as it is in the former case, when both ends of the bar are fixed. Thus also the note due to the longitudinal vibrations of an elastic bar fixed at both ends is an octave higher than that due to it when fixed at one end and having the other end free.

Other problems on elastic strings and bars will be found in Donkin's Acoustics; a work to which the student is referred, as he will find in it the consideration of many questions bearing on the theory of music.

465.] The last problem of the motion of elastic bodies which I propose to consider is that of the molecules of a thin elastic plate, which is fixed along one edge, and is otherwise free from external constraint. The statical conditions of rest of such a plate when bent by the action of external forces have been considered in Section 3, Chap. V, Vol. III. I shall use the same symbols and the same diagrams as in that Section; and shall assume the plate in its original state of rest to be rectangular.

The plate in its natural state is supposed to be plane; and to have been bent, as is assumed in those Articles, by the action of one or more external forces, and then left to itself. It subsequently vibrates by virtue of the elastic forces which have been brought into action by the original displacement. The problem

in Vol. III is the determination of the surface which the plate takes when it is bent by certain given forces: the problem herein to be treated is the subsequent motion of the plate by reason of its elastic forces, when the original bending forces have ceased to act. For although the plate is assumed to be very thin, yet its thickness is such that elastic forces are brought into action upon its being bent.

Now take the Figure 73 in Vol. III to be an enlarged representation of the plate at the time t, except that there are no forces X and Y: then the equation of motion of the particles of the plate may be formed as follows. Take any transverse section of it, as P″PP′, the projection of the mean fibre of which intersects the plane of (x, y) in the point (x, y). Then, if we suppose the fibre to be at rest for an instant, and L to be the moment of the elastic forces which are due to the part P″AP′ of the plate, and act on the section about an axis perpendicular to the plane of the paper; by (126) Art. 176, Vol. III,

$$\text{L} = \frac{2kb\tau^3}{3\text{R}}; \tag{84}$$

where R is the radius of curvature of the bent lamina at the point P. But the forces, of which this is the moment, produce the motion of the part P″AP′ relative to the mean fibre of the section P″P′. Now let us take a thin slice of the plate at a point between P and A; and as the plate is very thin, we may consider its mass, which is equal to $2\rho\tau b\,ds'$ (ρ being the density), to be condensed into a particle at (x', y'), the place of its mean fibre; so that the expressed momentum-increments of this slice parallel to the axes of x and y are respectively

$$2\rho\tau b\,ds'\frac{d^2x'}{dt^2} \quad \text{and} \quad 2\rho\tau\,ds'\frac{d^2y'}{dt^2}. \tag{85}$$

Let us moreover assume the displacement of all particles of the plate to be so small, that all velocity-increments parallel to the x-axis may be neglected; and we shall also assume the inclination to the x-axis of all elements of its curved fibres which are straight in their natural state, to be so small, that powers of $\frac{dy}{dx}$ above the first may be neglected. Thus

$$\frac{1}{\text{R}} = \frac{d^2y}{dx^2}, \qquad ds' = dx'. \tag{86}$$

Hence also the moment of the expressed velocity-increments of

the part P″AP′ about an axis perpendicular to the plane of the paper, and passing through the mean fibre of P″P′ is

$$-2\rho\tau b\int_x^a \frac{d^2y'}{dt^2}(x'-x)\,dx'; \tag{87}$$

and this in equilibrium with L; so that we have

$$\frac{2kb\tau^3}{3}\frac{d^2y}{dx^2} = -2\rho\tau b\int_x^a \frac{d^2y'}{dt^2}(x'-x)\,dx'. \tag{88}$$

Let us take the x-differential of this equation; then x being the inferior limit of the definite integral in the right-hand member,

$$\frac{k\tau^2}{3\rho}\frac{d^3y}{dx^3} = \int_x^a \frac{d^2y'}{dt^2}\,dx';$$

and again taking the x-differential

$$\frac{k\tau^2}{3\rho}\frac{d^4y}{dx^4} = -\frac{d^2y}{dt^2}; \tag{89}$$

so that the equation to the vibrating plate is of the form

$$\frac{d^2y}{dt^2} + b^2\frac{d^4y}{dx^4} = 0. \tag{90}$$

This equation is not capable of integration in finite terms; but it is evident that

$$y = \{A\cos m^2bt + B\sin m^2bt\}\sin(mx+a) \tag{91}$$

satisfies it; A, B, m, and a being undetermined constants: and since $y=0$, when $x=0$, whatever is the value of t, $a = n\pi$, where n is any integer number; so that

$$y = \{A_n\cos m^2bt + B_n\sin m^2bt\}\sin(mx+n\pi). \tag{92}$$

And as n may be any whole number, the complete solution will be

$$y = \Sigma.\{A_n\cos m^2bt + B_n\sin m^2bt\}\sin(mx+n\pi); \tag{93}$$

wherein the sign of summation denotes the sum of a series of similar quantities given by the several values of n, which admits of all integers.

All the undetermined coefficients must be found by means of the initial circumstances of the plate, in the same way as similar questions have been treated in the preceding Articles.

CHAPTER X.

THEORETICAL DYNAMICS.

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(*Note in the First Edition.*)

[It will be observed that many terms and symbols employed in this Chapter differ from the corresponding ones of the previous parts of the Treatise; this arises in part from the fact that Professor Donkin had not seen the previous Chapters when this was written. In the unavoidable absence of Professor Donkin, it has not been thought desirable to change either the one or the other. The advanced student indeed, for whom especially this Chapter is intended, will not require any alteration: he will understand the different terms and symbols by means of either the context or the explanations which are given. Whatever has been added to Professor Donkin's work is enclosed in square brackets. It should also be noticed that his MS. bears date Sept. 6, 1860.]

466.] The object of this chapter is to give some account of the recent progress of theoretical dynamics. But no attempt will be made to follow accurately a historical order, or to assign every step in detail to its proper author. Such a plan would be hardly in accordance with the design of this Treatise, and it is moreover rendered unnecessary by Mr. Cayley's "Report on Dynamics," lately published*.

The reader's attention is requested to the following explanations of notation:

Throughout this Chapter total differentiation with respect to t (the time) will be denoted by accents; and accents will be used for no other purpose. Thus, instead of $\frac{d^2u}{dt^2}+\frac{d^2v}{dt^2}$ we shall write either $u''+v''$ or $(u+v)''$,..., and if u be a function containing

* Report of the British Association for 1857.

t explicitly, and also involving $x, y, \ldots$, which are functions of t, we shall have

$$u' = \frac{du}{dt} + \frac{du}{dx}x' + \frac{du}{dy}y' + \ldots;$$

where $\frac{du}{dt}$ signifies the partial differential coefficient of u with respect to t, taken so far as t appears explicitly in u.

But no other distinction will in general be made, by means of notation, between the various possible meanings of differential coefficients; the interpretation of the symbols, if not clear from the context, will be explained in each case.

Secondly, expressions of the form $\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx}$ are of such frequent occurrence, that it is desirable to have a recognised abbreviation for them. The following has been found convenient, and will be adopted, namely*,

$$\frac{du}{dx}\frac{dv}{dy} - \frac{du}{dy}\frac{dv}{dx} = \frac{d(u,v)}{d(x,y)}.$$

467.] The theorem of D'Alembert reduces the mathematical statement of every dynamical problem to the expression of conditions of equilibrium; and when these conditions are put in the general form, assigned by the principle of virtual velocities, there results a single formula, which may be written thus:

$$\Sigma . m(x''\delta x + y''\delta y + z''\delta z) = \Sigma(\text{X}\,\delta x + \text{Y}\,\delta y + \text{Z}\,\delta z): \qquad (1)$$

in which x, y, z are the coordinates of the mass m, referred to rectangular axes fixed in space, and X, Y, Z are the components of the force applied at the point x, y, z.

It is necessary to observe, that the force here meant is not the so-called "accelerating force", that is, the force which would act on a unit of mass; but the total force, of whatever kind, which is impressed at the point (x, y, z). Otherwise the formula would not include the case in which all or any of the forces are to be considered as pressures acting merely on mathematical

* More generally, the symbol $\frac{d(u, v, w, \ldots)}{d(x, y, z, \ldots)}$ stands for the "Jacobian" determinant, of which the constituents are

$$\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}, \ldots; \qquad \frac{dv}{dx}, \frac{dv}{dy}, \frac{dv}{dz}, \ldots; \qquad \frac{dw}{dx}, \frac{dw}{dy}, \frac{dw}{dz}, \ldots.$$

This notation was proposed some time ago by the writer of this chapter (see Phil. Trans. for 1854, p. 72), and has since received the sanction of Mr. Cayley (Report on Dynamics, p. 5).

surfaces, lines, or points; and having no relation to the magnitude of the masses which they tend to move. It is true that such forces are only mathematical fictions; but so are the conditions of almost all mechanical problems, treated as we are at present obliged to treat them.

It would however be out of place to enter into the questions suggested by this remark, because, for the purposes of this chapter, we are not concerned with the nature of the problem which gives rise to the formula (1), further than is necessary for a clear understanding of the meaning of the symbols.

In the most general case which occurs in practice, the values of X, Y, Z, at the time t, may depend in a given manner upon the time, the positions of all the points of the system, and the velocities and directions of their movements at the instant considered. In other words, X, Y, Z may be given functions of t, of all the coordinates $x, y, z, \ldots$, and of their first differential coefficients $x', y', z', \ldots$. This most general case has not, except in special problems, as yet been treated successfully, and we shall find it necessary to limit the significations of X, Y, Z; but the problems excluded by the limitation are comparatively unimportant.

468.] The meaning of the symbol of variation δ may be explained as follows: If $x, y, z, \ldots$ be the values of the coordinates in the actual position of the system at the time t, then $x+\delta x$, $y+\delta y$, $z+\delta z, \ldots$ are the values belonging to any other position which the system might have had at that time without violating the conditions by which its possible displacements are limited; provided only that the two positions be infinitely near to one another, so that $\delta x, \delta y, \ldots$ are infinitesimal.

Whatever be the nature of the conditions just mentioned, by which the motion of the system is constrained, they may always be supposed to be expressed by means of a certain number of equations of condition

$$L_1 = 0, \quad L_2 = 0, \ldots L_m = 0; \tag{2}$$

in which $L_1, L_2, \ldots$ are given functions of any or all of the coordinates, and may also contain t explicitly. Suppose n to be the whole number of coordinates involved in the formula (1), and m the number of equations of condition, m will be in all cases less than n; otherwise those equations would imply either a determinate fixed position of the system, or a determinate motion independent of the forces.

The formula (1) must be satisfied for every set of values of $\delta x, \delta y, \ldots$ which satisfy the m equations,

$$\left.\begin{aligned} &\frac{d\mathrm{L}_1}{dx}\,\delta x + \frac{d\mathrm{L}_1}{dy}\,\delta y + \ldots, = 0,\\ &\frac{d\mathrm{L}_2}{dx}\,\delta x + \frac{d\mathrm{L}_2}{dy}\,\delta y + \ldots = 0,\\ &\quad\cdots\cdots\cdots\cdots\\ &\frac{d\mathrm{L}_m}{dx}\,\delta x + \frac{d\mathrm{L}_m}{dy}\,\delta y + \ldots = 0. \end{aligned}\right\} \qquad (3)$$

By means of these last equations any m out of the n quantities $\delta x, \delta y, \ldots$ may be expressed in terms of the rest, and when their values so expressed are substituted in (1), that formula will involve only the remaining $n-m$ variations; and since the values of these may be taken arbitrarily, without violating the m equations of condition, the coefficient of each must be separately equated to zero; and thus we obtain $n-m$ simultaneous linear differential equations of the second order, that is, as many as there are independent coordinates.

This process has been described briefly, because the reader is supposed to be already familiar with it, at least in principle, and because we shall not have to perform it actually.

469.] Instead however of eliminating the variations in the manner just explained, we are of course at liberty to make use of any equation that can be obtained from (1) by substituting for $\delta x, \delta y, \ldots$ any admissible set of values; that is, any set which satisfies the m equations (3).

Among the infinite number of ways in which admissible values may be chosen for the variations $\delta x, \delta y, \ldots$, there are two which require particular notice.

First, let us suppose (what is usually, but not necessarily, the case) that every position of the system which is possible at any one time, is possible at any other time; which is the same thing as supposing that none of the equations of condition involve t explicitly. In this case, it is evident, that among the positions which the system might have had at the time t is that which it actually has at the time $t+dt$; but at the time $t+dt$, the coordinates $x, y, \ldots$ have become $x+dx, y+dy, \ldots$, where $dx = x'dt$, $dy = y'dt, \ldots$; so that we are at liberty to take as an admissible set of variations the displacements which actually happen in the

time dt. In fact, since we are now supposing that the equations $L_1 = 0, \dots$ do not involve t, the equations $L'_1 = 0, \dots$, are

$$\frac{dL_1}{dx} x' + \frac{dL_1}{dy} y' + \dots = 0, \dots;$$

and comparing these with (3), we see that $\delta x, \delta y, \dots$, may be taken proportional to $x', y', \dots$, that is, to $dx, dy, \dots$. But if L_1 contained t explicitly we should have

$$L'_1 = \frac{dL_1}{dt} + \frac{dL_1}{dx} x' + \frac{dL_1}{dy} y' + \dots = 0,$$

which cannot be made to coincide with the equation

$$\delta L_1 = \frac{dL_1}{dx} \delta x + \frac{dL_1}{dy} \delta y + \dots = 0$$

by any values whatever of $\delta x, \delta y, \dots$.

In the case now supposed, however, when the values $x'dt$, $y'dt, \dots$, or $dx, dy, \dots$, are substituted for $\delta x, \delta y, \dots$ in (1), that formula becomes

$$\Sigma . m (x'x'' + y'y'' + z'z'') dt = \Sigma (X dx + Y dy + Z dz);$$

or, if we put T for the vis viva of the system, that is,

$$T = \frac{1}{2} \Sigma . m (x'^2 + y'^2 + z'^2) = \frac{1}{2} \Sigma . m v^2;$$

we obtain
$$dT = \Sigma (X dx + Y dy + Z dz),$$

an equation which we shall meet with afterwards in a somewhat different form.

470.] The second way of choosing admissible values for δx, $\delta y, \dots$, is applicable in all cases without exception, and may be explained as follows:

The n coordinates $x, y, \dots$ being subject to m equations of condition, it follows that any $n-m$ of the coordinates may be considered as absolutely independent; that is, their values might at any time t be assumed arbitrarily without violating the laws of constraint expressed by the equations of condition; and since the same thing is true at the time $t + dt$, it follows, that not only the $n-m$ coordinates, but also their first differential coefficients $x', y', \dots$ might be arbitrarily assumed at the time t; but the values of these $2(n-m)$ quantities being given, those of the remaining m coordinates and of their first differential coefficients are determinate. In fact, if $\xi, \eta, \dots$ be these remaining coordinates, the m equations of condition would suffice to express

each of them in terms of $x, y, \ldots$, and t, so that we should have m equations, such as

$$\xi = f(x, y, \ldots t);$$

from which we should get by differentiation

$$\xi' = \frac{df}{dt} + \frac{df}{dx}x' + \frac{df}{dy}y' + \ldots;$$

where it is evident that the values of both ξ and ξ' are given at the time t, if those of $x, y, \ldots, x', y', \ldots$ are given.

Now the motion of the system, under the action of the given forces, is completely determined if the positions of all its points, and the velocities and directions of their movements, be given at a determinate time; that is, if the values of all the coordinates $x, y, \ldots$, and their first differential coefficients $x', y', \ldots$, be given at that time; and since it has just been seen that all these quantities are given if any $n-m$ of the coordinates, with their first differential coefficients, be given, we conclude that the whole motion is determined if the values of these $2(n-m)$ quantities be given at any one time; it is convenient to take the instant when $t = 0$ for the time in question, and we may call the values of any quantities at that time their initial values.

From these considerations, it is easy to conclude that the final integral equations of the problem must contain $2(n-m)$ arbitrary constants, and no more; that is, that the values of all the coordinates must be expressible in terms of $2(n-m)$ arbitrary constants and t; for otherwise, the number of coordinates and first differential coefficients which it would be possible to assume arbitrarily at a given time would be either more or less than $2(n-m)$. The same conclusion follows from the theory of differential equations.

Hence we are in all cases at liberty to suppose that the actual value of every one of the coordinates at the time t is expressible by an equation of the form

$$x = f(a, b, \ldots t);$$

where $a, b, \ldots$ are the arbitrary constants, of which the number is twice that of the independent coordinates.

These constants may be the initial values of some set of independent coordinates and of their first differential coefficients, and must be expressible as functions of such initial values.

471.] Now if we suppose the initial positions and velocities to receive infinitesimal alterations, or if, which comes to the same

thing, we suppose the constants $a, b, \ldots$ to be changed into $a+\delta a$, $b+\delta b, \ldots$, the values of $x, y, \ldots$, at the time t, will be changed into $x+\delta x$, $y+\delta y, \ldots$, where

$$\delta x = \frac{dx}{da}\delta a + \frac{dx}{db}\delta b + \ldots,$$

$$\delta y = \frac{dy}{da}\delta a + \frac{dy}{db}\delta b + \ldots;$$

the partial differential coefficients $\frac{dx}{da}, \ldots$ being taken on the hypothesis that $x. y, \ldots$ are expressed, as above supposed, in terms of $a, b, \ldots, t$.

The values of $\delta x, \delta y, \ldots$, thus formed, are distinguished from other admissible sets of values by important properties, which we proceed to point out.

First, if we suppose $a, b, \ldots$ to have the values belonging to the actual motion of the system, so that the point (x, y, z) actually describes the path defined by the equations $x = f(a, b, \ldots, t), \ldots$, then the values $a+\delta a$, $b+\delta b, \ldots$ correspond to a motion which does not actually take place, but which might take place under the action of the existing forces, and would take place if the initial circumstances were suitably altered; so that the substitution, at every instant, of $x+\delta x$ for $x, \ldots$ would change the actual paths and velocities of all the points of the system into others not merely consistent with the given equations of condition, but consistent also with the action of the forces. All such paths and velocities may be called "dynamically possible." But if the values of $\delta x, \delta y, \ldots$ were merely chosen so as to be consistent with the equations of condition, without any further limitation, then the substitution at every instant of $x+\delta x$ for x, ..., would change the actual paths and velocities into others, which, though not inconsistent with the given laws of constraint of the system, could not be produced by the action of the existing forces. Such paths and velocities may be called "geometrically possible*" though dynamically impossible.

472.] But there is another, and in some respects more important, distinction.

The values of $\delta a, \delta b, \ldots$ are arbitrary infinitesimal constants. Let $\Delta a, \Delta b, \ldots$ be any other set of similar values; and let δu, Δu

* The expressions "dynamically possible" and "geometrically possible" are Sir W. R. Hamilton's.

be the increments of any function u, corresponding to the two sets of increments of the constants; so that if u be expressed as a function of $a, b, c, \ldots$, with or without t, we shall have

$$\delta u = \frac{du}{da}\delta a + \frac{du}{db}\delta b + \ldots,$$

$$\Delta u = \frac{du}{da}\Delta a + \frac{du}{db}\Delta b + \ldots.$$

Now suppose that in the above value of δu we change $a, b, \ldots$ into $a+\Delta a$, $b+\Delta b, \ldots$, without altering the values of δa, $\delta b, \ldots$; the corresponding increment of δu will be

$$\Delta\delta u = \frac{d^2 u}{da^2}\Delta a . \delta a + \frac{d^2 u}{da\,db}(\delta a . \Delta b + \delta b . \Delta a) + \ldots;$$

but the same expression will be obtained for $\delta\Delta u$ from the second of the above equations; consequently,

$$\Delta\delta u = \delta\Delta u. \tag{4}$$

In this equation u may evidently be any function of the coordinates $x, y, z, \ldots$ and their differential coefficients of all orders; the property expressed by it distinguishes those variations of u which are due to variations of the constants $a, b, \ldots$ from those which, though otherwise admissible, arise in a different manner. In fact, it will be found in general, either that the symbols $\Delta\delta$, $\delta\Delta$ are unmeaning, or that the above equation is not true.

473.] Lastly, we must notice a property which belongs to the variations denoted by δ or Δ without any limitation, namely, that the operations δ or Δ, and $\frac{d}{dt}$ are commutative; that is,

$$\delta(u') = (\delta u)'; \tag{5}$$

where u is any function of $x, y \ldots$ (with their differential coefficients) and t, and δx, $\delta y, \ldots$ may be considered as perfectly arbitrary functions of t, subject to the sole restriction of being infinitesimal. In fact, the meaning of $\delta(u')$ is $(u+\delta u)' - u'$, that is, is $(\delta u)'$.

474.] Now let $x, y, \ldots$ be any n variables, functions of t, and subject to m equations of condition. Also let $\xi, \eta, \ldots$ be other variables, of which the number is not less than $n-m$, and so connected with the former set of variables by given equations, which may involve t explicitly but may not involve the differential coefficients of either set, that any variable of one set may

be expressed as a function of variables of the other set, with or without t, by equations such as

$$\left.\begin{aligned} x &= \phi(\xi, \eta, \ldots, t), \\ \xi &= \psi(x, y, \ldots, t). \end{aligned}\right\} \qquad (6)$$

It is to be observed, that the expressions on the right of these equations are, to a certain extent, indeterminate in form; for any function of $x, y, \ldots$ and t may be variously transformed by means of the given equations of condition; and the same may be said of any function of $\xi, \eta, \ldots$, if the number of the latter variables be greater than $n-m$.

Suppose then that u is any function of $t, x, y, \ldots, x', y', \ldots, x'', y'', \ldots$; and let r be the order of the highest differential coefficient contained in u. By means of the equations, such as the first of (6), $x, y, \ldots$ can be expressed as functions of $t, \xi, \eta, \ldots$; x', y', as functions of $t, \xi, \eta, \ldots, \xi', \eta', \ldots$; and so on; so that u can be transformed into a function of $t, \xi, \eta, \ldots, \xi', \eta', \ldots$, in which the highest differential coefficient will still be of the order r. Also, $\delta x, \delta y, \ldots$ can be transformed by means of the equations

$$\delta x = \frac{dx}{d\xi}\delta\xi + \frac{dx}{d\eta}\delta\eta + \ldots;$$

where $\dfrac{dx}{d\xi}, \dfrac{dx}{d\eta}, \ldots$ are given functions of $t, \xi, \eta, \ldots$.

Let $E_x u$ be for a moment an abbreviation for

$$\frac{du}{dx} - \left(\frac{du}{dx'}\right)' + \left(\frac{du}{dx''}\right)'' - \ldots,$$

the series being continued until it terminates of itself. Then the expression

$$E_x u \,.\, \delta x + E_y u \,.\, \delta y + E_z u \,.\, \delta z + \ldots$$

can be transformed into another, involving the variables $\xi, \eta, \ldots$, with their variations and differential coefficients, instead of $x, y, \ldots$.

It is a known theorem in the Calculus of Variations that the result of this transformation is an expression of the same form, namely,
$$E_\xi u \,.\, \delta\xi + E_\eta u \,.\, \delta\eta + \ldots;$$
where u only differs from the former u in being expressed in terms of the new variables.

The direct demonstration* of this theorem in its general form is somewhat complicated, and need not be given here, because the only case with which we are concerned is that in which u

* For an indirect demonstration, see Lagrange, Mécanique Analytique, 2[de] partie, 4[me] section, 6, or De Morgan's Diff. Calc., p. 519.

contains no differential coefficient of a higher order than the first, or $$u = \text{F}(t, x, y, \ldots x', y', \ldots).$$
The theorem may then be conveniently written thus:

$$\Sigma . \left\{ \left(\frac{du}{dx'}\right)' - \frac{du}{dx} \right\} \delta x = \Sigma . \left\{ \left(\frac{du}{d\xi'}\right)' - \frac{du}{d\xi} \right\} \delta \xi; \qquad (7)$$

and this admits of the following simple demonstration, due in principle to Sir W. R. Hamilton.

475.] Since u, when expressed in terms of the new variables, $\xi, \eta, \ldots$, will contain $\xi', \eta', \ldots$, only because it originally contains $x', y', \ldots$, we shall have

$$\frac{du}{dx'} = \frac{du}{d\xi'}\frac{d\xi'}{dx'} + \frac{du}{d\eta'}\frac{d\eta'}{dx'} + \ldots;$$

where the differentiations in $\frac{d\xi'}{dx'}, \ldots$ are performed on the supposition that $\xi', \eta', \ldots$ are expressed in terms of $t, x, y, \ldots, x', y', \ldots$: now ξ being expressed in terms of $t, x, y, \ldots$, we have

$$\xi' = \frac{d\xi}{dt} + \frac{d\xi}{dx}x' + \frac{d\xi}{dy}y' + \ldots;$$

and since $\frac{d\xi}{dt}, \frac{d\xi}{dx}, \ldots$, do not contain $x', y' \ldots$, we obtain at once by differentiation

$$\frac{d\xi'}{dx'} = \frac{d\xi}{dx}, \qquad \frac{d\xi'}{dy'} = \frac{d\xi}{dy}, \ldots; \qquad (8)$$

in like manner we should find

$$\frac{d\eta'}{dx'} = \frac{d\eta}{dx}, \qquad \frac{d\eta'}{dy'} = \frac{d\eta}{dy}, \ldots:$$

hence the above expression for $\frac{du}{dx'}$ becomes

$$\frac{du}{dx'} = \frac{du}{d\xi'}\frac{d\xi}{dx} + \frac{du}{d\eta'}\frac{d\eta}{dx} + \ldots:$$

similarly we should have

$$\frac{du}{dy'} = \frac{du}{d\xi'}\frac{d\xi}{dy} + \frac{du}{d\eta'}\frac{d\eta}{dy} + \ldots;$$

and so on: whence, multiplying the first of these equations by δx, the second by $\delta y, \ldots$, and observing that

$$\frac{d\xi}{dx}\delta x + \frac{d\xi}{dy}\delta y + \ldots = \delta\xi,$$

we obtain by addition,

$$\frac{du}{dx'}\delta x + \frac{du}{dy'}\delta y + \ldots = \frac{du}{d\xi'}\delta\xi + \frac{du}{d\eta'}\delta\eta + \ldots;$$

and differentiating this last equation with respect to t, and observing (Art. 473) that $(\delta x)' = \delta x', \ldots$,

$$\left(\frac{du}{dx'}\right)' \delta x + \frac{du}{dx'} \delta x' + \ldots = \left(\frac{du}{d\xi'}\right)' \delta \xi + \frac{du}{d\xi'} \delta \xi' + \ldots.$$

Now δu may be expressed in either of the following ways:

$$\delta u = \frac{du}{dx} \delta x + \frac{du}{dx'} \delta x' + \frac{du}{dy} \delta y + \ldots,$$

$$\delta u = \frac{du}{d\xi} \delta \xi + \frac{du}{d\xi'} \delta \xi' + \frac{du}{d\eta} \delta \eta + \ldots;$$

and if the first of these values be subtracted from the left-hand member, and the second from the right-hand member of the above equation, the result is the equation (7), which was to be established.

476.] We now return to the dynamical formula (1), Art. 467. In that formula, the position of the system at the time t is assigned by means of the rectangular coordinates $x, y, z, \ldots$; but it is evident that any other set of variables, $\xi, \eta, \zeta, \ldots$, connected with $x, y, z, \ldots$, in the manner supposed in Art. 474, would answer the same purpose. We may extend the meaning of the word "coordinates" so as to include all sets of variables of which the values at the time t determine the position of the system at that time. For instance, the position of a rigid system which has one fixed point may be defined in several ways by means of three angles, which may be called the "coordinates" of the system.

Now the theorem (7), Art. 474, enables us to express the left-hand member of the formula (1) in a form adapted to any system of coordinates whatever, in the following manner:

Let T* denote as before the vis viva of the system; then, in terms of the original coordinates, we have

$$2\mathrm{T} = \Sigma . m (x'^2 + y'^2 + z'^2);$$

but when T is expressed in terms of any other coordinates, $\xi, \eta, \ldots$, it will become in general a function of $\xi, \eta, \ldots, \xi', \eta', \ldots$, with or without t, not containing any differential coefficients of a higher order than the first. Hence we may put T for u in the equation (7), and, observing that

$$\frac{d\mathrm{T}}{dx'} = \Sigma . m x', \quad \frac{d\mathrm{T}}{dx} = 0,$$

* [Vis viva, as used here, is one-half of the quantity which has heretofore been called by that name.]

we obtain $$\Sigma . m x'' \delta x = \Sigma . \left\{ \left(\frac{d\text{T}}{d\xi'}\right)' - \frac{d\text{T}}{d\xi} \right\} \delta \xi;$$
which is the required form.

The terms $\text{X}\delta x + \text{Y}\delta y + \ldots$, on the right of (1), when expressed in terms of the new coordinates, will take the form $\text{P}\delta\xi + \text{Q}\delta\eta + \ldots$, where P, Q, ... are functions of $\xi, \eta, \ldots$ with or without t; so that the equation (1) is finally reducible to the form
$$\Sigma . \left\{ \left(\frac{d\text{T}}{d\xi'}\right)' - \frac{d\text{T}}{d\xi} \right\} \delta\xi = \Sigma . \text{P}\delta\xi. \qquad (9)$$
In the most usual and important problems, X, Y, Z, ... are the partial differential coefficients with respect to $x, y, z, \ldots$ of a function U, called the force-function, which may also contain t, but does not contain $x', y', \ldots$: in this case we have
$$\text{X}\delta x + \text{Y}\delta y + \ldots = \delta\text{U};$$
and the right-hand member of (9) is obtained by deriving δU from U expressed in terms of the new variables: thus
$$\delta\text{U} = \frac{d\text{U}}{d\xi}\delta\xi + \frac{d\text{U}}{d\eta}\delta\eta + \ldots;$$
and the equation may then be written in the form
$$\Sigma . \left\{ \left(\frac{d\text{T}}{d\xi'}\right)' - \frac{d\text{T}}{d\xi} - \frac{d\text{U}}{d\xi} \right\} \delta\xi = 0: \qquad (10)$$
this may be abridged by putting $\text{T} + \text{U} = \text{W}$; for, since U does not contain $\xi', \ldots, \frac{d\text{T}}{d\xi'} = \frac{d\text{W}}{d\xi'}, \ldots$; so that the formula becomes
$$\Sigma . \left\{ \left(\frac{d\text{W}}{d\xi'}\right)' - \frac{d\text{W}}{d\xi} \right\} \delta\xi = 0. \qquad (11)$$

If the coordinates $\xi, \eta, \ldots$ be independent, that is, subject to no equations of condition, the coefficients of $\delta\xi, \delta\eta, \ldots$ must separately vanish; so that (11) is equivalent to the system of equations
$$\left(\frac{d\text{W}}{d\xi'}\right)' = \frac{d\text{W}}{d\xi}, \qquad \left(\frac{d\text{W}}{d\eta'}\right)' = \frac{d\text{W}}{d\eta}, \ldots. \qquad (12)$$
We shall refer to these as the "Lagrangian" equations; a name given to them by Mr. Cayley.

477.] It will be desirable to illustrate the preceding formulæ by some examples before proceeding further. First then, let it be required to express the equations of motion of a single material point by means of polar coordinates.

Let m be the mass of the point, x, y, z its rectangular coordinates, X, Y, Z the components of the force acting on m, so that

the equations of motion in their primitive form are included in the formula

$$m(x''\delta x+y''\delta y+z''\delta z) = \text{X}\delta x+\text{Y}\delta y+\text{Z}\delta z. \qquad (13)$$

In polar coordinates we have, according to the usual notation,

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\theta\,;$$

and we must express $2\text{T} = m(x'^2+y'^2+z'^2)$ in terms of r, θ, ϕ, r', θ', ϕ'. Differentiating, we find

$$x' = r'\sin\theta\cos\phi+\theta' r\cos\theta\cos\phi-\phi' r\sin\theta\sin\phi,$$
$$y' = r'\sin\theta\sin\phi+\theta' r\cos\theta\sin\phi+\phi' r\sin\theta\cos\phi,$$
$$z' = r'\cos\theta \qquad -\theta' r\sin\theta\,;$$

hence we easily obtain

$$2\text{T} = m\{r'^2+r^2\theta'^2+r^2(\sin\theta)^2\phi'^2\}\,;$$

and we know (Art. 476) that the left-hand member of (13) will become

$$\left\{\left(\frac{d\text{T}}{dr'}\right)'-\frac{d\text{T}}{dr}\right\}\delta r+\left\{\left(\frac{d\text{T}}{d\theta'}\right)'-\frac{d\text{T}}{d\theta}\right\}\delta\theta+\left\{\left(\frac{d\text{T}}{d\phi'}\right)'-\frac{d\text{T}}{d\phi}\right\}\delta\phi.$$

Now $\frac{d\text{T}}{dr'} = mr'$, $\qquad \frac{d\text{T}}{dr} = mr\{\theta'^2+(\sin\theta^2)\phi'^2\}$,

$$\frac{d\text{T}}{d\theta'} = mr^2\theta', \qquad \frac{d\text{T}}{d\theta} = mr^2\sin\theta\cos\theta\,\phi'^2,$$

$$\frac{d\text{T}}{d\phi'} = mr^2(\sin\theta)^2\phi', \quad \frac{d\text{T}}{d\phi} = 0\,;$$

and these values reduce the above expression to the following:

$$m\{r''-r\theta'^2-r(\sin\theta)^2\phi'^2\}\delta r$$
$$+m\{(r^2\theta')'-r^2\sin\theta\cos\theta\,\phi'^2\}\delta\theta+m\{r^2(\sin\theta)^2\phi'\}'\delta\phi.$$

The right-hand member of (13) will always be reducible to the form $\text{P}\delta r+\text{Q}\delta\theta+\text{R}\delta\phi$; and in the case in which X, Y, Z are of the form $\frac{d\text{U}}{dx}$, $\frac{d\text{U}}{dy}$, $\frac{d\text{U}}{dz}$, then also P, Q, R will be of the form $\frac{d\text{U}}{dr}$, $\frac{d\text{U}}{d\theta}$, $\frac{d\text{U}}{d\phi}$.

If the motion of m be unconstrained, δr, $\delta\theta$, $\delta\phi$ are all arbitrary, and the formula (13) breaks up into three separate equations.

478.] As a second example, let us consider the transformation from fixed rectangular axes to moving rectangular axes with the same origin.

*Let x, y, z be the coordinates of the material point m, referred

* [This is the problem which has already been investigated in the first Section of Chapter VIII. The difference of notation will be observed.]

to the fixed axes; ξ, η, ζ the coordinates of the same point, referred to the moving axes; and let the position of the moving system at the time t be defined in the usual manner by the equations

$$x = a_1\xi + a_2\eta + a_3\zeta, \quad y = b_1\xi + b_2\eta + b_3\zeta, \quad z = c_1\xi + c_2\eta + c_3\zeta;$$

where the nine direction-cosines a_1, a_2, ... are given functions of t. It will be convenient to introduce the usual symbols ω_1, ω_2, ω_3 for the angular velocities of the moving system of axes, estimated about the axes of ξ, η, ζ respectively. Let a rotation about the axis of ζ be positive when its direction is such that the axis of ξ is following the axis of η; then, with similar conventions as to the other axes, we shall have

$$a_3a_2' + b_3b_2' + c_3c_2' = -(a_2a_3' + b_2b_3' + c_2c_3') = \omega_1,$$
$$a_1a_3' + b_1b_3' + c_1c_3' = -(a_3a_1' + b_3b_1' + c_3c_1') = \omega_2,$$
$$a_2a_1' + b_2b_1' + c_2c_1' = -(a_1a_2' + b_1b_2' + c_1c_2') = \omega_3.$$

Now differentiating the equations $x = a_1\xi + a_2\eta + a_3\zeta$, ... we obtain

$$x' = a_1\xi' + a_2\eta' + a_3\zeta' + a_1'\xi + a_2'\eta + a_3'\zeta,$$
$$y' = b_1\xi' + b_2\eta' + b_3\zeta' + b_1'\xi + b_2'\eta + b_3'\zeta,$$
$$z' = c_1\xi' + c_2\eta' + c_3\zeta' + c_1'\xi + c_2'\eta + c_3'\zeta;$$

and hence, observing the above values of ω_1 ..., and the known relations between the nine direction-cosines, including the equations $a_1a_1' + b_1b_1' + c_1c_1' = 0$, ...,

$$a_1x' + b_1y' + c_1z' = \xi' + \omega_2\zeta - \omega_3\eta,$$
$$a_2x' + b_2y' + c_2z' = \eta' + \omega_3\xi - \omega_1\zeta,$$
$$a_3x' + b_3y' + c_3z' = \zeta' + \omega_1\eta - \omega_2\xi;$$

and finally, by adding the squares of these expressions on each side

$$x'^2 + y'^2 + z'^2 = (\xi' + \omega_2\zeta - \omega_3\eta)^2 + (\eta' + \omega_3\xi - \omega_1\zeta)^2 + (\zeta' + \omega_1\eta - \omega_2\xi)^2.$$

The meaning of the terms in this expression is easily seen: thus, ξ' is the velocity of the point (ξ, η, ζ) relative to the moving axes, estimated parallel to the axis of ξ; and $\omega_2\zeta - \omega_3\eta$ is the velocity, relative to fixed space, and estimated in the same direction, which the same point would have if (ξ, η, ζ) were invariable; the sum is the total component of the velocity relative to fixed space. Hence the value of T, expressed in terms of the new coordinates, becomes

$$\mathrm{T} = \frac{1}{2}\Sigma . m\{(\xi' + \omega_2\zeta - \omega_3\eta)^2 + (\eta' + \omega_3\xi - \omega_1\zeta)^2 + (\zeta' + \omega_1\eta - \omega_2\xi)^2\}; \quad (14)$$

which is the expression to be used in forming the left-hand member of the equation (9), Art. 476.

The right-hand member of that equation is to be obtained by transforming the expression $\Sigma(X\,\delta x + Y\,\delta y + Z\,\delta z)$.

Now from the equations

$$x = a_1\xi + a_2\eta + a_3\zeta, \dots,$$

we have
$$\delta x = a_1\delta\xi + a_2\delta\eta + a_3\delta\zeta, \dots$$

$a_1, a_2, \dots,$ being treated as invariable in the differentiation denoted by δ, because $\delta x, \dots$ refer to displacements in space which might subsist at the time t. Hence

$$X\,\delta x + Y\,\delta y + Z\,\delta z =$$
$$(a_1 X + b_1 Y + c_1 Z)\delta\xi + (a_2 X + b_2 Y + c_2 Z)\delta\eta + (a_3 X + b_3 Y + c_3 Z)\delta\zeta;$$

now $a_1 X + b_1 Y + c_1 Z$ is the component of the force acting at (x, y, z), estimated in the direction of the axis of ξ; so that if we call this component Ξ, the term involving $\delta\xi$ becomes $\Xi\,\delta\xi$; and similar conclusions will result for the other terms. Thus the right-hand member of the equation (9) may be represented by $\Sigma . \Xi\,\delta\xi$; the summation referring to all the coordinates as well as to all the points, and the coefficient of each variation being the corresponding component of force.

479.] As an illustration of the general formulæ of the preceding Article, we may take the following problem:

To find the modification introduced into the treatment of dynamical problems, referring to motion near the earth's surface, when the earth's rotation is taken into account.

Neglecting the curvature of the path of the earth's centre, and assuming that the forces concerned in the problem are independent of the earth's position in its orbit, we may consider the centre as fixed.

Let the primitive axes of coordinates then have their origin at the earth's centre, the positive axis of z being directed to the north pole, the axes of x and y being in the plane of the equator, but fixed in space; the positive axis of y being to the east of that of x.

We wish to take as new axes a system fixed relatively to the earth, and having its origin at a given point on the earth's surface.

As an intermediate step, let ξ, η, ζ refer to axes fixed in the earth and parallel to the required axes, but having their origin

at the centre. If then we call ω the angular velocity of the earth's rotation, and $\cos a$, $\cos\beta$, $\cos\gamma$ the direction-cosines of the polar axis referred to the axes of ξ, η, ζ, we shall have

$$\omega_1 = \omega\cos a, \quad \omega_2 = \omega\cos\beta, \quad \omega_3 = \omega\cos\gamma;$$

and the expression (14), Art. 478, becomes

$$\text{T} = \frac{1}{2}\Sigma . m\,[\{\xi' + \omega(\zeta\cos\beta - \eta\cos\gamma)\}^2 + \{\eta' + \omega(\xi\cos\gamma - \zeta\cos a)\}^2 + \{\zeta' + \omega(\eta\cos a - \xi\cos\beta)\}^2];$$

and it only remains to remove the origin to the required point on the surface by writing $\xi+\xi_0$, $\eta+\eta_0$, $\zeta+\zeta_0$ instead of ξ, η, ζ, where ξ_0, η_0, ζ_0 are the coordinates of the point in question, referred to the axes of ξ, η, ζ with centre as origin. This being done, the following values will be found without difficulty:

$$\frac{d\text{T}}{d\xi'} = m\xi' + m\omega\{(\zeta+\zeta_0)\cos\beta - (\eta+\eta_0)\cos\gamma\},$$

$$\frac{d\text{T}}{d\xi} = m\omega(\eta'\cos\gamma - \zeta'\cos\beta) + m\omega^2(\xi+\xi_0) - m\omega^2\cos a\{(\xi+\xi_0)\cos a + (\eta+\eta_0)\cos\beta + (\zeta+\zeta_0)\cos\gamma\};$$

consequently

$$\left(\frac{d\text{T}}{d\xi'}\right)' - \frac{d\text{T}}{d\xi} = m\xi'' + 2m\omega(\zeta'\cos\beta - \eta'\cos\gamma) - m\omega^2(\xi+\xi_0) + m\omega^2\cos a\{(\xi+\xi_0)\cos a + (\eta+\eta_0)\cos\beta + (\zeta+\zeta_0)\cos\gamma\};$$

from which the forms of the terms referring to η and ζ are obvious.

If we call l the latitude of the place at which the origin is fixed, and take the plane of (ξ, η) horizontal, the axis of ξ being directed to the south and that of η to the east, we shall have

$$\cos a = -\cos l, \quad \cos\beta = 0, \quad \cos\gamma = \sin l;$$

also $\xi_0 = 0$, and η_0, ζ_0 are given quantities, the former being small, of which the values are easily assigned in terms of the earth's axes and of l.

Thus we obtain*

$$\left(\frac{d\text{T}}{d\xi'}\right)' - \frac{d\text{T}}{d\xi} = m\xi'' - 2m\omega\sin l\,\eta' - m\omega^2(\sin l)^2\xi - m\omega^2\sin l\cos l(\zeta+\zeta_0),$$

$$\left(\frac{d\text{T}}{d\eta'}\right)' - \frac{d\text{T}}{d\eta} = m\eta'' + 2m\omega(\xi'\sin l + \zeta'\cos l) - m\omega^2(\eta+\eta_0),$$

$$\left(\frac{d\text{T}}{d\zeta'}\right)' - \frac{d\text{T}}{d\zeta} = m\zeta'' - 2m\omega\cos l\,\eta' - m\omega^2\sin l\cos l\,\xi - m\omega^2(\cos l)^2(\zeta+\zeta_0);$$

* [The equations given in (55), Art. 408, are identical with these when the signs of ω and η are changed.]

so that, finally, the general equations of motion of any system under the circumstances supposed are comprised in the following formula:

$$\begin{aligned}&\Sigma . m\,(\xi''\delta\xi+\eta''\delta\eta+\zeta''\delta\zeta)\\&\quad+2\,\omega\sin l\,\Sigma . m\,(\xi'\delta\eta-\eta'\delta\xi)+2\,\omega\cos l\,\Sigma . m\,(\zeta'\delta\eta-\eta' d\zeta)\\&\quad-\omega^2(\sin l)^2\,\Sigma . m\,\xi\delta\xi-\omega^2\cos^2 l\,\Sigma . m\,(\zeta+\zeta_0)\,\delta\zeta-\omega^2\,\Sigma . m\,(\eta+\eta_0)\,\delta\eta\\&\qquad-\omega^2\sin l\cos l\,\Sigma . m\,\{(\zeta+\zeta_0)\,\delta\xi+\xi\delta\zeta\}=\Sigma . m\,\Xi\delta\xi.\end{aligned}$$

480.] It is not intended in this Chapter to discuss particular problems; and the examples given in the last three Articles have been inserted only because the Lagrangian formulæ, if left in their general shape without illustration, would probably fail to convey precise notions to the mind of a reader coming to them for the first time.

We proceed now to an important transformation of these formulæ, due* to Sir W. R. Hamilton, without stopping to introduce at this stage the consequences derived from them by Lagrange; because these, with many other results, are more easily obtained from the Hamiltonian form.

Conforming to the notation of recent writers, we will denote the "coordinates" in any dynamical problem by $q_1, q_2, \ldots,$ so that the general formula (11), Art. 476, becomes

$$\Sigma . \left\{\left(\frac{d\mathrm{w}}{dq'}\right)' - \frac{d\mathrm{w}}{dq}\right\}\delta q = 0\,; \qquad (15)$$

in which we shall suppose that w may be any function whatever of $q_1, q_2, \ldots, q_1', q_2', \ldots,$ and t.

If now we put

$$\frac{d\mathrm{w}}{dq_1'} = p_1, \qquad \frac{d\mathrm{w}}{dq_2'} = p_2, \ldots, \qquad (16)$$

we may suppose $q_1', q_2', \ldots$ to be expressed, by means of these equations, in terms of $p_1, p_2, \ldots, q_1, q_2, \ldots,$ with or without t; and when these values of $q_1', q_2', \ldots$ are introduced in the formula (15), that formula, together with (16), will give a set of equations involving the two sets of variables $q_1, q_2, \ldots, p_1, p_2, \ldots,$ with their first differential coefficients, instead of the one set $q_1, q_2, \ldots$ with their first and second differential coefficients.

Thus, if the coordinates $q_1, q_2, \ldots, q_n,$ be an independent set, instead of n differential equations of the second order we shall have $2n$ of the first order.

* A first step towards this transformation was made by Poisson; but we have not space for details on this point.

The general form of these $2n$ equations was first assigned by Sir W. R. Hamilton. His demonstration depends upon the particular character of the function T in most actual problems; and the following, which is slightly different and more general, is therefore substituted.

481.] The principle of the demonstration may be most clearly exhibited independently, in the form of the following theorem:

If P be any function of the n quantities $x_1, x_2, \ldots x_n$, and if n other quantities $y_1, y_2, \ldots y_n$ be defined by the equations

$$y_1 = \frac{dP}{dx_1}, \quad y_2 = \frac{dP}{dx_2}, \ldots, \quad y_n = \frac{dP}{dx_n}; \tag{17}$$

then, if by means of these equations, $x_1, \ldots, x_n$ be expressed in terms of $y_1, y_2, \ldots, y_n$, their values will be of the form

$$x_1 = \frac{dQ}{dy_1}, \quad x_2 = \frac{dQ}{dy_2}, \ldots, \quad x_n = \frac{dQ}{dy_n}; \tag{18}$$

where
$$Q = -P + x_1 y_1 + x_2 y_2 + \ldots + x_n y_n: \tag{19}$$
in which $x_1, \ldots, x_n$ on the right are supposed to be expressed in terms of $y_1, \ldots, y_n$.

Also if P contain any other quantities, $\xi, \ldots$, besides $x_1, \ldots, x_n$, then
$$\frac{dP}{d\xi} = -\frac{dQ}{d\xi};$$
the differentiation with respect to ξ being in each case performed only so far as ξ appears explicitly.

To prove this we have, if the symbol d operate only on $x_1, \ldots, x_n, y_1, \ldots, y_n$,
$$dP = y_1 dx_1 + y_2 dx_2 + \ldots + y_n dx_n, \quad \text{by (17)};$$
but
$$d(x_1 y_1 + \ldots + x_n y_n) = y_1 dx_1 + \ldots + y_n dx_n + x_1 dy_1 + \ldots + x_n dy_n;$$
hence, by subtraction,
$$d(x_1 y_1 + \ldots + x_n y_n - P) = x_1 dy_1 + \ldots + x_n dy_n;$$
an equation which must be identical if both sides be expressed in the same way. If therefore we put, as above, $Q = x_1 y_1 + \ldots - P$, and suppose $x_1, \ldots$, on each side, expressed in terms of $y_1, \ldots$, since $dQ = \frac{dQ}{dy_1} dy_1 + \ldots$, we must have $x_1 = \frac{dQ}{dy_1}, \ldots$, which proves the first part of the theorem.

To prove the second part, we observe that the value (19) of Q will contain ξ explicitly, partly because it is contained explicitly in P, as originally expressed, and partly because the values of

$x_1, \ldots$, in terms of $y_1, \ldots$, when substituted in P and the other terms, will introduce it again. Hence we shall have

$$\frac{d\text{Q}}{d\xi} = -\frac{d\text{P}}{d\xi} - \frac{d\text{P}}{dx_1}\frac{dx_1}{d\xi} - \ldots - \frac{d\text{P}}{dx_n}\frac{dx_n}{d\xi}$$
$$+ y_1 \frac{dx_1}{d\xi} + \ldots + y_n \frac{dx_n}{d\xi};$$

but since $y_1 = \frac{d\text{P}}{dx_1}, \ldots$, this equation becomes simply $\frac{d\text{Q}}{d\xi} = -\frac{d\text{P}}{d\xi}$, which was to be proved.

Let us now apply this theorem to transform the formulæ (15) and (16), Art. 480.

The equations (16) being exactly similar to (17) of this Article, it follows, that if we put

$$\text{H} = -\text{W} + p_1 q_1' + p_2 q_2' + \ldots + p_n q_n',$$

and express $q_1', \ldots q_n'$ on the right in terms of $p_1, \ldots, p_n, \ldots$, we shall have

$$q_1' = \frac{d\text{H}}{dp_1}, \ldots, \quad q_n' = \frac{d\text{H}}{dp_n};$$

and moreover, since, besides $q_1', q_2', \ldots$, W contains also the quantities $q_1, q_2, \ldots$, analogous to $\xi, \ldots$, we shall have also

$$\frac{d\text{W}}{dq_1} = -\frac{d\text{H}}{dq_1}, \quad \frac{d\text{W}}{dq_2} = -\frac{d\text{H}}{dq_2}, \ldots;$$

so that the formula (15) will become

$$\Sigma\left(p' + \frac{d\text{H}}{dq}\right)\delta q = 0.$$

These results may be summed up as follows:

If W be any function of $q_1, q_2, \ldots, q_1', q_2', \ldots, t$, the formula

$$\Sigma\left\{\left(\frac{d\text{W}}{dq'}\right)' - \frac{d\text{W}}{dq}\right\}\delta q = 0$$

is transformed into the system

$$\left.\begin{aligned} \Sigma\left(p' + \frac{d\text{H}}{dq}\right)\delta q = 0, \\ q_1' = \frac{d\text{H}}{dp_1}, \quad q_2' = \frac{d\text{H}}{dp_2}, \quad \ldots \end{aligned}\right\} \quad (20)$$

by the following substitutions:

$$\frac{d\text{W}}{dq_1'} = p_1, \quad \frac{d\text{W}}{dq_2'} = p_2, \ldots,$$
$$\text{H} = p_1 q_1' + p_2 q_2' + \ldots - \text{W};$$

where, in forming the expression for H, we are to express q_1', q_2',

... in terms of $p_1, p_2, \ldots, q_1, q_3, \ldots$, so that H is in general a function of $p_1, p_2, \ldots, q_1, q_2, \ldots$, and t.

One case deserves particular notice, because it occurs in most actual dynamical problems. If W be of the form T + U, where T is homogeneous and of the second degree in $q_1', q_2', \ldots$, and U does not contain $q_1', q_2', \ldots$, then $p_1 = \frac{d\text{T}}{dq_1'}, \ldots$, and therefore

$$p_1 q_1' + p_2 q_2' + \ldots = q_1' \frac{d\text{T}}{dq_1'} + \ldots = 2\text{T};$$

hence, in this case,

$$\text{H} = 2\text{T} - \text{W} = \text{T} - \text{U},$$

where T is to be expressed in terms of $p_1, p_2, \ldots, q_1, q_2, \ldots$.

If $q_1, q_2, \ldots$ be a set of independent coordinates, say n in number, then the system (20) gives $2n$ separate equations, namely, those obtained by giving to i all integer values from 1 to n inclusive in the two following:

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i}. \tag{21}$$

We shall call these, as Mr. Cayley has done, the "Hamiltonian" equations*. In treating of their general properties it is usually unnecessary to take any account of the nature of the problems which give rise to such a system. H is to be considered merely as a given function of $p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n$, and t.

482.] The complete solution of the $2n$ simultaneous differential equations of the first order, represented by the formula (21), would consist of $2n$ equations involving the variables $p_1, \ldots, q_1, \ldots$, and t, with $2n$ arbitrary constants. Any one such equation may be called an "integral equation;" but it is desirable to distinguish by a separate name that particular form of integral equation in which a function of variables only is equated to an arbitrary constant. We shall call such an equation an "integral." Thus the general form of an integral will be

$$c = f(p_1, \ldots p_n, q_1, \ldots q_n, t);$$

* The Lagrangian equations may be considered as a particular case of a more general form, upon which the solution of a class of problems in the Calculus of Variations depends; and it has been shewn by M. Ostrogradsky, that this more general form is susceptible of a transformation which includes that of Sir W. R. Hamilton as a particular case. See "Mémoire sur les équations différentielles relatives au problème des isopérimètres, 1848." Mém. de l'Acad. Impér. des Sciences de St. Pétersbourg. Sciences Math. et Phys. t. iv, 1850.

where the function on the right contains no arbitrary constant; and it is a convenient abbreviation to speak of such an integral as* "the integral c."

Thus a complete solution of the system (21) may be supposed to consist of $2n$ integrals. But in order that $2n$ integrals may constitute a complete solution, it is necessary that they should be independent; that is, that no identical relations should subsist between the functions equated to the arbitrary constants. If such relations did subsist, the variables might be eliminated, and one or more equations be obtained involving the constants only, so that the constants would not be all arbitrary.

Hence the problem of integrating the system of equations (21) may be stated as follows:

"To find $2n$ independent functions of $p_1, p_2, \ldots p_n, q_1, q_2, \ldots q_n$, and t, each of which is constant by virtue of the differential equations (21)."

On the other hand, the same problem might be regarded as having for its object "to express each of the $2n$ variables, $p_1, \ldots$, $q_1, \ldots$, as a function of $2n$ arbitrary constants and t."

If a complete solution were obtained in either of these forms, it is evident that algebraical processes only would be required to deduce from it a solution in the other form, as well as an infinite variety of "integral equations."

The consideration of the two forms just mentioned is of the greatest theoretical importance, though neither of them is in general obtained as a direct result of existing methods of integration.

Inasmuch as all complete solutions of the same system of differential equations must be equivalent to one another, it follows that any arbitrary constant belonging to one solution must be capable of being expressed as a function of the arbitrary constants belonging to any other solution.

483.] Any $2n$ functions of the variables $p_1, \ldots, q_1, \ldots$, and t, may be called *elements*, provided that the equations by which they are defined are algebraically sufficient to determine conversely the $2n$ variables $p_1, \ldots, q_1, \ldots$, as functions of the ele-

* This expression however "the integral c" is, to avoid circumlocution, used not only to signify the equation $c = f(p_1, \ldots)$, but also to denote either side of that equation separately, viz., either the constant c, or the function $f(p_1, \ldots)$, which has that constant value. The last is the most usual meaning.

ments and t. Thus, if the elements $u_1, u_2, \dots u_{2n}$ be defined by $2n$ equations, such as

$$u_1 = f(p_1, \dots, q_1, \dots, t),$$

then it must not be possible to eliminate all the $2n$ variables, $p_1, \dots, q_1, \dots,$ from these equations.

From the above definition, it is evident that a complete solution of the differential equations would be obtained if any set of elements were expressed in terms of arbitrary constants and t.

It is also evident, that the functions which are equated to arbitrary constants in any complete set of integrals are "elements." Thus elements may be either variable or constant.

484.] It may be useful to exhibit at this stage, for the sake of clearness, the equations of a simple dynamical problem in the Hamiltonian form. For this purpose we may take the case of motion of a single material point about a fixed centre of force. Let m be the mass of the moving point; then, taking the origin of the polar coordinates r, θ at the fixed centre, and the plane of the motion for the plane of the angle θ, we shall have

$$2\text{T} = m(r'^2 + r^2\theta'^2);$$

and the force-function U will be a given function of r, say $\text{U} = \phi(r)$. Then, writing q_1 instead of r, and q_2 instead of θ, we have

$$2\text{T} = m(q_1'^2 + q_1^2 q_2'^2);$$

consequently,

$$p_1 = \frac{d\text{T}}{dq_1'} = mq_1', \qquad p_2 = \frac{d\text{T}}{dq_2'} = mq_1^2 q_2';$$

from which we have

$$q_1' = \frac{p_1}{m}, \qquad q_2' = \frac{p_2}{mq_1^2},$$

and therefore H, which in this case is T$-$U expressed in terms of p_1, p_2, q_1, q_2, becomes, see equation (20),

$$\text{H} = \frac{1}{2m}\Big(p_1^2 + \frac{p_2^2}{q_1^2}\Big) - \phi(q_1);$$

and the four equations (21), Art. 481, become

$$p_1' = \frac{p_2^2}{mq_1^3} + \phi'(q_1), \qquad q_1' = \frac{p_1}{m};$$

$$p_2' = 0, \qquad q_2' = \frac{p_2}{mq_1^2}.$$

485.] If u, v be any functions whatever, containing the vari-

ables $p_1, \dots p_n, q_1, \dots q_n$, then it is convenient to employ the following symbol: let

$$(u,v) = \Sigma\left(\frac{du}{dp_i}\frac{dv}{dq_i} - \frac{du}{dq_i}\frac{dv}{dp_i}\right),$$

or in the notation of Art. 466,

$$(u,v) = \Sigma\frac{d(u,v)}{d(p_i,q_i)};$$

the summation extending to all values of i from 1 to n.

The reader will hardly require to be reminded that the symbols (u, v), (p_i, q_i) on the right of the last equation, have only an accidental resemblance to the (u, v) on the left, without any connexion of meaning.

For example, if u, v contain p_1, p_2, q_1, q_2 only, then

$$(u,v) = \frac{du}{dp_1}\frac{dv}{dq_1} - \frac{du}{dq_1}\frac{dv}{dp_1} + \frac{du}{dp_2}\frac{dv}{dq_2} - \frac{du}{dq_2}\frac{dv}{dp_2}.$$

From the above definition the following consequences are easily deduced by means of the elementary principles of differentiation:

$$(u,v) = -(v,u), \qquad (u,u) = 0,$$
$$(p_i,q_i) = 1, \qquad (q_i,p_i) = -1,$$

and $(p_i, q_j) = 0$, if j be different from i.

Also, if a be any function of $v, w, \dots$, then

$$(u,a) = \frac{da}{dv}(u,v) + \frac{da}{dw}(u,w) + \dots.$$

Again, if v contain $p_1, q_1, \dots$ explicitly, and also $a, \beta, \dots$ functions of $p_1, q_1, \dots$, then

$$(u,v) = (u,\bar{v}) + \frac{dv}{da}(u,a) + \frac{dv}{d\beta}(u,\beta) + \dots,$$

where $(u, \bar{v})$ represents the expression formed by differentiating v only so far as it contains $p_1, q_1, \dots$ explicitly.

Lastly, if u, v contain explicitly any other quantity, say z, besides the variables $p_1, q_1, \dots$, then the partial differential coefficient of (u, v), taken explicitly with respect to z, is

$$\frac{d}{dz}(u,v) = \left(\frac{du}{dz}, v\right) + \left(u, \frac{dv}{dz}\right).$$

486.] The following theorem will be of use afterwards:

Let u, v, w be any three functions whatever, containing $p_1, q_1, \dots$, with or without other quantities, then

$$\{u,(v,w)\} + \{v,(w,u)\} + \{w,(u,v)\} = 0. \qquad (22)$$

For if this expression were developed, each term would, irre-

spective of sign, consist of the product of one second differential coefficient, and two first differential coefficients. Thus we should have terms in which u is twice differentiated, arising from $\{v, (w, u)\}$ in the three forms

$$\frac{dv}{dp_i}\frac{dw}{dp_j}\frac{d^2u}{dq_i dq_j}, \quad \frac{dv}{dq_i}\frac{dw}{dq_j}\frac{d^2u}{dp_i dp_j}, \quad \frac{dv}{dp_i}\frac{dw}{dq_j}\frac{d^2u}{dq_i dp_j},$$

including the case of $j = i$; but the same terms would arise from $\{w, (u, v)\}$ with the contrary signs, as the reader will easily verify. The same thing may be said of the terms in which v and w are twice differentiated. Hence the equation (22) is satisfied identically, as was to be shewn.

The properties established in this and the preceding Articles are independent of any suppositions as to the meanings of p_1, $q_1, \ldots$, and of the relations established by the differential equations (21), to the consideration of which we now return.

487.] Suppose a complete solution of the equations (21), namely,

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i},$$

to have been obtained, so that each of the $2n$ variables $p_1, \ldots p_n$, $q_1, \ldots q_n$, is a given function of t, and of $2n$ arbitrary constants $c_1, c_2, \ldots c_{2n}$.

Also let two independent sets of arbitrary infinitesimal variations be attributed to the constants, and denoted by the symbols δ, Δ, so that we should have

$$\delta p_i = \frac{dp_i}{dc_1}\delta c_1 + \frac{dp_i}{dc_2}\delta c_2 + \ldots,$$

$$\Delta p_i = \frac{dp_i}{dc_1}\Delta c_1 + \frac{dp_i}{dc_2}\Delta c_2 + \ldots;$$

then the expression

$$\delta p_1 \Delta q_1 - \Delta p_1 \delta q_1 + \delta p_2 \Delta q_2 - \Delta p_2 \delta q_2 + \ldots,$$

or

$$\Sigma(\delta p_i \Delta q_i - \Delta p_i \delta q_i) \qquad (23)$$

is constant. That is, if the above values of $\delta p_i, \ldots$, in terms of the constants, their variations, and t, be introduced, t will disappear from the result, and the expression (23) will become a function of the constants $c_1, c_2, \ldots$, and their variations $\delta c_1, \Delta c_1, \ldots$ only.

This remarkable theorem was discovered by Lagrange, employing his own form of the differential equations. The following simple demonstration of it is due to Professor Boole.

H is a given function of $p_1, \ldots p_n, q_1, \ldots q_n, t$; and we have

$$\delta H = \frac{dH}{dp_1}\delta p_1 + \frac{dH}{dq_1}\delta q_1 + \ldots;$$

so that by the equations (21)

$$\delta H = q_1'\delta p_1 - p_1'\delta q_1 + \ldots;$$

and consequently

$$\delta H = \Sigma(q_i'\delta p_i - p_i'\delta q_i).$$

Now performing the operation Δ on each side of this equation, we have

$$\Delta\delta H = \Sigma(\Delta q_i'\delta p_i - \Delta p_i'\delta q_i + q_i'\Delta\delta p_i - p_i'\Delta\delta q_i);$$

in like manner we should find

$$\delta\Delta H = \Sigma(\delta q_i'\Delta p_i - \delta p_i'\Delta q_i + q_i'\delta\Delta p_i - p_i'\delta\Delta q_i);$$

hence, subtracting and observing that $\Delta\delta = \delta\Delta$,

$$0 = \Sigma(\Delta q_i'\delta p_i + \Delta q_i\delta p_i' - \Delta p_i'\delta q_i - \Delta p_i\delta q_i');$$

now $\Delta q_i' = (\Delta q_i)', \ldots$; and thus this equation is equivalent to

$$0 = \Sigma(\delta p_i\Delta q_i - \Delta p_i\delta q_i)';$$

that is, the total differential coefficient with respect to t of the expression (23) vanishes, and that expression is therefore constant; which was to be proved.

488.] Suppose now that the $2n$ constants $c_1, c_2, \ldots$ are the initial values of the variables, which we will denote by $\lambda_1, \lambda_2, \ldots \lambda_n$, $\mu_1, \mu_2, \ldots \mu_n$; where $\lambda_1, \lambda_2, \ldots$ are the initial values of $p_1, p_2, \ldots$, and $\mu_1, \mu_2, \ldots$ of $q_1, q_2, \ldots$.

Since the value of $\Sigma(\delta p_i\Delta q_i - \Delta p_i\delta q_i)$ is independent of t, it is not altered by supposing $t = 0$; but when $t = 0$, the values of $\delta p_i, \delta q_i, \ldots$ are $\delta\lambda_i, \delta\mu_i, \ldots$; consequently,

$$\Sigma(\delta p_i\Delta q_i - \Delta p_i\delta q_i) = \Sigma(\delta\lambda_i\Delta\mu_i - \Delta\lambda_i\delta\mu_i). \tag{24}$$

By the help of this equation we can shew that the initial values $\lambda_1, \mu_1, \ldots$, considered as constant elements, possess certain remarkable properties.

Each of these elements may be supposed to be expressed as a function of the variables, $p_1, q_1, \ldots$, and t. Let this supposition be called Hypothesis I.

On the other hand each of the variables, $p_1, q_1, \ldots$, may be supposed to be expressed in terms of the elements $\lambda_1, \mu_1, \ldots$, and t. Let this supposition be called Hypothesis II.

Now in equation (24) let $\delta p_i, \delta q_i$, on the left-hand side be expressed in terms of $\delta\lambda_1, \delta\mu_1, \ldots$; thus,

$$\delta p_i = \frac{dp_i}{d\lambda_1}\delta\lambda_1 + \frac{dp_i}{d\mu_1}\delta\mu_1 + \ldots,$$

$$\delta q_i = \frac{dq_i}{d\lambda_1}\delta\lambda_1 + \frac{dq_i}{d\mu_1}\delta\mu_1 + \ldots;$$

and let $\Delta\lambda_i$, $\Delta\mu_i$, on the right, be expressed in terms $\Delta p_1, \Delta q_1, \ldots$; thus,

$$\Delta\lambda_i = \frac{d\lambda_i}{dp_1}\Delta p_1 + \frac{d\lambda_i}{dq_1}\Delta q_1 + \ldots,$$

$$\Delta\mu_i = \frac{d\mu_i}{dp_1}\Delta p_1 + \frac{d\mu_i}{dq_1}\Delta q_1;$$

so that on both sides of the equation all terms involving δ will be variations of constants, and those involving Δ will be variations of variables; and each of these sets of variations may have arbitrary values assigned to them; hence the coefficients of corresponding terms on the two sides of the equation must be equal. Thus we obtain by comparing the coefficients of

$$\left.\begin{array}{ll}
\delta\lambda_j\Delta q_i, \text{ the equation} & \dfrac{dp_i}{d\lambda_j} = \dfrac{d\mu_j}{dq_i}, \\
\delta\mu_j\Delta q_i, \text{ - - - -} & \dfrac{dp_i}{d\mu_j} = -\dfrac{d\lambda_j}{dq_i}, \\
\delta\lambda_j\Delta p_i, \text{ - - - -} & \dfrac{dq_i}{d\lambda_j} = -\dfrac{d\mu_j}{dp_i}, \\
\delta\mu_j\Delta p_i, \text{ - - - -} & \dfrac{dq_i}{d\mu_j} = \dfrac{d\lambda_j}{dp_i};
\end{array}\right\} \quad (25)$$

in which equations the differentiation refers to Hypothesis II on the left, and Hypothesis I on the right; and in each of them i may be equal to j.

Now suppose any one of the constants, say λ_j, to be expressed, according to Hypothesis I, in terms of the variables, thus

$$\lambda_j = f(p_1, \ldots p_n, q_1, \ldots q_n, t);$$

if on the right of this equation each of the variables were expressed, according to Hypothesis II, in terms of the constants and t, the equation would become identical; that is, the right-hand side would become identically λ_j; hence, if we differentiate each side with respect to λ_j, on Hypothesis II, the result on the right must $= 1$; but if we differentiate with respect to any other of the constants, the result must $= 0$; thus,

$$\frac{d\lambda_j}{dp_1}\frac{dp_1}{d\lambda_j} + \frac{d\lambda_j}{dq_1}\frac{dq_1}{d\lambda_j} + \frac{d\lambda_j}{dp_2}\frac{dp_2}{d\lambda_j} + \frac{d\lambda_j}{dq_2}\frac{dq_2}{d\lambda_j} + \ldots = 1;$$

and if c be any one of the constants except λ_j,

$$\frac{d\lambda_j}{dp_1}\frac{dp_1}{dc}+\frac{d\lambda_j}{dq_1}\frac{dq_1}{dc}+\frac{d\lambda_j}{dp_2}\frac{dp_2}{dc}+\frac{d\lambda_j}{dq_2}\frac{dq_2}{dc}+\ldots=0.$$

Now in the first of these equations let the values of $\frac{dp_1}{d\lambda_j}, \frac{dq_1}{d\lambda_j}, \ldots$ given by (25) be substituted; and it will be seen that the result is

$$(\lambda_j, \mu_j)=1;$$

see Art. 485. But if in the second of the above equations we take for c either λ_i, where i is not $=j$, or μ_i, where i may be either equal to j or not, the result in the first case is

$$(\lambda_j, \mu_i)=0;$$

and in the second it is

$$-(\lambda_j, \lambda_i)=0.$$

By supposing μ_j expressed in terms of $p_1, q_2, \ldots$, and reasoning in the same way, we should obtain the equation

$$(\mu_j, \mu_i)=0.$$

Thus we see that the elements $\lambda_1, \ldots \lambda_n, \mu_1, \ldots \mu_n$, possess the properties expressed by the equations

$$(\lambda_i, \mu_i)=1, \quad (\lambda_i, \mu_j)=0, \quad (\lambda_i, \lambda_j)=0, \quad (\mu_i, \mu_j)=0. \qquad (26)$$

If we call any pair, such as λ_i, μ_i, conjugate elements, the above properties may be briefly stated by saying, that if f, g be any two of the elements, then $(f, g)=\pm 1$ if f, g be conjugate, and $=0$ in every other case.

489.] If $a_1, a_2, \ldots a_n, b_1, b_2, \ldots b_n$ be elements, such that, f, g representing any pair, the value of (f, g) is ± 1 or 0, according as f, g are a conjugate pair (that is, a pair such as a_i, b_i) or not, then these elements are called *canonical elements*.

It has been shewn in the preceding Articles that when the $2n$ variables, $p_1, q_1, \ldots$, are determined as functions of t and of $2n$ arbitrary constants, by means of the Hamiltonian equations (21), there exists one set of arbitrary constants, namely, the initial values, $\lambda_1, \mu_1, \ldots$, of the variables, which form a system of canonical constant elements. We shall now prove that the number of such systems is infinite.

In fact if $a_1, \ldots a_n, b_1, \ldots b_n$ be determined as functions of $\lambda_1, \mu_1, \ldots$ by the $2n$ equations

$$\frac{d\text{A}}{da_i}=b_i, \qquad \frac{d\text{A}}{d\mu_i}=\lambda_i, \qquad (27)$$

where A is any arbitrary function of $a_1, \ldots a_n, \mu_1, \ldots \mu_n$, then $a_1, b_1, \ldots$ will be canonical elements. For we have

$$\delta A = \frac{dA}{da_1}\delta a_1 + \frac{dA}{d\mu_1}\delta\mu_1 + \ldots,$$
$$= \Sigma\left(\frac{dA}{da_i}\delta a_i + \frac{dA}{d\mu_i}\delta\mu_i\right);$$

and consequently, by (27),

$$\delta A = \Sigma(b_i\delta a_i + \lambda_i\delta\mu_i);$$

and performing the operation Δ on each side of this equation,

$$\Delta\delta A = \Sigma(\Delta b_i\delta a_i + \Delta\lambda_i\delta\mu_i) + \Sigma(b_i\Delta\delta a_i + \lambda_i\Delta\delta\mu_i):$$

similarly we should find

$$\delta\Delta A = \Sigma(\delta b_i\Delta a_i + \delta\lambda_i\Delta\mu_i) + \Sigma(b_i\delta\Delta a_i + \lambda_i\Delta\delta\mu_i);$$

whence, subtracting and observing that $\delta\Delta = \Delta\delta$,

$$\Sigma(\delta a_i\Delta b_i - \Delta a_i\delta b_i) = \Sigma(\delta\lambda_i\Delta\mu_i - \Delta\lambda_i\Delta\mu_i).$$

But it has already been proved, see Art. 488, that the right-hand member of this equation is $= \Sigma(\delta p_i\Delta q_i - \Delta p_i\delta q_i)$, so that we have

$$\Sigma(\delta a_i\Delta b_i - \Delta a_i\delta b_i) = \Sigma(\delta p_i\Delta q_i - \Delta p_i\delta q_i);$$

from which it follows, that all the consequences deduced in Art. 488 from equation (24) will be true, if we substitute $a_1, b_1, \ldots$ for $\lambda_1, \mu_1, \ldots$, and in particular that the conditions

$$(a_i, b_i) = 1, \qquad (a_i, a_j) = (a_i, b_j) = (b_i, b_j) = 0,$$

will subsist.

490.] The equations last written are particular cases of the following general theorem, discovered by Poisson.

If f, g be any two integrals whatever of the Hamiltonian equations, then (f, g) is constant. Poisson's demonstration was obtained by means of the Lagrangian form of the equations. The following, founded on the Hamiltonian form, is much simpler:

If u be any function whatever of $p_1, q_1, \ldots$, and t, we have

$$u' = \frac{du}{dt} + \frac{du}{dp_1}p_1' + \frac{du}{dq_1}q_1' + \ldots;$$

but as $p_1, q_1, \ldots$ are supposed to satisfy the differential equations (21), we have $p_1' = -\frac{dH}{dq_1}, q_1' = \frac{dH}{dp_1}, \ldots$, so that

$$u' = \frac{du}{dt} + \Sigma\left(\frac{dH}{dp_i}\frac{du}{dq_i} - \frac{dH}{dq_i}\frac{du}{dp_i}\right);$$
$$= \frac{du}{dt} + (H, u); \text{ (see Art. 485).}$$

Now if we take $u = (f, g)$, we have, (Art. 485),

$$\frac{du}{dt} = \left(\frac{df}{dt}, g\right) + \left(f, \frac{dg}{dt}\right),$$

and therefore

$$(f, g)' = \left(\frac{df}{dt}, g\right) + \left(f, \frac{dg}{dt}\right) + \{\text{H}, (f, g)\};$$

now since f and g are integrals, $f' = 0$, and $g' = 0$; that is,

$$\frac{df}{dt} + (\text{H}, f) = 0, \qquad \frac{dg}{dt} + (\text{H}, g) = 0:$$

these equations are identically true; so that we may substitute $-(\text{H}, f)$, $-(\text{H}, g)$, for $\frac{df}{dt}$, $\frac{dg}{dt}$ respectively in the above expression for $(f, g)'$, and the result may be written thus,

$$(f, g)' = \{g, (\text{H}, f)\} + \{f, (g, \text{H})\} + \{\text{H}, (f, g)\};$$

but by the theorem proved in Art. 486, the expression on the right of this equation vanishes identically, and therefore $(f, g)' = 0$; or (f, g), is constant, which is the theorem to be demonstrated.

Here it is to be observed, that f and g represent given functions of the variables $p_1, q_1, \ldots$, and t, which are constant by virtue of the differential equations. But the constancy of the expression (f, g) may subsist in two different ways:

First, (f, g) may be identically constant, that is, a determinate numerical constant, or zero: this always happens when f and g belong to a set of canonical elements.

Secondly, (f, g) may be constant, not identically, but by virtue of the differential equations; and in this case

$$c = (f, g)$$

will be an integral of the equations; but here again there are two cases, for c may be either an independent arbitrary constant or a function of f and g; in the latter case, the integral c is a combination of the integrals f, g, but in the former case, it is a distinct independent integral. Thus it may happen that the theorem will lead to the discovery of a new integral when two are known. For example, the problem of motion about a fixed centre of force, leads, as will be seen afterwards, to three integrals of the forms

$$\begin{aligned} h &= p_1^2 + p_2^2 + p_3^2 - \phi(q_1^2 + q_2^2 + q_3^2), \\ e &= q_2 p_3 - q_3 p_2, \\ f &= q_3 p_1 - q_1 p_3; \end{aligned}$$

and it will be found on trial that $(h, e) = 0$, $(h, f) = 0$; but that $(e, f) = q_2 p_1 - p_2 q_1$, which is neither identically constant nor expressible as a function of the other integrals: hence we may affirm that

$$g = p_2 q_1 - q_2 p_1$$

is a new integral. But if we attempt to discover more integrals by the same method, we shall fail; for it will be found that

$$(h, g) = 0, \qquad (e, g) = f, \qquad (f, g) = -e.$$

It is to be observed also that the integral g is as easily discoverable by ordinary methods as the integrals e and f; so that in this case, and probably in general, the theorem is of no practical use as a means of obtaining new integrals, though very important in other points of view.

491.] We now come to a most important discovery, due to Sir W. R. Hamilton.

Suppose the solution of the system of equations (21), namely,

$$p_i' + \frac{d\mathrm{H}}{dq_i} = 0, \qquad q_i' - \frac{d\mathrm{H}}{dp_i} = 0,$$

to be given in the form of $2n$ integral equations involving the initial values, $\lambda_1, \mu_1, \ldots$, of the variables, as arbitrary constants. By means of these $2n$ equations each of the $2n$ variables could be expressed in terms of the $2n$ constants and t; and therefore the differential coefficients of the variables with respect to t could be expressed in the same way. Consider then the expression

$$p_1 q_1' + p_2 q_2' + \ldots + p_n q_n' - \mathrm{H};$$

this being a given function of the variables, their first differential coefficients, and t, might be expressed as above supposed, and would become a function of the $2n$ constants, $\lambda_1, \mu_1, \ldots$, and t. Suppose this function to be integrated with respect to t from $t = 0$, and let the result be called S; so that

$$\mathrm{S} = \int_0^t \{\Sigma(p_i q_i') - \mathrm{H}\}\, dt. \tag{28}$$

The value of S, obtained in this way, would be also a function of $\lambda_1, \mu_1, \ldots$, and t. But by means of the $2n$ integral equations we might express the $2n$ quantities, $\lambda_1, \lambda_2, \ldots \lambda_n, p_1, p_2, \ldots p_n$, in terms of the $2n$ quantities $\mu_1, \mu_2, \ldots \mu_n, q_1, q_2, \ldots q_n$, and t; and if the values of $\lambda_1, \ldots \lambda_n$, thus expressed, were substituted in the above value of S, the result would be of the form

$$\mathrm{S} = \mathrm{F}(q_1, q_2, \ldots q_n, \mu_1, \mu_2, \ldots \mu_n, t). \tag{29}$$

Now, taking the form (28), and using the symbol δ in the same sense as before, so that $\delta t = 0$, we have, by the rules of the calculus of variations,

$$\delta s = \int_0^t [\Sigma\{p_i(\delta q_i)'\} + \Sigma(q_i'\delta p_i) - \delta H]\, dt:$$

let H in this equation be supposed to be expressed in its original form as a function of $p_1, \dots p_n, q_1, \dots q_n, t$; then

$$\delta H = \Sigma\left(\frac{dH}{dp_i}\delta p_i + \frac{dH}{dq_i}\delta q_i\right);$$

but by the differential equations (21),

$$\frac{dH}{dp_i} = q_i', \qquad \frac{dH}{dq_i} = -p_i',$$

hence

$$\delta H = \Sigma(q_i'\delta p_i - p_i'\delta q_i);$$

and if this be substituted in the above value of δs, the result is

$$\delta s = \int_0^t [\Sigma\{p_i(\delta q_i)'\} + \Sigma(p_i'\delta q_i)]\, dt$$

$$= \int_0^t \Sigma(p_i\delta q_i)'\, dt.$$

Thus δs turns out to be expressible as the integral of a perfect differential with respect to t. Performing the integration from $t = 0$, and observing that when $t = 0$ the values of $p_i, \delta q_i$ are $\lambda_i, \delta\mu_i$, we obtain

$$\delta s = \Sigma(p_i\delta q_i) - \Sigma(\lambda_i\delta\mu_i).$$

But if we suppose s to be expressed as in equation (29), we have

$$\delta s = \Sigma\left(\frac{ds}{dq_i}\delta q_i\right) + \Sigma\left(\frac{ds}{d\mu_i}\delta\mu_i\right).$$

Now these two values of δs involve the same set of $2n$ variations, $\delta q_1, \dots \delta q_n, \delta\mu_1, \dots \delta\mu_n$, which may all be considered as arbitrary and independent, because the $2n$ variables and $2n$ constants are only subject to $2n$ equations; so that the values of any set of $2n$ out of the $4n$ quantities could be assumed arbitrarily without contradicting the equations; consequently the coefficients of like variations must be equal, that is, the $2n$ equations,

$$\frac{ds}{dq_i} = p_i, \qquad \frac{ds}{d\mu_i} = -\lambda_i, \tag{30}$$

must be true; but these equations are obviously not true identically, and they contain the $2n$ arbitrary constants, $\lambda_1, \mu_1, \dots$. Hence they can only be a particular form of the integral equations of the problem.

492.] It appears from the equations (30), just established,

that if the single function S, expressed in the form (29), were known, a complete set of integral equations could be deduced from it by mere differentiation. Thus the complete integration of the system of differential equations (21) is made to depend upon finding the form of a single function. This is the most essential part of Sir W. R. Hamilton's discovery; but it must not be supposed that the above brief account of it represents the original form and manner of the author's investigation, much less that it gives any notion at all of the general contents of his two elaborate memoirs "On a General Method in Dynamics," contained in the Philosophical Transactions for 1834 and 1835.

493.] We proceed to examine the function S more closely. Recollecting that S is supposed to be expressed in terms of $q_1, \dots q_n, \mu_1, \dots \mu_n, t$, we have

$$\begin{aligned} S' &= \frac{dS}{dt} + \frac{dS}{dq_1} q_1' + \dots \\ &= \frac{dS}{dt} + \Sigma(p_i q_i'), \text{ by (30)}; \end{aligned}$$

on the other hand, equation (28) gives by differentiation with respect to t, $\quad S' = \Sigma(p_i q_i') - H$;

comparing these two values of S', we obtain

$$\frac{dS}{dt} + H = 0. \tag{31}$$

Now H is given as a function of $p_1, \dots, q_1, \dots, t$; say

$$H = f(p_1, p_2, \dots p_n, q_1, q_2, \dots q_n, t);$$

also, by (30), $p_i = \dfrac{dS}{dq_i}$; hence (31) may be written in the form

$$\frac{dS}{dt} + f\left(\frac{dS}{dq_1}, \frac{dS}{dq_2}, \dots \frac{dS}{dq_n}, q_1, q_2, \dots q_n, t\right) = 0. \tag{32}$$

But this equation contains, besides t, only the n variables $q_1, \dots q_n$, and the n constants $\mu_1, \dots \mu_n$; it cannot therefore be any combination of the integral equations (30), because those $2n$ equations would in general be insufficient to eliminate the $2n$ quantities $p_1, \dots p_n, \lambda_1, \dots \lambda_n$; hence it must be satisfied identically; in other words, it is a partial differential equation of which S is a solution.

494.] The above result is due to Sir W. R. Hamilton, and so is also the following: Differentiating H with respect to t, we have

$$H' = \frac{dH}{dt} + \Sigma\left(\frac{dH}{dp_i} p_i'\right) + \Sigma\left(\frac{dH}{dq_i} q_i'\right);$$

but, by the differential equations,

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i};$$

so that the above equation becomes simply

$$\text{H}' = \frac{d\text{H}}{dt}.$$

Now if H does not contain t explicitly, which is the case in ordinary* dynamical problems, then $\frac{d\text{H}}{dt} = 0$; therefore $\text{H}' = 0$, or,

$$\text{H} = \text{constant};$$

and this is an integral of the problem, called the "integral of vis viva." In this case the equation (31) is of the form

$$\frac{d\text{S}}{dt} + f(p_1, p_2, \ldots p_n, q_1, q_2, \ldots q_n) = 0;$$

and from this we may deduce an equation like (32) as before; but since H is now constant, its value will not be altered by substituting for p_1, q_1, ... their initial values; hence we may write

$$\frac{d\text{S}}{dt} + f(\lambda_1, \lambda_2, \ldots \lambda_n, \mu_1, \mu_2, \ldots \mu_n) = 0;$$

or, by (30),

$$\frac{d\text{S}}{dt} + f\Big(-\frac{d\text{S}}{d\mu_1}, -\frac{d\text{S}}{d\mu_2}, \ldots -\frac{d\text{S}}{d\mu_n}, \mu_1, \mu_2, \ldots \mu_n\Big) = 0; \quad (33)$$

which is a second partial differential equation satisfied by S in the case supposed.

495.] We must now notice an important extension of the preceding theory, discovered by Jacobi, and contained in the following theorem:

If H be $f(p_1, p_2, \ldots p_n, q_1, q_2, \ldots q_n, t)$, and if S be any complete solution of the partial differential equation,

$$\frac{d\text{S}}{dt} + f\Big(\frac{d\text{S}}{dq_1}, \frac{d\text{S}}{dq_2}, \ldots \frac{d\text{S}}{dq_n}, q_1, q_2, \ldots q_n, t\Big) = 0, \quad (34)$$

* Ordinary dynamical problems may be defined as those in which the law of "action and reaction" is maintained. This law is violated when, for instance, in the planetary theory, the disturbing planet is considered as being itself undisturbed; or when, in investigating the effect of the earth's rotation upon the motion of a pendulum, the pendulum is supposed not to affect the motion of the earth; or, speaking generally, whenever any part of a system is supposed to be subject to an "obligatory motion." In all these cases H will contain t explicitly.

containing the n arbitrary constants $a_1, a_2, \dots a_n$, then the $2n$ equations

$$\frac{d\text{S}}{dq_i} = p_i, \qquad \frac{d\text{S}}{da_i} = b_i$$

are a complete set of integral equations of the system

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i}. \tag{35}$$

Here $b_1, b_2, \dots b_n$ are new arbitrary constants; and by a complete solution of the equation (34) is meant a solution containing, besides any constant merely added to S, n arbitrary constants $a_1, a_2, \dots a_n$, in such a manner that they cannot be all eliminated so as to produce a partial differential equation of the first order, without employing all the $n+1$ differential coefficients

$$\frac{d\text{S}}{dq_1}, \dots, \frac{d\text{S}}{dq_n}, \frac{d\text{S}}{dt}.$$

Before proving the theorem we must investigate a criterion by which it may be known whether any solution of such an equation as (34) be complete or not.

Suppose then $\text{S} = \psi(a_1, \dots a_n, q_1, \dots q_n, t)$ to be a solution; and this to give

$$\frac{d\text{S}}{dq_1} = \phi_1(a_1, \dots a_n, q_1, \dots q_n, t),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\frac{d\text{S}}{dq_n} = \phi_n(a_1, \dots a_n, q_1, \dots q_n, t):$$

now if it be impossible to eliminate the n quantities $a_1, \dots a_n$, between these equations, the solution is evidently complete, because the additional equation obtained by differentiating with respect to t will then be required.

But it is known * that the supposed elimination is always impossible unless the determinant formed with the n^2 constituents

* If the elimination be possible, it will lead to an identical relation between $\phi_1, \dots \phi_n$, considered as functions of $a_1, \dots a_n$, say $\text{F}(\phi_1, \phi_2, \dots \phi_n) = 0$; differentiating this equation with respect to a_i we obtain

$$\frac{d\text{F}}{d\phi_1}\frac{d\phi_1}{da_i} + \dots + \frac{d\text{F}}{d\phi_n}\frac{d\phi_n}{da_i} = 0,$$

and if from the n equations given by this formula, on putting $i=1, \dots, i=n$, we eliminate the n terms $\frac{d\text{F}}{d\phi_1}, \dots \frac{d\text{F}}{d\phi_n}$, the result is $\Delta = 0$, Δ being the determinant mentioned in the text.

$$\begin{matrix} \frac{d\phi_1}{da_1}, & \frac{d\phi_1}{da_2}, & \dots & \frac{d\phi_1}{da_n} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{d\phi_n}{da_1}, & \frac{d\phi_n}{da_2}, & \dots & \frac{d\phi_n}{da_n} \end{matrix}$$

vanishes identically.

We may therefore express the required criterion as follows:

In order that the solution s may be complete, it is only necessary that the determinant formed with the n^2 constituents

$$\begin{matrix} \frac{d^2 s}{da_1\,dq_1}, & \frac{d^2 s}{da_2\,dq_1}, & \dots & \frac{d^2 s}{da_n\,dq_1} \\ \cdot & \cdot & \cdot & \cdot \\ \frac{d^2 s}{da_1\,dq_n}, & \frac{d^2 s}{da_2\,dq_n}, & \dots & \frac{d^2 s}{da_n\,dq_n} \end{matrix}$$

shall not vanish identically. In the abridged notation, explained in the note of Art. 466, this determinant would be written

$$\frac{d\left(\frac{ds}{dq_1}, \frac{ds}{dq_2}, \dots \frac{ds}{dq_n}\right)}{d(a_1, a_2, \dots a_n)}.$$

496.] This being premised, let

$$s = \psi(a_1, a_2, \dots a_n, q_1, q_2, \dots q_n, t)$$

be a complete solution of the equation (34). We have to prove that if $p_1, p_2, \dots, p_n$ be defined by the n equations $\frac{ds}{dq_i} = p_i$, and if the $2n$ variables $p_1, \dots p_n, q_1, \dots q_n$, be determined as functions of t, and constants, by those equations joined with the n further equations $\frac{ds}{da_i} = b_i$, then p_i, q_i will satisfy the differential equations (35).

Putting p_i for $\frac{ds}{dq_i}$ in (34), that equation becomes

$$\frac{ds}{dt} + f(p_1, p_2, \dots p_n, q_1, q_2, \dots q_n, t) = 0, \qquad (36)$$

which is identical on the supposition that $p_1, \dots p_n$ are expressed in the form $\frac{ds}{dq_1}, \dots \frac{ds}{dq_n}$. Differentiating the equation with respect to a_i on this supposition, and observing that

$$\frac{dp_1}{da_i} = \frac{d^2 s}{da_i\,dq_1}, \dots,$$

we have

$$\frac{d^2 s}{da_i\,dt} + \frac{df}{dp_1}\frac{d^2 s}{da_i\,dq_1} + \frac{df}{dp_2}\frac{d^2 s}{da_i\,dq_2} + \ldots + \frac{df}{dp_n}\frac{d^2 s}{da_i\,dq_n} = 0.$$

Again, differentiating the equation $\frac{ds}{da_i} = b_i$ totally, with respect to t, we find

$$\frac{d^2 s}{dt\,da_i} + \frac{d^2 s}{dq_1\,da_i}q_1' + \frac{d^2 s}{dq_2\,da_i}q_2' + \ldots + \frac{d^2 s}{dq_n\,da_i}q_n' = 0;$$

and subtracting the former equation from the latter,

$$\frac{d^2 s}{da_i\,dq_1}\left(q_1' - \frac{df}{dp_1}\right) + \frac{d^2 s}{da_i\,dq_2}\left(q_2' - \frac{df}{dp_2}\right) + \ldots + \frac{d^2 s}{da_i\,dq_n}\left(q_n' - \frac{df}{dp_n}\right) = 0.$$

Suppose the n equations obtained from this by giving i all its values $1, 2, \ldots n$, to be written down; then it is evident, from the theory of algebraic linear equations, that one of two things must be true; either the determinant formed with the n^2 coefficients $\frac{d^2 s}{da_i\,dq_j}$ must vanish, or else each of the n terms $q_i' - \frac{df}{dp_i} = 0$. But it was shewn in the last Article that the former supposition would imply that s was not a complete solution of (34). Hence the latter alternative is alone admissible; and since $f(p_1, \ldots p_n, q_1, \ldots q_n, t)$ is the same as H, we have $q_i' = \frac{dH}{dp_i}$; which is one part of the conclusion to be proved.

To establish the other part, differentiate the equation (36) on the same supposition as before, with respect to q_i, observing that $p_i = \frac{ds}{dq_i}$, and $\frac{dp_i}{dq_j} = \frac{dp_j}{dq_i}$; the result may then be written,

$$\frac{dp_i}{dt} + \frac{df}{dq_i} + \frac{df}{dp_1}\frac{dp_i}{dq_1} + \frac{df}{dp_2}\frac{dp_i}{dq_2} + \ldots + \frac{df}{dp_n}\frac{dp_i}{dq_n} = 0;$$

on the other hand, differentiating the equation $p_i = \frac{ds}{dq_i}$ totally with respect to t, we obtain

$$p_i' = \frac{dp_i}{dt} + \frac{dp_i}{dq_1}q_1' + \frac{dp_i}{dq_2}q_2' + \ldots + \frac{dp_i}{dq_n}q_n';$$

and combining these two equations,

$$p_i' + \frac{df}{dq_i} = \frac{dp_i}{dq_1}\left(q_1' - \frac{df}{dp_1}\right) + \ldots + \frac{dp_i}{dq_n}\left(q_n' - \frac{df}{dp_n}\right):$$

but it has just been shewn that every term on the right of this

equation vanishes; hence $p_i' + \frac{df}{dq_i} = 0$, or $p_i' = -\frac{d\text{H}}{dq_i}$, which completes the demonstration.

497.] The equations $\frac{d\text{S}}{dq_i} = p_i$, $\frac{d\text{S}}{da_i} = b_i$, give

$$\delta\text{S} = \Sigma(p_i\,\delta q_i) + \Sigma(b_i\,\delta a_i);$$

and if from this we form the expression for $\Delta\delta\text{S}$, and subtract from it the analogous expression for $\delta\Delta\text{S}$, equating the result to zero, see the process of Art. 489, we obtain the equation

$$\Sigma(\delta p_i\,\Delta q_i - \Delta p_i\,\delta q_i) = \Sigma(\delta a_i\,\Delta b_i - \Delta a_i\,\delta b_i);$$

and this being similar to equation (24), Art. 488, it follows, as in Art. 489, that $(a_i, b_i) = 1$, $(a_i, a_j) = (a_i, b_j) = (b_i, b_j) = 0$; so that $a_1, \ldots a_n, b_1, \ldots b_n$, are canonical elements.

498.] The formulæ established in Arts. 495, 496 may be somewhat modified in the case in which H does not contain t explicitly. We have seen, Art. 494, that in this case $h = \text{H}$ is one of the integrals of the problem; h being an arbitrary constant, called the "constant of vis viva."

Suppose then that

$$\text{H} = f(p_1, \ldots p_n, q_1, \ldots q_n);$$

and let V be a complete solution of the equation

$$f\left(\frac{d\text{V}}{dq_1}, \ldots \frac{d\text{V}}{dq_n}, q_1, \ldots q_n\right) = h; \tag{37}$$

that is, a solution containing, besides h, $n-1$ arbitrary constants $a_1, a_2, \ldots a_{n-1}$, in such a manner that these $n-1$ constants cannot be all eliminated without employing all the n differential coefficients $\frac{d\text{V}}{dq_1}, \ldots \frac{d\text{V}}{dq_n}$. Then, if we take

$$\text{S} = -ht + \text{V},$$

it is evident that S will be a solution of the equation

$$\frac{d\text{S}}{dt} + f\left(\frac{d\text{S}}{dq_1}, \ldots \frac{d\text{S}}{dq_n}, q_1, \ldots q_n\right) = 0;$$

for we have $\frac{d\text{S}}{dt} = -h$; and since $\frac{d\text{S}}{dq_i} = \frac{d\text{V}}{dq_i}$, the second term in the equation just written $= h$, by (37), so that the two terms destroy one another. Also S is a complete solution, for it is evident that h cannot be eliminated in addition to $a_1, \ldots a_{n-1}$, without employing the differential coefficient $\frac{d\text{S}}{dt}$.

If then the function V is known, the integral equations of the problem will be expressible as follows: since $\frac{d\text{S}}{dq_i} = \frac{d\text{V}}{dq_i}$, we shall have n equations $\frac{d\text{V}}{dq_i} = p_i$. Also, since S contains $a_1, \ldots a_{n-1}$ only as they are contained in V, $\frac{d\text{S}}{da_i} = \frac{d\text{V}}{da_i}$, so that there will be $n-1$ equations,

$$\frac{d\text{V}}{da_i} = b_i. \tag{38}$$

The remaining integral equation is $\frac{d\text{S}}{dh} = \text{constant}$. But

$$\frac{d\text{S}}{dh} = -t + \frac{d\text{V}}{dh},$$

and therefore, putting τ for the constant, we may write the equation

$$\frac{d\text{V}}{dh} = t + \tau. \tag{39}$$

In this way we should obtain a solution involving the canonical elements

$$\left.\begin{array}{l} h,\ a_1,\ a_2,\ \ldots\ a_{n-1}, \\ \tau,\ b_1,\ b_2,\ \ldots\ b_{n-1}. \end{array}\right\} \tag{40}$$

It is to be observed, that the only one of the integral equations containing t is the equation (39). And it is evident, that if the $2n$ integral equations were solved algebraically, so as to express each of the $2n$ elements (40) in terms of the variables, that is, so as to obtain $2n$ "integrals" properly so called, only one of these integrals would contain t, and would be of the form

$$\tau = -t + \phi(p_1, \ldots p_n, q_1, \ldots q_n).$$

Lastly, it is to be noticed, that whenever the constant of vis viva is one of a set of canonical elements, the element conjugate to it is the constant τ, which is added to t.

499.] Although the preceding theory assigns in a very remarkable manner certain forms in which the integral equations of the system (21) are capable of being expressed, yet it gives no assistance whatever towards the actual integration of that system. For the discovery of the function S, according to Sir W. R. Hamilton's definition of it, would require a knowledge of a complete set of integrals, and according to Jacobi's definition, would depend on the solution of a partial differential equation (34), which is a problem as difficult as the integration of the system (21) itself. In fact, the most hopeful way of attempt-

ing the integration of (34) would in general be to make it depend upon that of the system (21). See Boole's Differential Equations, Chapter XIV, Art. 14, and also page 475.

It will be seen however, from a theorem now to be demonstrated, that a knowledge of half the integrals of the system (21) will, when certain conditions are fulfilled, lead to the discovery of the function S, and therefore enable us to complete the integration.

500.] *Theorem**. Suppose $a_1, a_2, \ldots a_n$ to be n integrals of the system

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i}, \qquad (i = 1 \text{ to } i = n),$$

where H is a given function of $p_1, \ldots p_n, q_1, \ldots q_n, t$; then, if the $\frac{n(n-1)}{2}$ conditions $(a_i, a_j) = 0$ subsist, where i, j is any pair of the n indices, the remaining integrals may be found as follows: By means of the given n integrals, let $p_1, p_2, \ldots p_n$ be expressed in terms of $q_1, \ldots q_n, a_1, \ldots a_n, t$; and let these values be introduced in H, so that H will be expressed in the same way. Then the values of $p_1, p_2, \ldots p_n, -\text{H}$, will be the partial differential coefficients with respect to $q_1, q_2, \ldots q_n, t$, of a certain function of $a_1, \ldots a_n, q_1, \ldots q_n, t$; and calling this function S, we may find it by integrating the expression

$$d\text{S} = p_1 dq_1 + p_2 dq_2 + \ldots - \text{H}\,dt; \qquad (41)$$

of which the right-hand member is a perfect differential. And the remaining integral equations will be

$$\frac{d\text{S}}{da_1} = b_1, \ldots, \qquad \frac{d\text{S}}{da_n} = b_n; \qquad (42)$$

where $b_1, \ldots b_n$ are n new arbitrary constants.

501.] The proof of this theorem consists of two parts. First, we have to shew that the above expression for dS is a perfect

* When this theorem was given by the writer as new, in the Phil. Trans. for 1854, p. 85, he had no means of knowing that it had been previously discovered by M. Liouville, and communicated to the Bureau des Longitudes in 1853; for no accessible notice of this communication appears to have been printed before 1855 (in Liouville's Journal). As M. Liouville has referred to the question of priority, it seemed necessary to mention the subject here. See note to p. 31 of Mr. Cayley's Report, in which for "second part" read "first part."

differential. Putting e, f for any two of the integrals $a_1, \ldots a_n$, suppose

$$e = \phi(p_1, p_2, \ldots p_n, q_1, q_2, \ldots q_n, t).$$

If in this equation the values of $p_1, \ldots p_n$ were expressed, as above supposed, in terms of $q_1, \ldots, a_1 \ldots$, it would become identical. Differentiating it therefore on this hypothesis with respect to q_i, we have

$$0 = \frac{de}{dq_i} + \frac{de}{dp_1}\frac{dp_1}{dq_i} + \ldots + \frac{de}{dp_n}\frac{dp_n}{dq_i};$$

and in like manner

$$0 = \frac{df}{dq_i} + \frac{df}{dp_1}\frac{dp_1}{dq_i} + \ldots + \frac{df}{dp_n}\frac{dp_n}{dq_i}:$$

and if we multiply the first of these equations by $\frac{df}{dp_i}$, and the second by $\frac{de}{dp_i}$, and subtract, there results an equation which may be written thus:

$$\frac{de}{dp_i}\frac{df}{dq_i} - \frac{de}{dq_i}\frac{df}{dp_i} = \Sigma_j\left\{\frac{dp_j}{dq_i}\left(\frac{de}{dp_j}\frac{df}{dp_i} - \frac{de}{dp_i}\frac{df}{dp_j}\right)\right\};$$

or, writing a_α, a_β, instead of e, f, and employing the notation of Art. 466,

$$\frac{d(a_\alpha, a_\beta)}{d(p_i, q_i)} = \Sigma_j\left\{\frac{dp_j}{dq_i}\frac{d(a_\alpha, a_\beta)}{d(p_j, p_i)}\right\}.$$

If now the terms on each side be summed with respect to i, the result on the left is (a_α, a_β); and observing that on the right the term multiplied by $\frac{dp_i}{dq_j}$ will only differ in sign from that multiplied by $\frac{dp_j}{dq_i}$, we obtain

$$(a_\alpha, a_\beta) = \Sigma\left\{\left(\frac{dp_j}{dq_i} - \frac{dp_i}{dq_j}\right)\frac{d(a_\alpha, a_\beta)}{d(p_j, p_i)}\right\};$$

the summation referring both to i and j, and extending to every binary combination.

From this equation it appears that in order that the condition $\frac{dp_j}{dq_i} = \frac{dp_i}{dq_j}$ may subsist for every pair of indices i, j, it is necessary that the condition $(a_\alpha, a_\beta) = 0$ shall be satisfied for every pair of the n integrals $a_1, \ldots a_n$. We ought in strictness to prove that no other conditions are necessary. As however the rigorous * proof of this involves certain preliminary theorems, which

* It might seem sufficient to say that the number of equations $\frac{dp_j}{dq_i} = \frac{dp_i}{dq_j}$ is the same as that of the equations $(a_i, a_j) = 0$, viz. $\frac{1}{2}n(n-1)$,

have not been given in this Chapter, it is here omitted; but it will be found in the Philosophical Transactions for 1854, pp. 84, 85.

Assuming then that $\frac{dp_j}{dq_i} = \frac{dp_i}{dq_j}$, in order to prove that the expression for $d\,\text{S}$, (41), is a perfect differential, it is necessary also to shew that $\frac{dp_i}{dt} = -\frac{d\,\text{H}}{dq_i}$. Here H is supposed to be expressed, like p_i, in terms of $a_1, \dots a_n, q_1, \dots q_n, t$. Putting for a moment (H) to denote H in its original form, namely, a function of $p_1, \dots p_n, q_1, \dots q_n, t$, we have

$$\frac{d\,\text{H}}{dq_i} = \frac{d\,(\text{H})}{dq_i} + \frac{d\,(\text{H})}{dp_1}\frac{dp_1}{dq_i} + \dots + \frac{d\,(\text{H})}{dp_n}\frac{dp_n}{dq_i};$$

now $$\frac{d\,(\text{H})}{dq_i} = -p_i', \quad \frac{d\,(\text{H})}{dp_1} = q_1' \dots, \quad \frac{dp_1}{dq_i} = \frac{dp_i}{dq_1}, \dots;$$

hence this equation becomes

$$\frac{d\,\text{H}}{dq_i} = -p_i' + q_1'\frac{dp_i}{dq_1} + \dots + q_n'\frac{dp_i}{dq_n};$$

but p_i being expressed as above supposed, we have

$$p_i' = \frac{dp_i}{dt} + \frac{dp_i}{dq_1}q_1' + \dots + \frac{dp_i}{dq_n}q_n';$$

whence the above value of $\frac{d\,\text{H}}{dq_i}$ becomes simply

$$\frac{d\,\text{H}}{dq_i} = -\frac{dp_i}{dt};$$

which is the condition that was to be established, and completes the proof that $p_1 dq_1 + \dots - \text{H}\,dt$ is a perfect differential.

Secondly, we have to shew that $\frac{d\,\text{S}}{da_i}$ is constant by virtue of the differential equations, or that $\left(\frac{d\,\text{S}}{da_i}\right)' = 0$.

Now*

$$\left(\frac{d\,\text{S}}{da_i}\right)' = \frac{d^2\,\text{S}}{dt\,da_i} + \frac{d^2\,\text{S}}{dq_1\,da_i}q_1' + \dots + \frac{d^2\,\text{S}}{dq_n\,da_i}q_n';$$

and that therefore the former set cannot imply any conditions not involved in the latter; but this reasoning is not absolutely conclusive.

* The reader will not understand this reasoning unless he be careful to recollect the way in which H, (H), S, $p_1 \dots p_n$, are supposed to be expressed.

and $\frac{ds}{dt} = -\text{H}$, whence $\frac{d^2s}{dt\,da_i} = -\frac{d\text{H}}{da_i}$; also $\frac{ds}{dq_1} = p_1$, whence $\frac{d^2s}{dq_1 da_i} = \frac{dp_1}{da_i}$; and $q_1' = \frac{d(\text{H})}{dp_1}, \ldots$; so that we obtain

$$\left(\frac{ds}{da_i}\right)' = -\frac{d\text{H}}{da_i} + \frac{d(\text{H})}{dp_1}\frac{dp_1}{da_i} + \ldots + \frac{d(\text{H})}{dp_n}\frac{dp_n}{da_i};$$

but from the relation between H and (H), we have

$$\frac{d\text{H}}{da_i} = \frac{d(\text{H})}{dp_1}\frac{dp_1}{da_i} + \ldots + \frac{d(\text{H})}{dp_n}\frac{dp_n}{da_i},$$

for H only contains a_i because it is introduced in the values of $p_1, p_2, \ldots p_n$. Hence the above equation becomes $\left(\frac{ds}{da_i}\right)' = 0$, or $\frac{ds}{da_i}$ = constant; which was to be proved.

502.] The formulæ (41), (42), Art. 500, admit of modification in the case in which the so-called "principle of vis viva" holds good; namely, that in which H, in its original form, does not contain t explicitly, so that $\text{H} = h$ is an integral.

In this case, suppose $h, a_1, a_2, \ldots a_{n-1}$ to be n integrals, satisfying the conditions $(h, a_i) = 0$, $(a_i, a_j) = 0$ for every pair. It is evident that when the n equations, expressing the values of h, $a_1, \ldots$, in terms of the variables, are solved for $p_1, \ldots p_n$ and the values of these latter quantities substituted in H, the result will be identically the constant h, so that the value of ds (41) will become
$$ds = -h\,dt + p_1 dq_1 + \ldots + p_n dq_n,$$
where $p_1, \ldots p_n$ cannot contain t, otherwise this expression would not be a perfect differential. Hence we shall have

$$s = -ht + \text{v};$$

where v is a function of $q_1, \ldots q_n, h, a_1, \ldots a_{n-1}$, not containing t, and given by the equation

$$d\text{v} = p_1 dq_1 + \ldots + p_n dq_n; \tag{43}$$

of which the right-hand member is a perfect differential.

We shall now have

$$\frac{ds}{dh} = -t + \frac{d\text{v}}{dh}, \qquad \frac{ds}{da_1} = \frac{d\text{v}}{da_1}, \ldots;$$

so that if we put τ for the constant conjugate to h, the equations corresponding to (42) may be written

$$\frac{d\text{v}}{dh} = t + \tau, \qquad \frac{d\text{v}}{da_1} = b_1, \ldots, \qquad \frac{d\text{v}}{da_{n-1}} = b_{n-1}. \tag{44}$$

503.] It is useful to observe, that if c be any integral whatever, not containing t explicitly, and h the integral of vis viva, the condition $(h, c) = 0$ necessarily subsists. For we have, see the general expression for u', Art. 490,

$$0 = c' = \frac{dc}{dt} + (\text{H}, c);$$

but $\frac{dc}{dt} = 0$ by hypothesis, and (H, c) is the same as (h, c); consequently $(h, c) = 0$.

From this proposition, combined with the results of the last Article, it is evident, that when the independent coordinates in any dynamical problem in which the principle of vis viva subsists are only two in number, if, besides the integral of vis viva, one other integral, not containing t explicitly, be given, the discovery of the two remaining integrals is reducible to quadratures. For V is given by the equation $d\text{V} = p_1 dq_1 + p_2 dq_2$, of which the right-hand member is necessarily a perfect differential. The reader will find it a good exercise to treat the problem of the simple conical pendulum in this manner, omitting the effect of the earth's rotation.

504.] In the practical use of the theorem explained in Arts. 500—502, the following circumstances should be attended to. Instead of first finding the function S by integrating the expression $p_1 dq_1 + p_2 dq_2 + \ldots - \text{H}\, dt$, and then forming the integral equations $\frac{d\text{S}}{da_1} = b_1, \ldots$, it is generally more convenient to perform the differentiations with respect to $a_1, \ldots$, first, and the integrations afterwards. Thus we should have

$$b_i = \int \left\{ \frac{dp_1}{da_i} dq_1 + \frac{dp_2}{da_i} dq_2 + \ldots - \frac{d\text{H}}{da_i} dt \right\};$$

but the solution thus obtained will not be canonical, unless care be taken so to assume the inferior limits of the integrals that the functions equated to $b_1, b_2, \ldots$, shall be, as they ought to be, the differential coefficients of one and the same function of $a_1, \ldots a_n$, $q_1, q_2, \ldots$.

To shew that this condition will not necessarily subsist, we may take the case which most frequently occurs, namely, that in which the value of p_i contains none of the variables except q_i, so that each term in the value of $d\text{S}$ will be, omitting indices, of

the form $\phi(q, a_1, a_2, \ldots a_n)\, dq$, and so far as one such term is concerned, we should have

$$S = \int_{A}^{q} \phi(q, a_1, a_2, \ldots a_n)\, dq,$$

where A is an arbitrary function of $a_1, \ldots a_n$. Differentiating this with respect to a_i, we obtain

$$\frac{dS}{da_i} = \int_{A}^{q} \frac{d\phi}{da_i}\, dq - \phi(A, a_1, \ldots a_n) \frac{dA}{da_i};$$

that is,

$$\int_{A}^{q} \frac{d\phi}{da_i}\, dq = \frac{d}{da_i} \int_{A}^{q} \phi . dq + \phi(A, a_1, \ldots a_n) \frac{dA}{da_i}:$$

from which it is evident that differentiating first and integrating afterwards will not in general give the same result as the converse process.

We see however that the values of

$$\int_{A}^{q} \frac{d\phi}{da_1}\, dq, \qquad \int_{A}^{q} \frac{d\phi}{da_2}\, dq, \ldots$$

will be the differential coefficients with respect to $a_1, a_2, \ldots$ of $\int_{A}^{q} \phi . dq$, provided that A, which must be the same in each of these integrals, be so assumed, that

$$\phi(A, a_1, \ldots a_n) \frac{dA}{da_i} = 0$$

for every value of i. This condition may be secured in either of two ways: first, by taking $A = 0$, or $A =$ any determinate constant; that is, not involving $a_1, \ldots$; or secondly, by taking for A any root of the equation

$$\phi(x, a_1, \ldots a_n) = 0;$$

or, which is the same thing, a value of q which would make p vanish.

505.] As an illustration of the principal formulæ hitherto established, we will take the important problem of the motion of a single particle about a fixed centre of force, in treating which our object will be to obtain a set of canonical elements. The advantage of this form of solution will appear afterwards.

Let m be the mass of the particle, x, y, z its rectangular coordinates at the time t, referred to fixed axes having their origin at the centre of force.

Then if r, θ, λ be polar coordinates, such that

$$x = r \cos\lambda \cos\theta, \qquad y = r \cos\lambda \sin\theta, \qquad z = r \sin\lambda,$$

r is the radius vector; and we may call θ the longitude, reckoned from the axis of x, in the plane of (x, y); and λ the latitude, or the angle between r and the plane of (x, y).

In rectangular coordinates $2\text{T} = m(x'^2 + y'^2 + z'^2)$; but if x', y', z' be expressed in terms of $r, r', \ldots$, it will be found that

$$2\text{T} = m\{r'^2 + r^2(\cos\lambda)^2\theta'^2 + r^2\lambda'^2\}.$$

We shall take r, θ, λ as the coordinates of the problem, corresponding to q_1, q_2, q_3, in the general formulæ. Let the variables conjugate to them, corresponding to p_1, p_2, p_3, be denoted, for greater clearness, by $r_{,}, \theta_{,}, \lambda_{,}$; so that we shall have

$$r_{,} = \frac{d\text{T}}{dr'} = mr', \quad \theta_{,} = \frac{d\text{T}}{d\theta'} = mr^2(\cos\lambda)^2\theta', \quad \lambda_{,} = \frac{d\text{T}}{d\lambda'} = mr^2\lambda';$$

and if the values of r', θ', λ' in terms of $r_{,}, \theta_{,}, \lambda_{,}$, given by these equations, be substituted in T, the result is

$$\text{T} = \frac{1}{2m}\left\{r_{,}^2 + \frac{(\sec\lambda)^2}{r^2}\theta_{,}^2 + \frac{\lambda_{,}^2}{r^2}\right\}. \tag{45}$$

Since the force is central, the force-function U is a function of r, say $\text{U} = \phi(r)$; so that we shall have

$$\text{H} = \text{T} - \phi(r);$$

in which T stands for the expression (45) just given. See Art. 481. The six differential equations of the problem in the Hamiltonian form would then be

$$r_{,}' = -\frac{d\text{H}}{dr}, \quad \theta_{,}' = -\frac{d\text{H}}{d\theta}, \quad \lambda_{,}' = -\frac{d\text{H}}{d\lambda},$$

$$r' = \frac{d\text{H}}{dr_{,}}, \quad \theta' = \frac{d\text{H}}{d\theta_{,}}, \quad \lambda' = \frac{d\text{H}}{d\lambda_{,}};$$

but they are not required for our present purpose.

506.] Since H does not contain t explicitly, the integral of vis viva subsists; namely,

$$h = \text{T} - \phi(r).$$

We have now to find two other integrals, say c and k, such that $(h, c) = 0$, $(h, k) = 0$, $(c, k) = 0$. We know that the first two of these conditions will be satisfied if c, k be any integrals not containing t explicitly. Let us take then for c one of the three integrals expressing the conservation of areas; namely, $c = m(xy' - x'y)$; and for k, the equation obtained from the three by squaring and adding, namely,

$$k = m\{(yz' - zy')^2 + (zx' - xz')^2 + (xy' - yx')^2\}^{\frac{1}{2}}:$$

these however must be expressed in terms of the new variables; thus,

$$c = mr^2 (\cos\lambda)^2 \theta' = \theta_,,$$

$$k = m\{r^2(x'^2+y'^2+z'^2) - r^2 r'^2\}^{\frac{1}{2}} = \{\theta_,^2 (\sec\lambda)^2 + \lambda_,^2\}^{\frac{1}{2}}.$$

Thus we have the three integrals

$$h = \frac{1}{2m}\left\{r_,^2 + \frac{(\sec\lambda)^2}{r^2}\theta_,^2 + \frac{1}{r^2}\lambda_,^2\right\} - \phi(r), \qquad \text{(i)}$$

$$c = \theta_,, \qquad \text{(ii)}$$

$$k = \{(\sec\lambda)^2\theta_,^2 + \lambda_,^2\}^{\frac{1}{2}}; \qquad \text{(iii)}$$

and if it be recollected that the meaning of (u, v) is now

$$\frac{d(u,v)}{d(r_,, r)} + \frac{d(u,v)}{d(\theta_,, \theta)} + \frac{d(u,v)}{d(\lambda_,, \lambda)},$$

it will be found on trial, not only that $(h, c) = 0$, $(h, k) = 0$, as we know a priori, but also that $(c, k) = 0$.

Hence we may apply the method of Art. 502 to the equations (i), (ii), (iii); that is, we may find v from the equation

$$d\mathrm{v} = r_, dr + \theta_, d\theta + \lambda_, d\lambda,$$

in which $r_,$, $\theta_,$, $\lambda_,$ are to be expressed in terms of r, θ, λ, h, c, k, by means of (i), (ii), (iii).

It will be found without difficulty that

$$r_, = \left[2m\{h + \phi(r)\} - \frac{k^2}{r^2}\right]^{\frac{1}{2}},$$

$$\theta_, = c,$$

$$\lambda_, = \{k^2 - c^2(\sec\lambda)^2\}^{\frac{1}{2}};$$

and it is obvious that $r_, dr + \theta_, d\theta + \lambda_, d\lambda$ is, as it ought to be, a perfect differential.

Supposing v to be found by integration, the three remaining integral equations would be, Art. 502,

$$\frac{d\mathrm{v}}{dh} = t + \tau, \qquad \frac{d\mathrm{v}}{dc} = \alpha, \qquad \frac{d\mathrm{v}}{dk} = \beta;$$

where τ, α, β are three new constants.

The conclusions of Art. 504, however, shew, that we may perform the differentiations before the integrations, if the inferior limits of the latter be properly chosen; and in accordance with the rule there laid down we may take 0 as the limit for θ and λ, and a root of the equation $2m\{h + \phi(r)\} - \frac{k^2}{r^2} = 0$, say ρ, as the limit for r. This being understood, we should have

$$\mathrm{v} = \int_\rho^r \left\{2m(h + \phi(r)) - \frac{k^2}{r^2}\right\}^{\frac{1}{2}} dr + c\theta + \int_0^\lambda \{k^2 - c^2(\sec\lambda)^2\}^{\frac{1}{2}} d\lambda;$$

and the three new integral equations will be obtained by differentiating under the signs of integration, and integrating afterwards. The integrations with respect to r cannot be performed till $\phi(r)$ is assigned, and in fact will not be actually needed; those with respect to λ are left to the reader. The results are

$$m\int_{\rho}^{r} r\,\{2mr^2(h+\phi(r))-k^2\}^{-\frac{1}{2}}dr = t+\tau, \qquad \text{(iv)}$$

$$\theta-\sin^{-1}\frac{c\tan\lambda}{(k^2-c^2)^{\frac{1}{2}}} = a, \qquad \text{(v)}$$

$$-k\int_{\rho}^{r}\frac{1}{r}\{2mr^2(h+\phi(r))-k^2\}^{-\frac{1}{2}}dr+\sin^{-1}\frac{k\sin\lambda}{(k^2-c^2)^{\frac{1}{2}}} = \beta. \qquad \text{(vi)}$$

The equations (i), (ii), (iii), (iv), (v), (vi) comprise a canonical solution of the problem.

507.] It is easy to interpret the constants τ, a, β.

The equation (iv) shews that $t+\tau = 0$ when $r = \rho$, so that $-\tau$ is the time at which $r = \rho$. Now if we differentiate the equation with respect to t, we obtain

$$mr' = \left\{2m(h+\phi(r))-\frac{k^2}{r^2}\right\}^{\frac{1}{2}};$$

and we took for ρ a value of r, which makes this expression vanish; hence, when $r = \rho$, $r' = 0$, and it is evident therefore that we may suppose ρ to be the minimum value of r. Thus $-\tau$ will be the time at which m is at the least distance from the centre of force.

Again, if we put ι for the inclination of the plane of the motion to the plane of (x, y), we have, by well-known principles, $c = k\cos\iota$; hence

$$\frac{c}{(k^2-c^2)^{\frac{1}{2}}} = \cot\iota, \qquad \frac{k}{(k^2-c^2)^{\frac{1}{2}}} = \frac{1}{\sin\iota};$$

equation (v) is therefore equivalent to

$$\sin(\theta-a) = \tan\lambda\cot\iota. \qquad (46)$$

Next let ϑ be the "argument of latitude;" that is, the angle between r and the ascending node, or line joining m with the centre at the instant when m passes through the plane of (x, y) from the negative to the positive side; considering then the right-angled spherical triangle, formed by the intersections with a sphere of the plane of (x, y), and the planes of the angles ϑ and λ, in which ϑ is the hypothenuse and ι the angle opposite to λ, we see, from (46), that $\theta-a$ is the side which is in the plane of

(x, y), and therefore a is evidently the longitude of the ascending node.

Lastly, equation (vi) shews that when $r = \rho$, $\sin\lambda = \sin\iota . \sin\beta$; but from the triangle above mentioned we have also $\sin\lambda = \sin\iota . \sin\vartheta$; hence we see that when $r = \rho$, $\vartheta = \beta$, from which it is evident that β is the angle between the least distance and the node.

508.] We shall briefly indicate the application of these results to the case of the undisturbed motion of a planet about the sun. Let μ = mass of sun + mass of planet; then, m being the mass of the planet, we shall have for the force-function $\phi(r) = \frac{m\mu}{r}$; also, from the ordinary theory of elliptic motion, we have

$$h = -\frac{m\mu}{2a}, \qquad k = m\{\mu a(1-e^2)\}^{\frac{1}{2}}, \qquad c = m\{\mu a(1-e^2)\}^{\frac{1}{2}}\cos\iota.$$

Hence we have the following set of canonical elements, arranged in conjugate pairs. The signs of the first pair have been changed, which is obviously allowable, and p is put for $a(1-e^2)$, that is, for the semi-latus-rectum;

$\frac{m\mu}{2a}$, time of perihelion passage;

$m(\mu p)^{\frac{1}{2}}\cos\iota$, longitude of node;

$m(\mu p)^{\frac{1}{2}}$, distance of perihelion from node.

These elements were first given by Jacobi.

509.] Variation of elements. Returning to the general form of the differential equations,

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i}, \qquad (i = 1, \text{ to } i = n),$$

let us suppose that a complete set of integrals $c_1, c_2, \ldots c_{2n}$ has been obtained, where $c_1, c_2, \ldots$ represent certain functions of the variables $p_1, q_1, \ldots$, and t, which, by virtue of the differential equations, have constant values; and conversely, $p_1, p_2, \ldots$ are given functions of $c_1, c_2, \ldots$, and t. Now let $c_1, \ldots, c_{2n}$ continue to represent the same functions of the variables and t, but let the variables be determined as functions of t by a different set of differential equations, say,

$$p_i' = -\frac{d\text{Q}}{dq_i}, \qquad q_i' = \frac{d\text{Q}}{dp_i}; \tag{47}$$

where Q is, like H, a given function of $p_1, \ldots p_n, q_1, \ldots q_n, t$,

but not the same function as H. Then the functions $c_1, \dots c_{2n}$ will no longer have constant values. In other words, the elements $c_1, c_2, \dots$, which were constant in the former problem, are variable in the latter problem.

The theory of the variation of elements has for its principal object to transform the equations (47) by taking as new variables the functions $c_1, \dots c_{2n}$, instead of the original variables $p_1, \dots p_n$, $q_1, \dots q_n$.

The integration of the equations after this transformation would give $c_1, \dots c_{2n}$ as functions of t and $2n$ constants; and the variables $p_1, p_2, \dots$ would then be known as functions of t, because they are the same functions of the elements $c_1, c_2, \dots$ and t, as they were in the original problem.

In practice, it is usually convenient to put $\text{Q} = \text{H} + \Omega$, where H is the same as before, and Ω is a given function of $p_1, \dots p_n$, $q_1, \dots q_n$, t, and is called technically the "disturbing function." Thus the equations (47) become

$$p_i' = -\frac{d\text{H}}{dq_i} - \frac{d\Omega}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i} + \frac{d\Omega}{dp_i};$$

and these are transformed as follows: we have

$$c_1' = \frac{dc_1}{dt} + \Sigma\left(\frac{dc_1}{dp_i}p_i' + \frac{dc_1}{dq_i}q_i'\right);$$

that is, introducing the above values of p_i', q_i',

$$c_1' = \frac{dc_1}{dt} + (\text{H}, c_1) + (\Omega, c_1):$$

now c_1 is by hypothesis a function of $p_1, \dots$ and t, which in the "undisturbed" problem, that is, when $\Omega = 0$, is constant; and this constancy would be expressed by the equation

$$\frac{dc_1}{dt} + (\text{H}, c_1) = 0,$$

which is identically true, all its terms being functions of $p_1, \dots,$ $q_1, \dots, t$. But since c_1 and H are the same functions of $p_1, \dots,$ $q_1, \dots, t$, in the disturbed, as they were in the undisturbed problem, the above equation is still identically true, and the expression for c_1' therefore becomes simply

$$c_1' = (\Omega, c_1):$$

here Ω is expressed in its original form as a function of $p_1, \dots,$ $q_1, \dots, t$; but if we suppose $p_1, \dots, q_1, \dots$ to be expressed in terms of $c_1, \dots c_{2n}, t$, Ω will become a function of the latter

quantities, and denoting, for a moment, the new form of Ω by $\bar{\Omega}$, we shall have, by the elementary properties of the symbol (u, v), Art. 466,

$$c_1' = (\Omega, c_1) = \frac{d\bar{\Omega}}{dc_1}(c_1, c_1) + \frac{d\bar{\Omega}}{dc_2}(c_2, c_1) + \ldots + \frac{d\bar{\Omega}}{dc_{2n}}(c_{2n}, c_1);$$

in which the first term vanishes, since $(c_1, c_1) = 0$.

The general form is evidently, omitting the line over Ω,

$$c_i' = \frac{d\Omega}{dc_1}(c_1, c_i) + \frac{d\Omega}{dc_2}(c_2, c_i) + \ldots + \frac{d\Omega}{dc_{2n}}(c_{2n}, c_i), \qquad (48)$$

involving $2n-1$ terms on the right-hand side. And by giving to i the values $1, 2, \ldots 2n$, we obtain a set of $2n$ differential equations, which will involve only the new variables $c_1, c_2, \ldots$ instead of $p_1, p_2, \ldots$, when the terms $(c_1, c_i) \ldots$, which are given functions of the old variables, are expressed in terms of the new. It follows, from what was proved in Art. 490, that when so expressed they will not contain t.

510.] But the formula (48) undergoes a remarkable simplification when $c_1, \ldots c_{2n}$, are a set of canonical elements; for in that case all the terms (c_1, c_i), (c_2, c_i), ... vanish except one, namely, that in which c_i is combined with its conjugate element, and then the term is either $+1$ or -1.

Suppose then that $a_1, \ldots a_n, b_1, \ldots b_n$ are canonical elements, so that $(a_i, b_i) = 1$, $(b_i, a_i) = -1$, and all other combinations give zero. The formula (48) will be seen at once to give

$$a_i' = -\frac{d\Omega}{db_i}, \qquad b_i' = \frac{d\Omega}{da_i}, \qquad (49)$$

which are the transformed equations in this case, and are of the same general form as the original equations.

These formulæ were discovered in a particular case by Lagrange; namely, that in which $a_1, b_1, \ldots$ are the initial values of $p_1, q_1, \ldots$. The extension to canonical elements in general is due partly to Sir W. R. Hamilton and partly to Jacobi.

There are many points of interest and importance connected with the further development and application of the theory of the variation of elements, but we cannot afford space for them here.

511.] We shall conclude this Chapter by a brief notice of an addition to the general theory of the Hamiltonian equations

made by M. Bour, and improved in accordance with a suggestion of M. Liouville*.

Let $a_1, a_2, \dots a_n, b_1, b_2, \dots b_n$, be, as before, a set of canonical integrals of the equations

$$p_i' = -\frac{d\text{H}}{dq_i}, \qquad q_i' = \frac{d\text{H}}{dp_i}. \qquad (50)$$

If we represent any integral whatever by ζ, the equation $\zeta' = 0$ gives $\frac{d\zeta}{dt} + (\text{H}, \zeta) = 0$; which, written in full, would be

$$\frac{d\zeta}{dt} + \frac{d\text{H}}{dp_1}\frac{d\zeta}{dq_1} - \frac{d\text{H}}{dq_1}\frac{d\zeta}{dp_1} + \dots + \frac{d\text{H}}{dp_n}\frac{d\zeta}{dq_n} - \frac{d\text{H}}{dq_n}\frac{d\zeta}{dp_n} = 0. \quad (51)$$

In this equation $\frac{d\text{H}}{dp_1}, \frac{d\text{H}}{dq_1}, \dots$ are given functions of $p_1, \dots p_n, q_1, \dots q_n, t$; so that with reference to ζ it is a linear partial differential equation of the first order, of which any solution whatever is an integral of the equations (50), and of which the general solution might be put in the form

$$\zeta = f(a_1, \dots a_n, b_1, \dots b_n),$$

where f is an arbitrary function.

Thus the complete integration of the equations (50) would be effected if we could find the general solution of (51). But this transformation of the problem is practically useless, since the only known general method of treating the equation (51) requires the integration of the system (50), as the reader will see at once on applying Lagrange's method to the former.

The proposition we are about to demonstrate is the following:

Suppose m of the n integrals $a_1, \dots a_n$, to be known; say $a_1, a_2, \dots a_m$ where m is of course less than n; and the condition $(a_i, a_j) = 0$ to subsist for every pair; then by means of these m integrals, m of the variables, say $p_1, p_2, \dots p_m$ might be expressed in terms of

$$a_1, a_2, \dots a_m, p_{m+1}, \dots p_n, q_1, \dots q_n, t; \qquad (52)$$

* See Liouville's Journal, t. xx. pp. 185–200, 201, 202. The writer is not aware whether M. Bour's investigations have yet been published complete in the Memoirs of the Academy. He believes that the process and results in the text must be substantially what those of M. Bour become when H contains t, which is the generalization proposed by M. Liouville; but he can only speak from conjecture on this point.

[The Memoir of M. Bour, to which the note refers, is published in Tome XIV of the Mémoires des Savants Etrangers, p. 792; having been presented to the Academy of Sciences at Paris on March 5th, 1855.]

and by the substitution of these values of $p_1, \dots p_m$, H and ζ might be expressed in the same way: we shall prove that after this substitution the equation (51) would still subsist, the differentiations being all performed only so far as the variables would appear explicitly. But since H and ζ would no longer contain $p_1, \dots p_m$ explicitly, all the terms in (51) which involve differentiation with respect to these m variables would disappear; and the equation would become

$$\frac{d\zeta}{dt}+\frac{d\text{H}}{dp_{m+1}}\frac{d\zeta}{dq_{m+1}}-\frac{d\text{H}}{dq_{m+1}}\frac{d\zeta}{dp_{m+1}}+\dots+\frac{d\text{H}}{dp_n}\frac{d\zeta}{dq_n}-\frac{d\text{H}}{dq_n}\frac{d\zeta}{dp_n}=0; \quad (53)$$

in which the number of terms is diminished by $2m$.

In this equation H is a given function of the quantities (52); but as it does not involve $\frac{d\zeta}{dq_1}, \dots \frac{d\zeta}{dq_m}$, the variables $q_1, \dots,$ would be treated as constants in integration. Any solution of it will be a function of the same quantities (52), and will be an integral of (50); but the number of distinct solutions will be less by $2m$ than those of (51). In fact (53) will be satisfied by $a_{m+1}, \dots a_n, b_{m+1}, \dots b_n$, but not by $a_1, \dots a_m, b_1, \dots b_m$; and it follows that any solution c of (53) will satisfy the conditions $(c, a_1) = 0$, $(c, a_m) = 0$, but any two solutions, c, e, will not necessarily satisfy the condition $(c, e) = 0$.

512.] We proceed to demonstrate what has been stated in the last Article. Let H, when transformed by substituting for $p_1, \dots p_m$ the values obtained from the given integrals $a_1, \dots a_m$ in terms of the quantities (52), be denoted by $\overline{\text{H}}$. The equation (51) is

$$\frac{d\zeta}{dt}+(\text{H}, \zeta) = 0;$$

but by the elementary property of the symbol (u, v), before referred to, we have

$$(\text{H}, \zeta) = (\overline{\text{H}}, \zeta)+\frac{d\overline{\text{H}}}{da_1}(a_1, \zeta)+\dots+\frac{d\overline{\text{H}}}{da_m}(a_m, \zeta);$$

in which the expression $(\overline{\text{H}}, \zeta)$ is to be formed by differentiating $\overline{\text{H}}$ with respect to the variables only so far as they appear explicitly. Thus the equation (51) becomes

$$\frac{d\zeta}{dt}+(\overline{\text{H}}, \zeta)+\frac{d\overline{\text{H}}}{da_1}(a_1, \zeta)+\dots+\frac{d\overline{\text{H}}}{da_m}(a_m, \zeta) = 0;$$

and this would still be satisfied by putting $\zeta =$ any integral of (50). But if ζ be any one of the integrals $a_1, \dots a_n, b_{m+1} \dots b_n$,

we shall have $(a_1, \zeta) = 0$, $(a_2, \zeta) = 0, \ldots (a_m, \zeta) = 0$, whereas one of these terms would be unity if ζ were one of the integrals $b_1, \ldots b_m$. Consequently the equation

$$\frac{d\zeta}{dt} + (\text{H}, \zeta) = 0 \tag{54}$$

will be satisfied by $a_1, \ldots a_n, b_{m+1}, \ldots b_n$, but not by $b_1, \ldots b_m$.

In this equation however ζ is still supposed to be expressed in terms of all the variables $p_1, \ldots, q_1, \ldots$, and t. But if we suppose ζ to be transformed in the same way as H, and then to be denoted by $\bar{\zeta}$ we shall have

$$\frac{d\zeta}{dt} = \frac{d\bar{\zeta}}{dt} + \frac{d\bar{\zeta}}{da_1}\frac{da_1}{dt} + \ldots + \frac{d\bar{\zeta}}{da_m}\frac{da_m}{dt},$$

$$(\bar{\text{H}}, \zeta) = (\bar{\text{H}}, \zeta) + \frac{d\bar{\zeta}}{da_1}(\bar{\text{H}}, a_1) + \ldots + \frac{d\bar{\zeta}}{da_m}(\bar{\text{H}}, a_m);$$

so that the equation (54) becomes

$$\frac{d\bar{\zeta}}{dt} + (\bar{\text{H}}, \bar{\zeta}) + \frac{d\bar{\zeta}}{da_1}\left\{\frac{da_1}{dt} + (\bar{\text{H}}, a_1)\right\} + \ldots + \frac{d\bar{\zeta}}{da_m}\left\{\frac{da_m}{dt} + (\bar{\text{H}}, a_m)\right\} = 0,$$

in which all the terms after the first two vanish; for, since a_1 is an integral, we have $a_1' = 0$, that is, $\frac{da_1}{dt} + (\text{H}, a_1) = 0$; but

$$(\text{H}, a_1) = (\bar{\text{H}}, a_1) + \frac{d\bar{\text{H}}}{da_1}(a_1, a_1) + \frac{d\bar{\text{H}}}{da_2}(a_2, a_1) + \ldots + \frac{d\bar{\text{H}}}{da_m}(a_m, a_1),$$

and $(a_1, a_1) = 0$, $(a_2, a_1) = 0, \ldots (a_m, a_1) = 0$, so that $(\text{H}, a_1) = (\bar{\text{H}}, a_1)$; and therefore $\frac{da_1}{dt} + (\bar{\text{H}}, a_1) = 0$; and the same reasoning applies to the other terms; finally, therefore, the equation becomes

$$\frac{d\bar{\zeta}}{dt} + (\bar{\text{H}}, \bar{\zeta}) = 0; \tag{55}$$

and this will be satisfied by the same integrals as (54) with the following exceptions. Suppose the integral a_i to be

$$a_i = \phi(p_1, \ldots p_n, q_1, \ldots q_n, t); \tag{56}$$

then (54) is satisfied by putting for ζ the function ϕ on the right of this equation. And if this function be transformed by putting for $p_1, \ldots p_m$ their values in terms of the quantities (52), a_i will in general be expressed as a function of the same quantities, and (55) will be satisfied by taking this function for $\bar{\zeta}$; but if i be one of the indices $1, 2, \ldots m$, it is evident that the transformation in question will reduce the right-hand member of (56) identically to a_i, so that ϕ will cease to contain any of the variables;

and (55) will then only be satisfied by $\overline{\zeta} = a_i$, in the sense that all the differential coefficients of $\overline{\zeta}$ vanish separately; that is, in the sense that the equation is satisfied by putting $\overline{\zeta} =$ any constant.

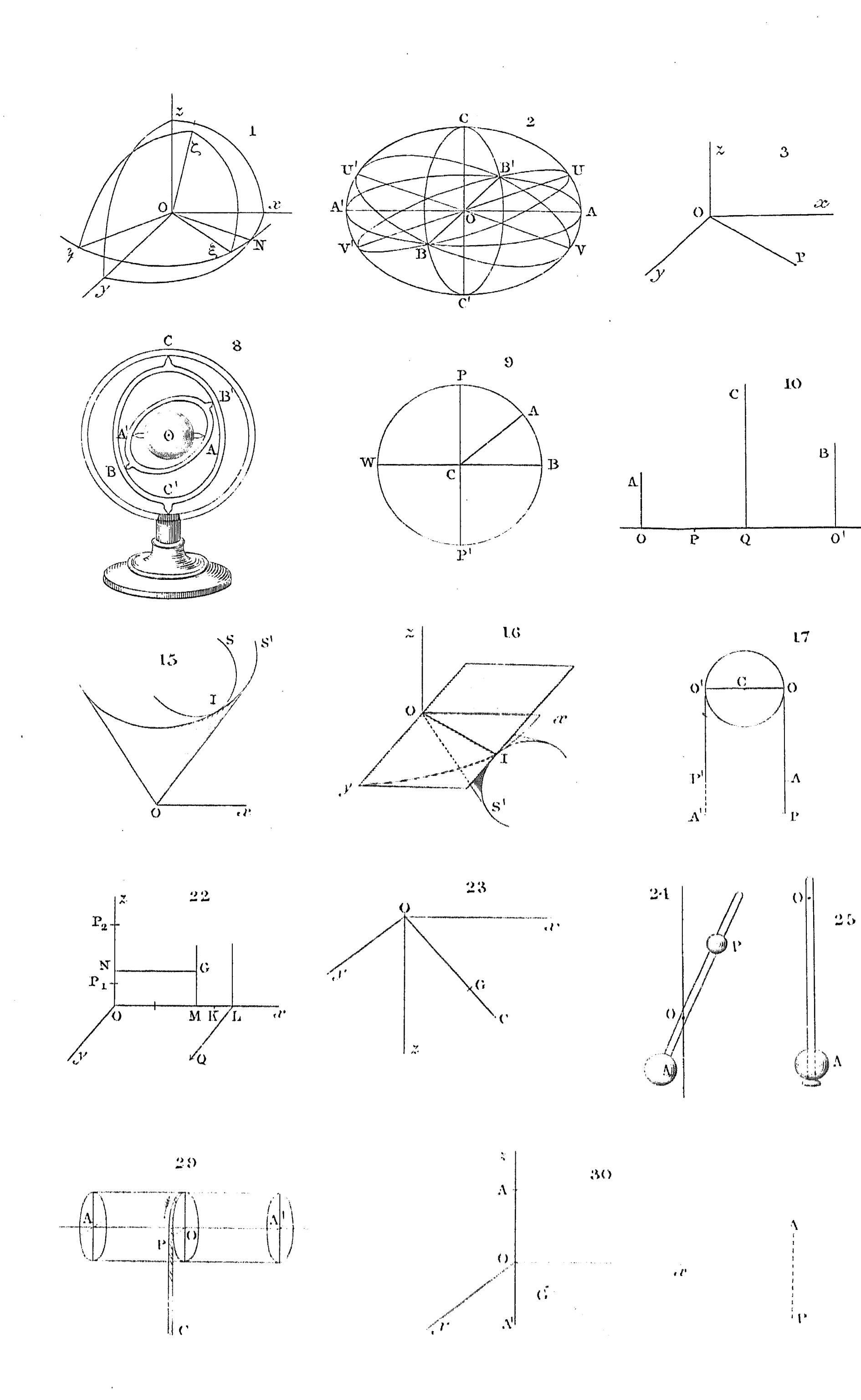
It follows therefore that, in the ordinary sense, (55) is satisfied by $a_{m+1}, \dots a_n, b_{m+1}, \dots b_n$, expressed in terms of the quantities (52), but not by the remaining integrals of the canonical set.

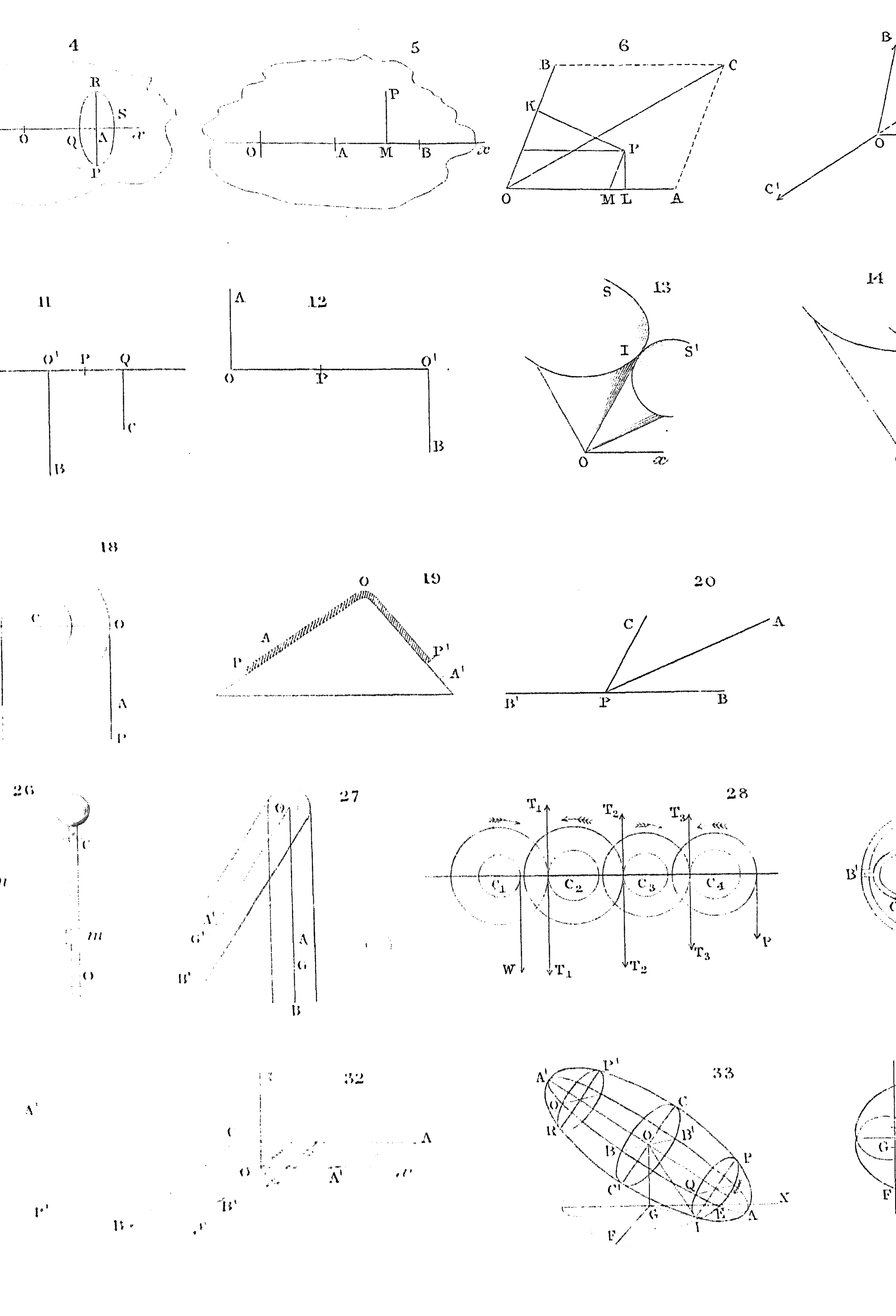
If the distinctive notation of (55) be omitted, and the equation be written at length, the result is the equation (53) of the last Article.

These formulæ require some alteration in the case in which the principle of vis viva subsists; but we have not space to enter further into the subject.

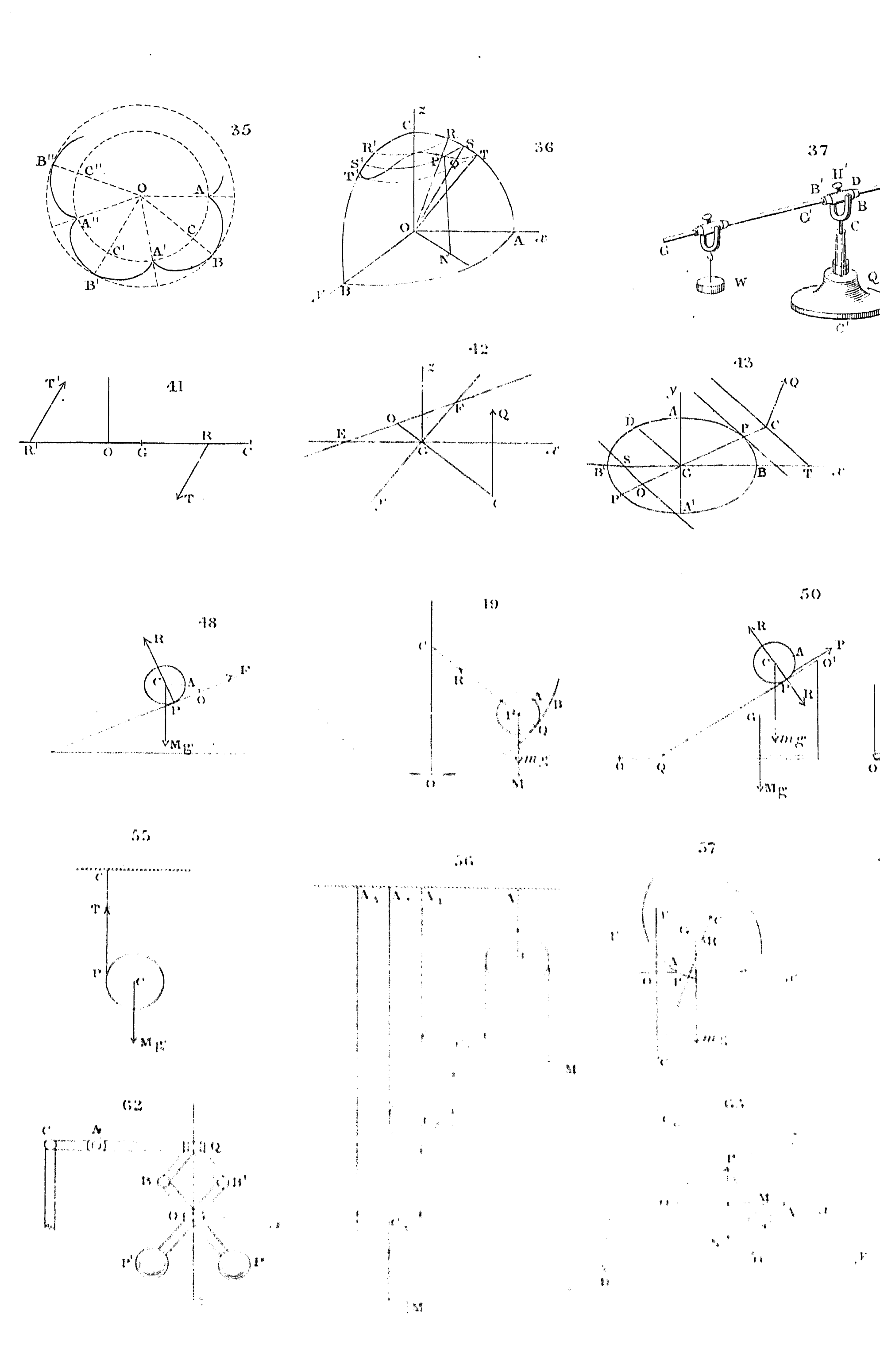
513.] In taking leave of the subject, it is proper to mention the investigations of Jacobi, contained in his Memoirs, entitled, "Theoria nova multiplicatoris systemati æquationum differentialium vulgarium applicandi," Crelle's Journal, Vols. XXVII and XXIX. These investigations are only so far connected with the subject of this Chapter, that they are applicable to the Hamiltonian equations as a particular case. But although the results are interesting and important, they are omitted here because the demonstration depends on the properties of functional determinants, and could not be given without a long digression.

THE END.

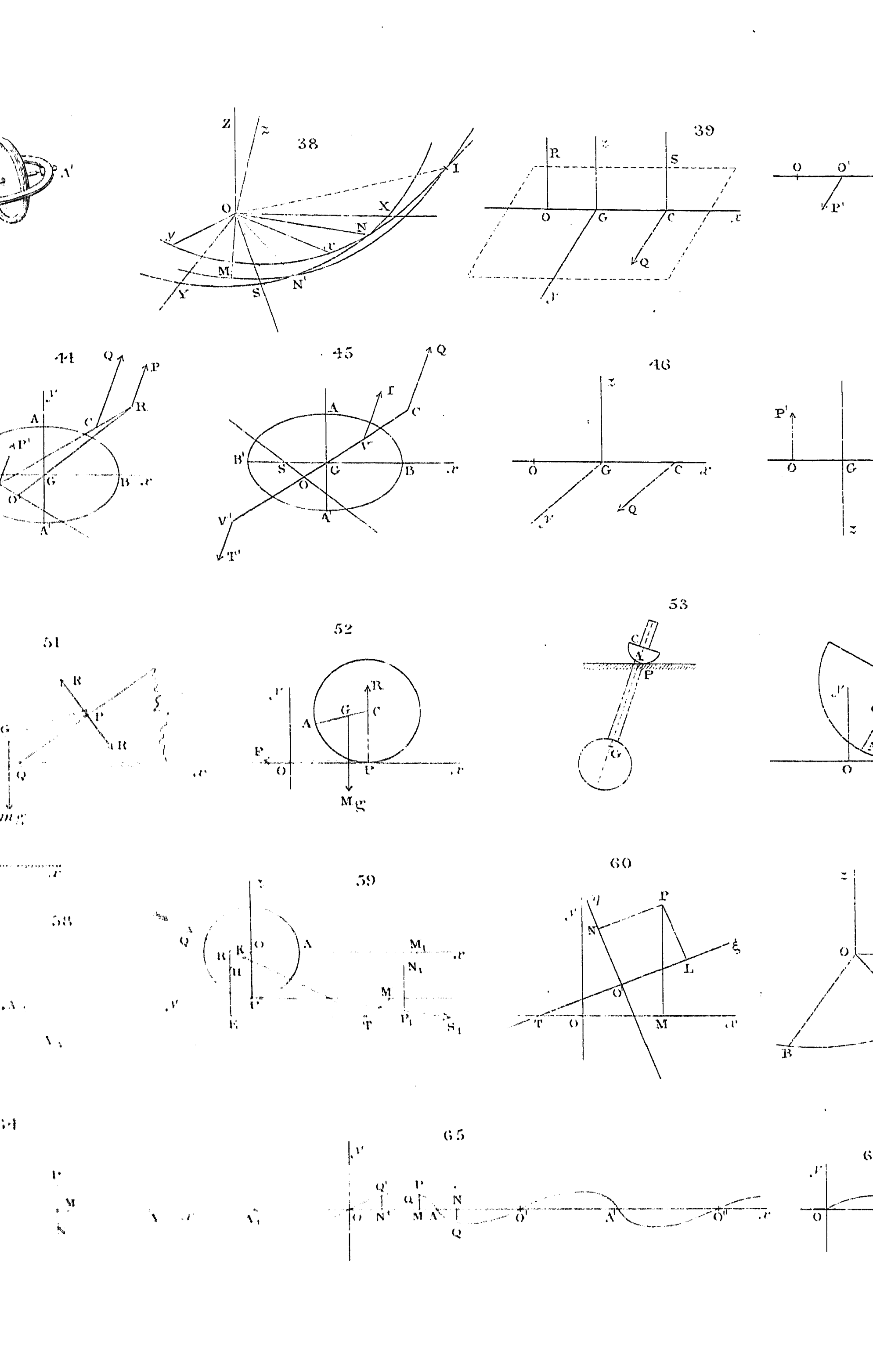




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